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OF

ELECTRICAL ENGINEERING

EDITED BY

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VOLUME I.

THEORY AND CALCULATION
OF
ELECTRIC CURRENTS

Red

BY

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VÄSTERÅS   TRONDHJEM

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PREFACE.

THE present volume is intended to serve as a text-book of that part of the theory of alternating-currents and the allied branches of the theory of electricity, which are necessary for a complete study of heavy electrical engineering. In the first chapters the phenomena in alternating-current circuits are treated at length. For the calculation of alternating-currents the symbolic method has been chiefly used, because this is the simplest and forms the best connecting link with the practical expressions for the watt and wattless components. Alongside the symbolic method, however, the graphic has also been systematically developed by substituting the corresponding graphic constructions for all analytic operations. Thus, expressing the well-known *Kirchhoff's Laws* symbolically, the equation of any circuit appears as the simplest possible analytical expressions, and these formulae at once supply the graphical method for the complete solution of the problem. In this way not only can every problem be expressed mathematically in the simplest possible manner, but also we have the great advantage that the result obtained by the graphical solution shews straight away the behaviour of the circuit under all conditions.

In the following chapters the measurement of electric currents, the magnetic properties of iron and the electric properties of dielectrics are fully dealt with. In the last chapter the constants of electric conductors and circuits are calculated.

The work has been carefully translated by Dr. S. P. Smith, Lecturer at City and Guilds (Engineering) College, London, and late Chief Designer at the General Electric Co., Witton; in addition,

Mr. B. P. Haigh, B.Sc., of the University of Glasgow, has greatly assisted in preparing the matter for the press. The customary English symbols and expressions have been substituted throughout in the text and diagrams.

Great credit is also due to the publishers for their valuable assistance

J. L. LA COUR.

O. S. BRAGSTAD.

February, 1913.

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INTRODUCTORY.

1. Continuous Currents. 2. The Magnetic Field. 3. Electromagnetism. 4. Electromagnetic Induction. 5. Energy, Work and Power. 6. Complex Quantities.

IN this introductory chapter, only the more important laws governing *electromagnetic* phenomena will be summarised. The *electrostatic* laws referred to in the later chapters will be found discussed in Chapter XIX.

1. Continuous Currents. If an electric *difference of potential* (P.D.) exist between the terminals of a conductor, in which there are no electromotive forces (E.M.F.'s) active, a *current* will flow along the conductor from the higher to the lower potential. If the potential-difference is maintained constant, the current-strength will also be constant.

Ohm was the first to prove that, with constant temperature, the current-strength in a conductor is directly proportional to the difference of potential at the terminals of the conductor.

The ratio of the terminal pressure p to the current i is defined as the electric or *ohmic resistance* of the circuit.

Thus $$r=\frac{p}{i}. \qquad (1)$$

The ohmic resistance r of a uniform conductor of constant cross-section is directly proportional to its length l and inversely proportional to its cross-section q, or

$$r=\rho\frac{l}{q};$$

ρ is called the *specific resistance* of the conductor.

In the electromagnetic system of units, r has the dimension

$$r=\text{dim.}\left(\frac{\text{E.M.F.}}{\text{current}}\right)=\text{dim.}\,(LT^{-1}),$$

and is measured in ohms.

Thus, $$\text{ohm}=\frac{\text{volt}}{\text{ampere}}=\frac{10^8}{10^{-1}}=10^9 \text{ C.G.S. units.}$$

Specific resistance is taken as the resistance between two opposite faces of a cube, the length of whose side is one cm. Expressed thus, the specific resistance of copper is:

$$\rho = 1{\cdot}6 \,.\, 10^{-6}(1 + 0{\cdot}004T^\circ)\Omega = 1{\cdot}6(1 + 0{\cdot}004T^\circ) \text{ microhms},$$

and of aluminium,

$$\rho = 2{\cdot}7(1 + 0{\cdot}004T^\circ) \text{ microhms},$$

where T denotes the temperature in degrees Centigrade and the factor 0·004 is the *temperature-coefficient*—this latter happens to be the same for both copper and aluminium.

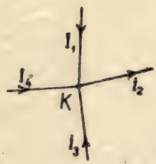


Fig. 1.

Ohm's Law can be written thus: *In any part of a circuit in which no active E.M.F. is present, the current equals the ratio of the fall of potential along this part of the circuit to its resistance.*

Now, since electricity cannot accumulate at a point K, where several wires are joined (Fig. 1), it is obvious that the sum of the currents flowing towards the junction must equal the sum of those flowing away from it. Regarding, then, the latter as positive and the former as negative, we have for every junction such as K:

$$\Sigma(i) = 0. \quad \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

This is known as *Kirchhoff's First Law*, which may be written: *The algebraic sum of all currents flowing towards or away from any junction is zero.*

Consider, now, a closed electric circuit, as for example in Fig. 2, in which the potentials of the several points are denoted by the letters P_1 to P_4. Then, according to Ohm's Law,

$$\begin{aligned} i_1r_1 &= e_a + P_1 - P_2, \\ i_2r_2 &= \phantom{e_a +{}} P_2 - P_3, \\ i_3r_3 &= e_b + P_4 - P_3, \\ i_4r_4 &= \phantom{e_a +{}} P_1 - P_4. \end{aligned}$$

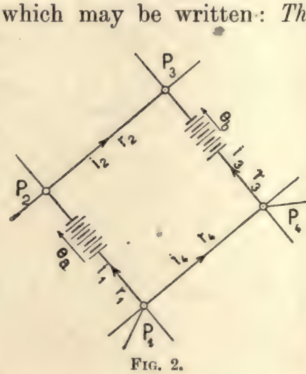


Fig. 2.

Taking the algebraic sum of ir, by regarding the current direction as positive when clockwise and as negative when counter-clockwise, we get:

$$\Sigma(ir) = i_1r_1 + i_2r_2 - i_3r_3 - i_4r_4 = e_a - e_b = \Sigma(e).$$

Thus $$\Sigma(ir) = \Sigma(e). \quad \dots\dots\dots\dots\dots\dots\dots\dots (2a)$$

This is *Kirchhoff's Second Law*, which reads as follows: *In every closed circuit, the algebraic sum of the products of the current and resistance in*

the several parts of the circuit equals the algebraic sum of the E.M.F.'s acting in the circuit.

If we consider the phenomena in an electric conductor from the electrostatic standpoint, then the current i corresponds to the passing of a quantity of electricity i per second from a point at potential P_1 to a point at lower potential P_2. By moving unit positive electric charge from potential P_1 to P_2, the work done by the electric forces is $P_1 - P_2$; hence the work done when current i flows for time t is

$$A = i(P_1 - P_2)t.$$

This energy is converted into heat. The work done in unit time is the *power* (W), whence

$$\mathrm{W} = i(P_1 - P_2) = i^2 r. \quad \text{(3)}$$

This law was first demonstrated experimentally by Joule, and reads: *The amount of heat produced in a conductor by a constant current in unit time varies directly as the resistance of the conductor and as the square of the current flowing in it.*

If a constant current i flows in a circuit in which an E.M.F. e—produced by a battery or generator—is acting, the work done per second equals ei, and we can say in general that in any part of a circuit where the E.M.F. e is present and a current i is flowing, the power $\mathrm{W} = ei$ will be given out. When e and i have the same direction, this energy must be supplied from the external sources which produce the current. When, on the contrary, e and i oppose one another, work will be done by the current and can be used outside in the form of mechanical or chemical energy, and so on.

2. The Magnetic Field. The space in which magnetic actions can be observed is called a magnetic field. Without forming any special hypothesis about the nature of magnetism, it is nevertheless possible to speak of a quantity of magnetism, or of *magnetic masses* which can be regarded as mathematically definite quantities, the magnitude of which can be determined by the forces they exert. Like magnetic masses repel, unlike attract one another.

Though actually there is no such thing as free magnetism, it is often convenient to substitute for magnetic fields, magnetic masses which are assumed capable of acting at a distance. For instance, the field of a long bar magnet can be replaced, with close approximation, by imaginary magnetic masses situated at two points symmetrically placed with regard to the axis of the magnet. These points—known as the poles of the magnet—are from 0·8 to 0·85 of the axial length apart.

The force exerted by two magnetic masses, each concentrated at a point, on one another, is expressed by *Coulomb's Law*,

$$K = f\frac{m_1 m_2}{r^2}, \quad \text{(4)}$$

where r is the distance between the two masses and f is a coefficient depending on the system of units and on the medium.

In the electromagnetic system of units (C.G.S. system) and for a gaseous medium or vacuum, $f=1$. The mechanical force K has the dimension

$$\dim.(K) = \dim.(LMT^{-2}),$$

and is measured in $\frac{\text{cm gm}}{\text{sec}^2}$ in absolute units.

The *unit of mechanical force is the dyne*, and is defined as that force which gives unit acceleration to unit mass.

The practical unit of force is a kilogram weight, 1 kg = 981000 dynes. The dimension of the product m_1m_2 is

$$\dim.\, m_1m_2 = Kr^2 = \dim.(L^3MT^{-2});$$

consequently magnetic mass has the dimension:

$$\dim.(m) = \dim.(L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}).$$

Unit magnetic mass is defined as that mass which, when placed in air, exerts a force of one dyne on a similar mass at a distance of 1 cm.

In general, the points in a field where magnetic masses appear to be concentrated are designated *poles.* Unit magnetic mass in a magnetic field is acted on by a mechanical force H. This force H is defined as the *field-strength* or *-intensity*, and has the dimension:

$$\dim.\left(\frac{\text{mechanical force}}{\text{magnetic mass}}\right) = \dim.\left(L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\right).$$

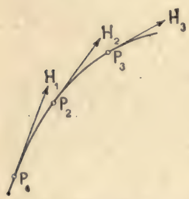


FIG. 3.—Line of Force.

By a *line of force* is understood that line the tangent to which at any point coincides in direction with the field-strength at any point (Fig. 3).

Lines of force can be represented by means of iron filings strewn on

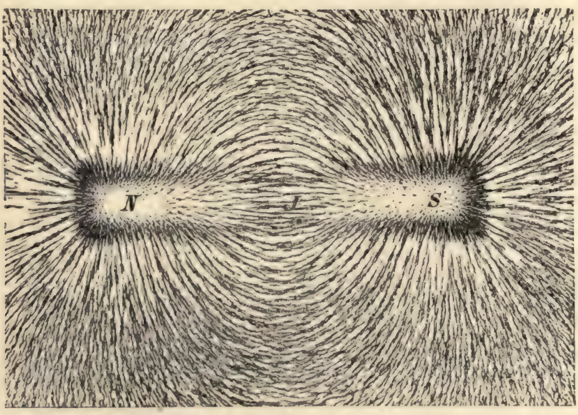


FIG. 4.—Field of Bar Magnet.

a sheet of paper placed in the plane of the field. The filings then arrange themselves in lines which approximate in direction to the lines

of force. Fig. 4 is from a photograph taken with a bar magnet, whilst Fig. 5 shows the lines of force of a horse-shoe magnet.

Constant magnetic forces have a potential, which, at any point in the field, is given by:

$$P = \Sigma\left(\frac{m}{r}\right), \qquad \ldots\ldots\ldots(5)$$

where m is the magnetic mass of the field and r the distance from the point considered. The summation is taken for all the magnetic masses producing the field.

A surface which, at every point, is perpendicular to the direction of the field is called an *equipotential surface.* Such a surface is the locus of all points having the same potential.

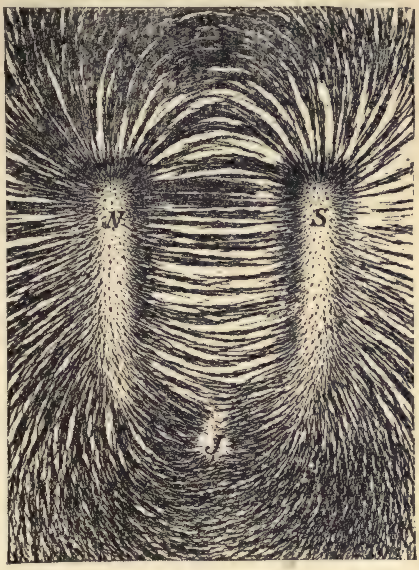


FIG. 5.—Field of Horse-shoe Magnet.

The *element of magnetic flux* passing through a surface-element is the product of the surface-element df and the normal component H_n of the field-strength, that is (Fig. 6):

$$d\Phi = H_n\,df = H\cos\alpha\,df,$$

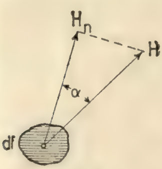


FIG. 6.

and

$$H_n = \frac{d\Phi}{df}.$$

If we split up any desired surface F into surface-elements and take the sum of the fluxes passing through the several elements, we get *the magnetic flux* Φ *passing through the surface* F;

or,

$$\Phi = \Sigma_F H\cos\alpha\,df$$

$$= \int_F H\cos\alpha\,df = \int_F H_n\,df.$$

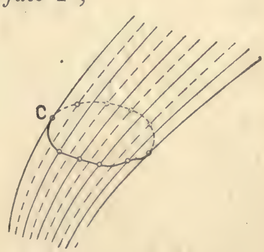


FIG. 7.—Tube of Force.

A *magnetic tube of force* (Fig. 7) is defined as the space which is bounded by lines of force passing through a closed curve C. If we draw a number of surfaces through any point in the tube, then the same flux will pass through all sections which the tube of force makes with these surfaces; for, in an infinitely small tube, for any section, we have

$$d\Phi = H_n\,df = H\cos\alpha\,df = H\,df_n,$$

where df_n denotes the section which the tube cuts on the equipotential-surface at the point considered.

Gauss and Green's Theorem can be deduced directly from Coulomb's Law, and may be written: *The total number of lines of force Φ passing through any closed surface F equals 4π times the sum of the magnetic masses m within that surface.* From this it follows the flux has the same dimension as magnetic mass.

$$\int_F H_n df = \Phi = 4\pi\Sigma(m). \quad \text{.........................(6)}$$

Since no flux can pass through the boundary of a tube of force, it follows, from Gauss and Green's Theorem, that the flux passing through any section of a tube is quite independent of the position of the section, i.e. *the flux inside a tube of force is constant.* A tube enclosing the flux $\Phi = 1$ (C.G.S. units) is defined as a *unit tube of force*, and any tube of force may be said to contain a certain number of unit tubes. In a strong field, the unit tubes have a very small cross-section. The field-strength at a point denotes the number of unit tubes of force of like section which pass through a square centimetre at the place in question.

The above properties of the magnetic field hold in general for a homogeneous medium, as for instance a vacuum. If a body be

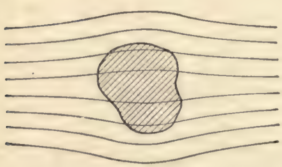


FIG. 8.—Weakening of Magnetic Field due to Introduction of Diamagnetic Body.

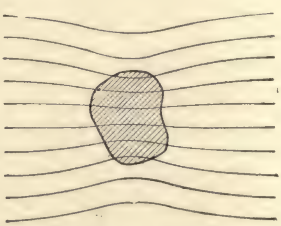


FIG. 9.—Strengthening of Magnetic Field due to Introduction of Paramagnetic Body.

brought into a vacuum where a magnetic field exists, the field in the body and its neighbourhood will, in general, change in shape and strength. If the field is weakened, i.e. if the tubes of force are widened out, the body is called *diamagnetic* (Fig. 8); if the field is strengthened, i.e. if the tubes are contracted, the body is called *paramagnetic* (Fig. 9), whilst if the field becomes strongly concentrated, the body is said to be *ferro-magnetic.*

The magnetic conductivity of a substance is called its *permeability*, and is denoted by μ.

When a body is brought into a magnetic field, it is said to be magnetised by *induction*, and the ratio

$$\frac{d\Phi}{df} = B$$

is called the *magnetic induction* or the *flux density*. $d\Phi$ is the flux passing through the elemental section df of an equipotential surface in the body.

In a ferro-magnetic substance situated in a uniform field, take two cylindrical cavities, whose axes lie in the direction of the magnetic force. The one cavity (Fig. 10a) is a narrow canal, so long that it may be considered as a tube of force, since the lines of force are parallel to the axis. If we bring unit magnetic mass into this cavity, in order to test the magnetic conditions, it will be acted on by a force equal to the field-strength H at this point; this force is much smaller than

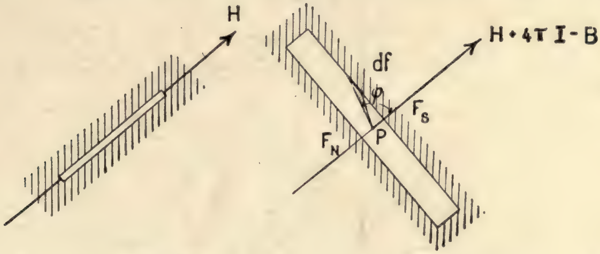


FIG. 10a. FIG. 10b.

Field-strength and Induction inside a Ferro-magnetic Body.

the above defined induction B; whence it follows that the magnetic force inside a ferro-magnetic substance or a magnet is not the same as that $\frac{d\Phi}{df}$ in a vacuum, but is defined thus:

The magnetic force, or field-strength, at a point inside a magnet is the force which acts on unit magnetic mass when placed at this point, when the same is taken in an infinitely thin cavity cut in the direction of the lines of magnetisation.

The second cavity (Fig. 10b) is an infinitely thin crevasse perpendicular to the direction of the magnetic force. The unit mass, when brought into this crevasse, will be acted on by the force B, although the magnetic force inside the magnet is, as shown, only H. In order to explain this phenomenon, we imagine the two end-surfaces F_N and F_S to be respectively charged with north and south magnetism. These magnetic masses exert a force on the unit mass at point P, which can be calculated from Coulomb's Law.

Denote the magnetic density of the two charges by $+I$ and $-I$. Then the force exerted on P by a surface-element df is $\frac{I\,df}{r^2}$. This can be split up into two components—one in the direction of the magnetic force, and the other normal to it. Component forces normal to the magnetic force obviously neutralise one another, whilst the resultant in the direction of the magnetic force is:

$$\frac{I\,df}{r^2}\cos\phi = I\,d\omega,$$

where $d\omega$ is the solid angle subtended by df at P. Summing up the components of all surface-elements of the surface F_N in the direction of H, we get

$$\int_{F_N} I\,d\omega = 2\pi I,$$

when the surface F_N is large compared with the height of the cylinder. The same result is obtained by considering the surface F_S, so that the resultant magnetic force of the two surface-charges is $4\pi I$, and as resultant force on the unit mass in the crevasse, we get

$$H + 4\pi I = B,$$

where I is defined as the *intensity of magnetisation.* I is also—as above assumed—equal to the *density* of the surface-charges assumed to exist on the boundary surfaces F_N and F_S.

The magnetic permeability is

$$\mu = \frac{B}{H},$$

and has the dimension of a number.

Consequently the magnetic induction B and the field-strength H have the same dimension. The unit of this dimension in the electromagnetic system of units is called a *Gauss.*

A distinction must be made between the H-flux and the B-flux. The B-flux, i.e. the flux due to induction, which passes through a closed surface F, is independent of the magnetic nature of the medium in which the surface is taken, that is, Gauss' theorem is, in general:

$$\int_F \mu H_n df = 4\pi\Sigma(m), \quad \text{..........................(6a)}$$

or, in other words, *the B-flux remains constant in passing from one medium to another.*

Take two points close to the boundary surface between the two substances K_1 and K_2 (Fig. 11). Then, since the B-flux remains the same in passing from one medium to the other, we have

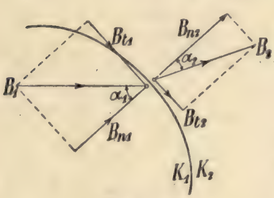


Fig. 11.

$$B_{n_1} = B_{n_2} \quad \text{or} \quad \mu_1 H_{n_1} = \mu_2 H_{n_2}.$$

If $\mu_1 \gtrless \mu_2$, then $H_{n_1} \lessgtr H_{n_2}$, that is, *in passing from one medium to the other, the components of the magnetic force, taken normal to the boundary surface, are discontinuous.*

The tangential components of the magnetic force are continuous in passing from one medium to another, that is,

$$H_{t_1} = H_{t_2},$$

whence

$$\frac{B_{t_1}}{B_{t_2}} = \frac{\mu_1}{\mu_2}.$$

i.e. *in passing from one medium to another the tangential components of the B-flux are discontinuous.*

From Fig. 11, $$\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\mu_2}{\mu_1}.$$

Hence, *in passing from one medium to another the induction or B-tubes are discontinuous.* In substances of high permeability therefore, like iron, the tubes enter and leave almost perpendicularly to the surface.

In order to treat magnetic problems mathematically in spite of the discontinuity of the H-tubes, we assume the boundary surface between the two bodies to be replaced by magnetic surface-charges from which tubes enter and leave. Where the flux passes out of a medium of higher permeability, e.g. iron, these magnetic charges have the positive sign (north-pole magnetism); and where it enters a medium of higher permeability, the negative sign (south-pole magnetism). Such imaginary charges are called poles.

3. Electromagnetism. A magnetic field is most easily produced by means of an electric current. *Oersted* was the first to discover that an electric current acted on a freely-suspended magnetic needle by tending to bring the same into a direction perpendicular to that of the current. According to the elemental-law of Laplace, the mechanical force K exerted by a current-element on the magnetic mass m at a distance r is:

$$K = \frac{mi\,ds}{r^2} \sin \phi. \qquad (7)$$

This force has a direction normal to the plane passing through the element ds and the mass m (Fig. 12a). Conversely, the current-element is acted on by the magnetic mass in the opposite direction.

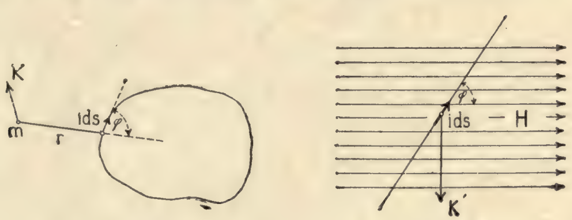


FIG. 12a. FIG. 12b.

Electromagnetic Forces.

Every electric current produces a magnetic field, which surrounds the conductor in which the current flows, and acts on all magnetic masses in the neighbourhood; conversely, every conductor which carries a current is acted on by a mechanical force when brought into a magnetic field. This force is expressed by

$$K' = Hi\,ds \sin \phi, \qquad (7a)$$

where ϕ denotes the angle between the current-element ds and the direction of the field H (Fig. 12b).

As mentioned above, the field at any point due to a current-element is perpendicular to the plane passing through the element and

the point considered. The *direction* of this field can at once be found from the following rule:

Place the palm of the right *hand along the conductor so that the fingers point in the direction in which the current is flowing—then the thumb points in the direction of the field-strength H at the point P* (Fig. 13).

If the conductor (Fig. 12*b*) is movable, it would be displaced by the force K' in the direction as given by following rule:

Place the left *hand along the conductor so that the flux enters the palm of the hand and the fingers point in the direction of the current—the thumb will then give the direction in which the conductor will tend to move.*

This rule can be used for determining the direction of rotation in the case of a motor.

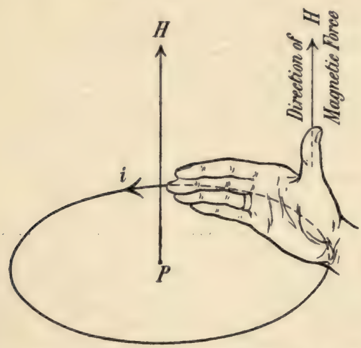


FIG. 13.—Determination of Direction of Field due to Electric Current.

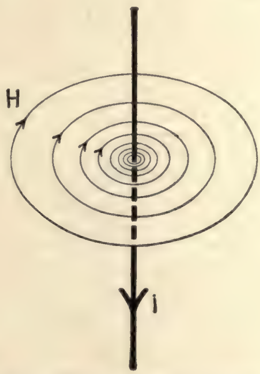


FIG. 14.—Magnetic Field produced by Current in a Straight Wire.

From formula (7) it is clear that the lines of force produced by a *straight-line current* (Fig. 14) are concentric circles, lying in planes normal to the conductor, and that the field-strength H at any point r cm away from the conductor is

$$H=\frac{2i}{r}.$$

For a *circular current* (Fig. 15) the field at the centre is

$$H=\frac{2\pi i}{R},$$

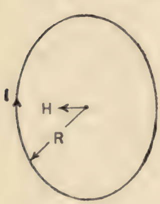


FIG. 15.

where R = radius of the circle.

From this we can express the dimension of current in electro-magnetic units,

$$i=\text{dim.}(\text{length}\times\text{field-strength})$$
$$=\text{dim.}(L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1});$$

and in the same system of units, unit current is that current which—flowing in a circle of unit radius—produces a field-strength 2π at the centre. An *ampere* is $\frac{1}{10}$ of this unit.

At the centre of a long *solenoid* (Fig. 16), the strength of the field is

$$H_m = \frac{4\pi i w}{\sqrt{L^2 + D^2}},$$

where w = no. of turns of the solenoid and i = current in each turn, measured in absolute units.

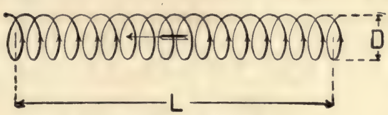


FIG. 16.—Solenoid.

When $\frac{D}{L}$ is small, the field-strength may be written:

$$H = \frac{4\pi i w}{L},$$

and is nearly constant at all points inside the solenoid. When the current is measured in amperes, we get

$$H = \frac{0{\cdot}4\pi i w}{L} = \frac{1{\cdot}25 i w}{L} = \frac{iw}{0{\cdot}8L};$$

iw is called the *ampere-turns* of the solenoid, and is of late referred to as the *magnetomotive force.**

This formula is still more exact if the solenoid be closed (Fig. 17) to form a ring.

The work done in carrying unit quantity of magnetism, placed inside this ring, round one complete turn of length L against the force H, is

$$HL = 0{\cdot}4\pi i w.$$

If the unit quantity is moved over any closed curve C, the work done is equal to the sum over the whole circuit of all the work-elements $H\,dl$, i.e.

$$\Sigma_c H\,dl = \int_c H\,dl.$$

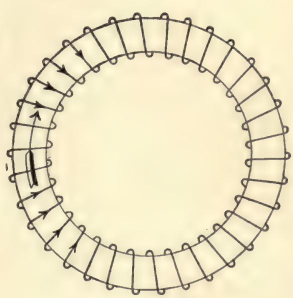


FIG. 17.—Simple Magnetic Circuit.

This summation is called *the line-integral of the magnetic force H over the curve C, and is equal to* $0{\cdot}4\pi$ *times the sum of all the ampere-turns linked with the curve C.*

Thus, $$\int_c H\,dl = 0{\cdot}4\pi i w. \quad \text{.............................(8)}$$

Of recent years, it has been customary to start from this as the

* This must not be confused with the obsolete conception of magnetomotive force (M.M.F.), which is used to denote $1{\cdot}25 iw$, i.e.

M.M.F. = $1\frac{1}{4}$ amp.-turns.

fundamental law of electromagnetism and not from the differential equation in formula (7); the former can be deduced from the latter.

Let the toroid in Fig. 17 have an iron core, and let a current pass through the coils, which are wound evenly on the core. Then at all points equidistant from the axis of the ring there will be—on account of symmetry—the same magnetic force; and, corresponding to this force H, there will be the induction B. Hence the tubes of induction produced by the current are concentric and have their path inside the ring. The whole body will be magnetically neutral to all other bodies, i.e. there are no poles, and is therefore termed a *closed magnetic circuit.*

Magnetic circuits as a rule have not a constant section as in the case of the above ring, and have not the same material throughout, so that the permeability varies from point to point.

Consider, however, one tube of induction of a magnetic circuit—we know that the flux Φ_x in the tube is constant, and practically symmetrically distributed over the small surface f_x; then

$$\Phi_x = Bf_x$$

and

$$B = \mu H;$$

hence

$$H = \frac{B}{\mu} = \frac{\Phi_x}{\mu f_x},$$

whence it follows that

$$\frac{iw}{0{\cdot}8} = \int_c H\,dl = \int_c \frac{\Phi_x}{\mu f_x} dl = \Phi_x \int_c \frac{dl}{\mu f_x}$$

or

$$\frac{iw}{\Phi_x} = \int_c \frac{0{\cdot}8 dl}{\mu f_x} = R_x,$$

where R_x is called the *magnetic resistance or reluctance* of the tube of force under consideration.

$$\lambda_x = \frac{1}{R_x},$$

is the *magnetic permeance* of the tube and has the dimension of a length. If several tubes are interlinked with the same ampere-turns, the permeance of all the tubes can be added and the reluctance R of the total magnetic circuit with which the ampere-turns iw are interlinked is

$$R = \frac{1}{\Sigma\lambda_x}.$$

The total flux in the circuit is then

$$\Phi = \Sigma\Phi_x = iw\Sigma\lambda_x = \frac{iw}{R},$$

or

$$\text{flux} = \frac{\text{ampere-turns}}{\text{reluctance}} \quad \text{.........................(9)}$$

The electromagnetic unit of flux is called a *weber.* Formula (9) is similar to Ohm's Law for electric currents. From this formula and

the fact that tubes of induction possess constant flux, it follows that Kirchhoff's two laws hold for magnetic circuits.

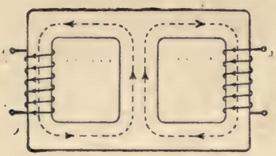


FIG. 18*a*.

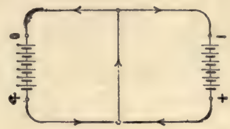


FIG. 18*b*.

Comparison between Interlinked Magnetic Circuits and Interlinked Electric Circuits.

Fig. 18*a* shows two interlinked circuits for which these laws hold, the magnetic circuits corresponding to the electric circuits of Fig. 18*b*.

4. Electromagnetic Induction. When a conductor forms a closed circuit in a magnetic field which is *varying*, an E.M.F. will be induced in the circuit. This phenomenon, discovered by *Faraday*, is known as *electromagnetic induction.* On the basis of Faraday's researches, *Maxwell* formulated the fundamental law of electromagnetic induction, which experience has completely verified. This law can also be developed from the fundamental laws of electromagnetism and the principle of the conservation of energy. Maxwell's Law is as follows:

The E.M.F. e induced in a closed conductor C equals the rate of change of the flux Φ which is interlinked with the conductor C.

Thus
$$e = -\frac{d\Phi}{dt}. \qquad (10)$$

The current produced in the circuit C by this induced E.M.F. is called an *induced* current, and the field which induces the E.M.F. is called the *inducing* field. The change of flux can take place in various ways, e.g. by a change of field-strength, whilst the conductor retains its position, or by a change of position of the conductor in a constant field. In the first case the direction of the current is always such as to oppose the change in the field-strength—hence the negative sign in formula (10). By means of the hand-rule, we get the directions of the induced E.M.F.'s as in Figs. 19*a* and *b* for increase and decrease of the field-strength. In the second case the E.M.F. is induced by a relative displacement of the conductor in the field.

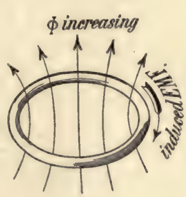


FIG. 19*a*.

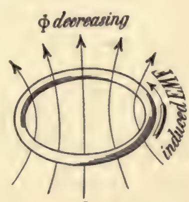


FIG. 19*b*.

When only a part of the conductor is in the field it is easier to determine the induced E.M.F. by means of the elemental-law of electromagnetic induction. Such a law cannot be proved, and it must suffice that from this the fundamental law can be deduced.

This elemental law is as follows:

If an element ds of a circuit be moved in a magnetic field, an E.M.F. will be induced equal to the flux cut by ds in unit time, i.e.

$$de = -\frac{d\Phi_{ds}}{dt}. \quad \text{.............................(11)}$$

To determine the positive direction of the induced E.M.F. the following hand-rule is convenient:

Place the right *hand in the magnetic field so that the flux enters the palm and the thumb points in the direction in which the conductor moves—the fingers will then point in the direction of the induced E.M.F. (or of the current), as in* Fig. 20.

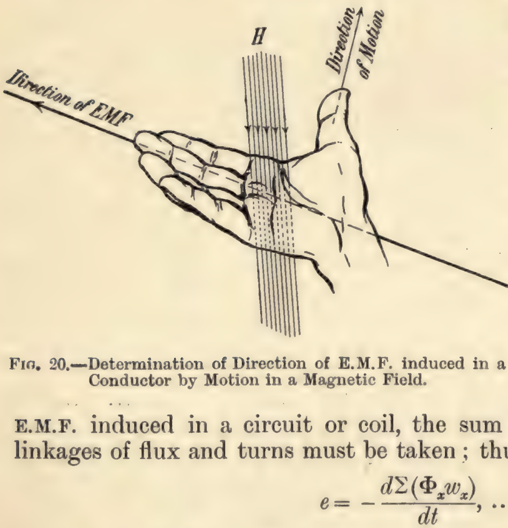


FIG. 20.—Determination of Direction of E.M.F. induced in a Conductor by Motion in a Magnetic Field.

Often the circuit C is not a simple curve, but consists of several turns, some of which do not embrace the total flux. In every case, the E.M.F. induced in a turn is proportional to the change of flux in that turn. Hence, to find the total E.M.F. induced in a circuit or coil, the sum $\Sigma(\Phi_x w_x)$ of all the interlinkages of flux and turns must be taken; thus, in general,

$$e = -\frac{d\Sigma(\Phi_x w_x)}{dt}, \quad \text{...........................(10a)}$$

that is, *the E.M.F. induced in a circuit equals the rate of change of the number of interlinkages of the flux with the circuit.*

E.M.F. has the dimension

$$\dim. e = \dim. \left(\frac{\text{field-strength} \times \text{surface}}{\text{time}}\right)$$

$$= \dim. (L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}).$$

The absolute unit of electromotive force is that E.M.F. which is induced in a circuit when the number of interlinkages is altered by unity in unit time. The practical unit has been chosen equal to 10^8 times this absolute unit, and is called a *volt*; hence

$$e = -\frac{d\Sigma(\Phi_x w_x)}{dt} 10^{-8} \text{ volts.}$$

5. Energy, Work and Power. Every mechanical system of forces possesses a certain potential energy. Such a system always tends

to assume a position of equilibrium, in which the potential energy will be a minimum. When the potential energy is decreased, work is done by the system; when the potential energy is increased, energy is taken from outside, i.e. work is given to the system.

Electromagnetic forces also possess potential energy, which can be determined from the fundamental law of electromagnetism. The potential energy of an electric current i interlinked with a magnetic field Φ independent of the current is $-i\Phi$, where Φ is the flux interlinked with the current i, the direction of the flux being the same as that of the flux due to the current (Fig. 21). If the conductor carrying the current i is displaced, or the field is varied, so that the interlinked flux changes from Φ_1 to Φ_2, the forces exerted by the field on the current will perform an amount of work A equal to the change of potential energy in the system.

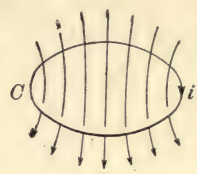


FIG. 21.

Thus, $$A = i(\Phi_2 - \Phi_1).$$

According as Φ_2 is greater or less than Φ_1, the energy of the system decreases or increases, and the work is done by the field forces or against them.

If the current is kept constant and the flux varied, the work done by the field on the current in the time-element dt is

$$dA = i\,d\Phi,$$

and the power exerted by the field at this instant will be

$$W = \frac{dA}{dt} = i \cdot \frac{d\Phi}{dt}$$

or $$W = -ei, \qquad (12)$$

where e is the E.M.F. induced in the direction in which the current flows.

If the flux Φ is increased, i.e. if $d\Phi$ is positive, an E.M.F. e will be induced which will tend to weaken the flux by opposing the current. Thus W is positive and work is done by the field. This is the case of a motor. On the other hand, if the flux Φ is decreased, an E.M.F. will be induced in the same direction as the current i and the power W is negative. The work is thus done against the field, and we have a generator. *We thus see that the current and induced E.M.F. have the same direction in a generator and opposite directions in a motor.*

From formula (12) and from section 1, it is seen that the work *supplied to* a circuit in the element of time dt is always

$$dA = ei\,dt, \qquad (13)$$

where e and i are to be taken positive when they have the same direction.

If current and E.M.F. have constant magnitudes, as is the case with continuous currents, the supplied power is

$$W = ei.$$

If the circuit is not a simple one, as in Fig, 21, but has several complicated branches, then the potential energy of this system is

$$-\Sigma i\{\Sigma(w_x\Phi_x)\},$$

where $\Sigma(w_x\Phi_x)$ denotes the number of interlinkages of tubes of force with the current i. The product of current and interlinkages $\Sigma(w_x\Phi_x)$ must be taken for each current of the system and the sum of the whole found.

If the circuit is movable in space, the electrodynamic forces which act on it tend to make the potential energy of the system a minimum. Conversely, if the circuit is fixed in space the distribution of the flux will be such that the number of interlinkages of tubes of force tends to become a maximum.

When the flux of the magnetic field is not independent of the current in the electric circuit, but its reluctance constant, then the potential energy of such a system is

$$-\tfrac{1}{2}\Sigma i\{\Sigma(w_x\Phi_x)\}.$$

The simplest form of such an electromagnetic system is an electric circuit together with the magnetic field produced by the current in the circuit. The energy which is necessary for the production of the magnetic field of the circuit is equal to the potential energy with opposite sign. Let us calculate this energy. The variation of the energy in the time dt is

$$dA = -ei\,dt = id\Sigma(w_x\Phi_x).$$

If the reluctance of the field is constant, Φ_x is proportional to i, and by integration we obtain

$$A = \int id\Sigma(w_x\Phi_x) = \tfrac{1}{2}i\Sigma(w_x\Phi_x),$$

which is the magnetic energy of an electric circuit with constant reluctance. Substituting in this formula the relation $\int H\,dl = 4\pi i w_x$, the energy of the field per unit volume is expressed by

$$A = \int\frac{H\,dB}{8\pi} = \int H\,dI,$$

which is quite analogous to the expression for the work of deformation in a purely elastic body. This formula for the field energy per unit volume holds quite generally for all magnetic fields.

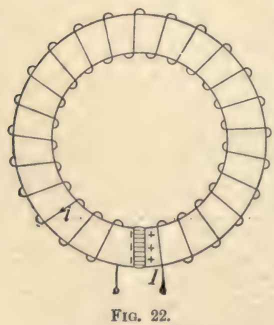


FIG. 22.

If an iron ring with an air gap, as shown in Fig. 22, is magnetised by means of a continuous current, the energy supplied to it will be $\tfrac{1}{2}i\Sigma(w_x\Phi_x)$, which will be stored in the magnetic circuit. This energy exerts a force on the magnetic circuit, which strives to reduce the reluctance of the latter. In the present case this could be accomplished by decreasing the air gap. The magnetic charges which we can suppose to exist on the boundary surfaces possess opposite polarity and attract one another. Thus the force of attraction between these two surfaces stresses the whole

ring like a spring, which condition only ceases when the current, and with it the magnetism and stored energy, disappears.

The attractive force between the two surfaces Q may be easily calculated. The magnetic charge on a surface exerts a force of $2\pi I$ on each of the IQ units of the opposite surface. Consequently, the force of attraction is

$$K = 2\pi I^2 Q,$$

or, if we put

$$B \simeq 4\pi I,$$

then

$$K = \frac{B^2 Q}{8\pi} = \frac{B\Phi}{8\pi} \text{ dynes.}$$

Power has the dimension

$$\text{dim.}(\text{power}) = \text{dim.}(\text{E.M.F.} \times \text{current}) = \text{dim.}(L^2 M T^{-3}).$$

The practical unit of power in the C.G.S. system is a *watt*.

Watt = volt × ampere = $10^8 \times 10^{-1} = 10^7$ units of power in the electromagnetic system.

The unit of work in the electromagnetic system is the *erg*:

$$1 \text{ erg} = 1 \text{ cm dyne};$$

and the practical unit is the *joule*:

$$1 \text{ joule} = 10^7 \text{ ergs}.$$

Thus the power of one watt corresponds to one joule per sec. The engineer's unit of work is the *kilogramme-metre* (kgm) or the *foot-pound* (ft.-lb.).

Since $1 \text{ kg} = 2{\cdot}205 \text{ lbs.} = 981000 \text{ dynes}$

or $1 \text{ lb.} = 0{\cdot}453 \text{ kg.} = 444000 \text{ dynes},$

and $1 \text{ metre} = 3{\cdot}28 \text{ ft.} \quad \text{or} \quad 1 \text{ ft.} = 30{\cdot}5 \text{ cm.},$

then $1 \text{ kgm} = 981000 \,.\, 100 \text{ ergs} = 9{\cdot}81 \text{ joules}$

and $1 \text{ ft.-lb.} = 444000 \,.\, 30{\cdot}5 \text{ ergs} = 1{\cdot}355 \text{ joules}.$

The practical unit of power is known as a *horse-power*.

The horse-power in the metric system as used on the Continent is

$$1 \text{ P.S.} = 75 \text{ kgm per second} = 75 \,.\, 9{\cdot}81 = 736 \text{ watts};$$

and in the English system,

$$1 \text{ H.P.} = 550 \text{ ft.-lbs. per second} = 550 \,.\, 1{\cdot}355 = 746 \text{ watts}.$$

The unit of heat is the *calorie*, and is equal to the mean amount of heat required to raise the temperature of unit mass of water by one Centigrade degree.

The *small* or *gm-calorie* is equivalent to 0·428 kgm; thus a gm-calorie is equivalent to 4·2 joules or the power of 4·2 watts for one second.

The *large* or *kg-calorie* is 1000 times as large as the gm-calorie.

6. Complex Quantities. It is well known that any given positive or negative number can be represented by a point in the abscissa-axis $\overline{OX}$, by taking the direction from the origin O towards X as positive

and the opposite direction as negative. We can extend this system of representation by letting the complex number $a+jb$, where $j=\sqrt{-1}$, be represented by a point in the plane of the co-ordinates, which is obtained by setting off the distance b along the ordinate at a in the X-axis, b being set off in the direction of the Y-axis when it is positive and in the opposite direction when it is negative.

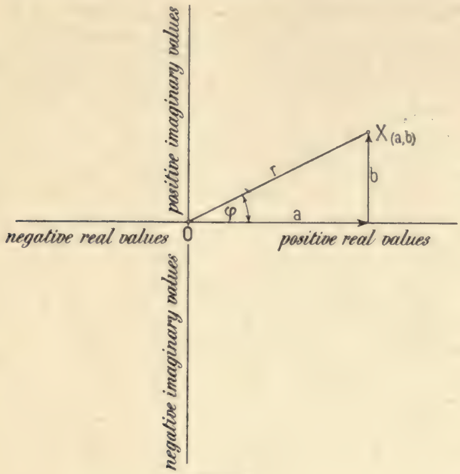


FIG. 23.

Thus every number, whether real or imaginary, has a corresponding point in the plane of the co-ordinates (Fig. 23); conversely, every point in the plane of the co-ordinates corresponds to a definite number.

In the following, *symbolic* expressions for complex quantities will be denoted by placing a dot below the letter. Thus, in Fig. 23, let

$$a=r\cos\phi \quad \text{and} \quad b=r\sin\phi,$$

where

$$r=\sqrt{a^2+b^2} \quad \text{and} \quad \tan\phi=\frac{b}{a};$$

then the symbolic expression for the point X is

$$\underset{\cdot}{X}=a+jb=r(\cos\phi+j\sin\phi)=r\epsilon^{j\phi},$$

where $\epsilon=2{\cdot}71828$ is the base of natural logarithms.

r is called the *absolute value* of the complex quantity X, and equals the length of the line joining the origin O to the point X. ϕ is defined as the *argument* of the complex quantity, and is the angle the vector $\overline{OX}$ makes with the axis of positive real values. Positive real numbers fall on the axis representing positive real values, i.e. to the right of O on the abscissa-axis (see Fig. 23), and have the argument zero, whilst negative real numbers fall to the left of O on the abscissa-axis and have the argument π.

Similarly, positive imaginary numbers have the argument $\frac{\pi}{2}$ and lie on the positive ordinate-axis; negative imaginary numbers have the argument $\frac{3\pi}{2}$ and lie on the negative part of the ordinate-axis.

Two complex numbers which have the same absolute value and whose arguments are equal but of opposite sign are called *conjugate* numbers, as, for example, $a+jb$ and $a-jb$. Two conjugate complex numbers correspond to points in the plane which are the images of one another with respect to the axis of real values.

We must now extend our conception of complex quantities and see how the same can be subjected to the process of calculation. This extension can be so effected, that complex magnitudes can be calculated by the same rules as those which govern the operation of real magnitudes, and the fundamental laws for real magnitudes can be taken as special cases of these rules. For this purpose, we deduce the following formulae :

ADDITION AND SUBTRACTION.

Let $X = a_1 + jb_1$ and $Y = a_2 + jb_2$.

Then $Z = X \pm Y = a + jb$

$$= (a_1 + jb_1) \pm (a_2 + jb_2) = a_1 \pm a_2 + j(b_1 \pm b_2).$$

Both X and Y represent a point or a vector in the plane of the co-ordinates.

Let a point P in the plane of the co-ordinates be represented by two complex expressions, e.g. $P = a + jb = c + jd$, then we must have

$$a = c \quad \text{and} \quad b = d,$$

for the point P has only one abscissa and one ordinate. Hence every complex equation such as $a + jb = c + jd$ can always be split up into two real equations. This is due to the fact that, strictly speaking, j is merely a symbol or index, which serves to distinguish between ordinate and abscissa magnitudes in analytical expressions.

From the above Theorem of Addition, it then follows directly that

$$a = a_1 \pm a_2 \quad \text{and} \quad b = b_1 \pm b_2,$$

when $X = a_1 + jb_1;\quad Y = a_2 + jb_2;$

and $Z = X \pm Y = a + jb.$

Hence Z is represented by a point whose co-ordinates are the sum of the co-ordinates of X and Y.

As seen from Fig. 24a, the radius-vector Z is the geometrical sum of the vectors X and Y; or, in other words, Z is the resultant of the two components X and Y.

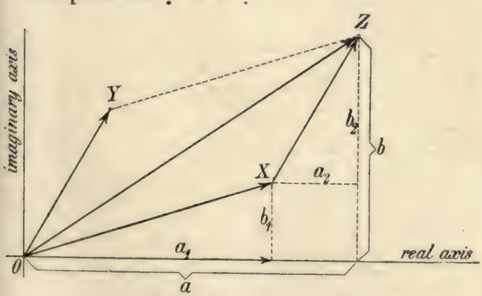


FIG. 24a.—Addition.

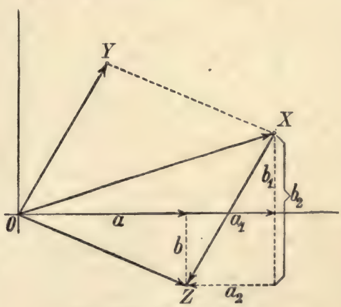


FIG. 24b.—Subtraction.

The point Z is obtained by drawing a line from the point X parallel and equal to $\overline{OY}$; or, in other words, starting from the one component X,

the sum Z is obtained in the same way as when the second component Y is found by starting from the origin O.

Similarly the diagram in Fig. 24*b* represents the process of subtraction.

MULTIPLICATION.

Let $$X = a_1 + jb_1 = r_1(\cos\phi_1 + j\sin\phi_1) = r_1\epsilon^{j\phi_1},$$
and $$Y = a_2 + jb_2 = r_2(\cos\phi_2 + j\sin\phi_2) = r_2\epsilon^{j\phi_2}.$$
Then $$Z = XY = a_1a_2 - b_1b_2 + j(a_1b_2 + b_1a_2)$$
or $$\begin{aligned} &= r_1r_2\{(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) \\ &\qquad + j(\sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2)\} \\ &= r_1r_2\{\cos(\phi_1+\phi_2) + j\sin(\phi_1+\phi_2)\} \\ &= r_1r_2\epsilon^{j(\phi_1+\phi_2)}, \end{aligned}$$

that is, *the multiplication of two complex numbers is effected by multiplying the absolute magnitude of the one by that of the other and taking the sum of their arguments.*

The product of two conjugate complex quantities is a real quantity and equals the sum of the squares of their absolute values; thus

$$(a+jb)(a-jb) = a^2 + b^2.$$

As seen from Fig. 25, the product of two vectors can be regarded as formed from one vector by multiplying the absolute value of one vector

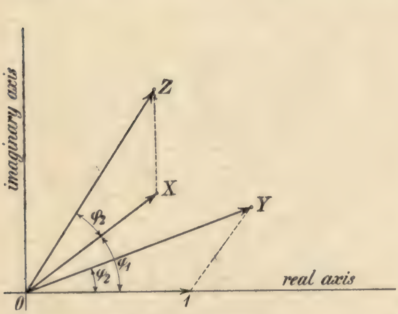


FIG. 25.—Multiplication.

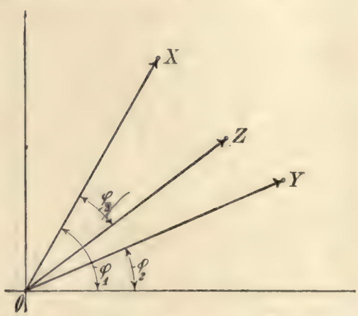


FIG. 26.—Division.

by that of the other, and at the same time turning the former vector through an angle equal to the argument of the latter vector. Such an operation is called *rotation* in geometry, for the vector Z is considered to result from the vector X by rotating and by increasing X by an amount given by the second vector $Y = r_2\epsilon^{j\phi_2}$. The rotation is counter-clockwise when ϕ_2 is positive and clockwise when ϕ_2 is negative.

Let the value $+1$ be set off along the abscissa-axis and join $1Y$. Then the triangles $O1Y$ and OXZ are similar, for we have

$$\frac{\overline{OZ}}{\overline{OX}}=\frac{\overline{OY}}{\overline{O1}}, \quad \text{and} \quad \angle(XOZ)=\phi_2=\angle(1OY),$$

that is to say, *the product $\dot{Z}$ is formed from one of the factors, e.g. from $\dot{X}$, in the same way as the second factor $\dot{Y}$ is formed from unity.*

Division.

The operation of division is the reverse of that of multiplication, as seen from Fig. 26, that is, *the division of two complex numbers is effected by dividing the absolute magnitude of the one by that of the other and taking the difference of their arguments.*

The denominator of a complex quotient is made real by multiplying both denominator and numerator by the conjugate quantity of the denominator, for example:

$$\dot{Z}=\frac{\dot{X}}{\dot{Y}}=\frac{a_1+jb_1}{a_2+jb_2}=\frac{(a_1+jb_1)(a_1-jb_1)}{a_2^2+b_2^2}$$

$$=\frac{a_1a_2+b_1b_2+j(b_1a_2-a_1b_2)}{a_2^2+b_2^2}$$

or

$$=\frac{r_1(\cos\phi_1+j\sin\phi_1)}{r_2(\cos\phi_2+j\sin\phi_2)}$$

$$=\frac{r_1}{r_2}\{\cos(\phi_1-\phi_2)+j\sin(\phi_1-\phi_2)\}$$

$$=\frac{r_1}{r_2}\epsilon^{j(\phi_1-\phi_2)}.$$

Involution.

From the formula for multiplication, we get

$$\dot{Z}=\dot{X}^n=(a+jb)^n=\{r(\cos\phi+j\sin\phi)\}^n$$
$$=r^n(\cos n\phi+j\sin n\phi)=r^n\epsilon^{jn\phi}.$$

Hence, to raise a complex number to any power, we must raise the absolute value to that power and multiply its argument by the index.

Fig. 27 represents this operation. We have thus, for example,

$$(a+jb)^2=a^2-b^2+j2ab.$$

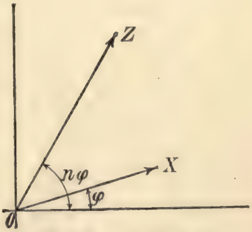


FIG. 27.—Involution.

Evolution.

$$Z = \sqrt[n]{X} = \sqrt[n]{a+jb}$$

$$= \sqrt[n]{r}\left(\cos\frac{\phi}{n} + j\sin\frac{\phi}{n}\right) = \sqrt[n]{r}\epsilon^{j\frac{\phi}{n}}.$$

Hence, to find the root of a complex number, we take the root of the absolute value and divide the argument by the index.

It may here be noted that in complex equations it is always allowable to substitute $-j$ for $+j$, provided all terms in the equation are similarly treated. For example, to calculate $\sqrt{a+jb}$, put

$$\sqrt{a+jb} = \alpha + j\beta,$$

then also

$$\sqrt{a-jb} = \alpha - j\beta.$$

Multiplying these two equations together, we get

$$\sqrt{a^2+b^2} = \alpha^2 + \beta^2.$$

By squaring the first equation,

$$a + jb = \alpha^2 - \beta^2 + j2\alpha\beta$$

or

$$a = \alpha^2 - \beta^2 \quad \text{and} \quad b = 2\alpha\beta;$$

whence

$$\alpha = \pm\sqrt{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)}$$

and

$$\beta = \pm\sqrt{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)}.$$

Since $b = 2\alpha\beta$, it is seen that α and β have the same sign when b is positive and unlike signs when b is negative. Hence

$$\sqrt{a \pm jb} = \pm\left\{\sqrt{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)} \pm j\sqrt{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)}\right\}.$$

Since the above theorems apply equally well to real numbers, it is obvious that they are therefore quite general.

CHAPTER I.

SIMPLE ALTERNATING-CURRENTS AND THEIR REPRESENTATION.

7. Sine Wave Currents. The simplest alternating-current is one whose momentary value can be expressed as a function of the time by a sine wave, e.g.

$$\begin{aligned} i &= I_{\max} \sin (2\pi ct + \phi) \\ &= I_{\max} \sin \left(2\pi \frac{t}{T} + \phi\right) \\ &= I_{\max} \sin (\omega t + \phi), \end{aligned}$$

where $I_{\max}$ is the *amplitude* of the current, T the *time in seconds the current takes to pass through a complete cycle or period*, whilst $\frac{1}{T} = c$ represents the number of such cycles the current passes through in one second, and

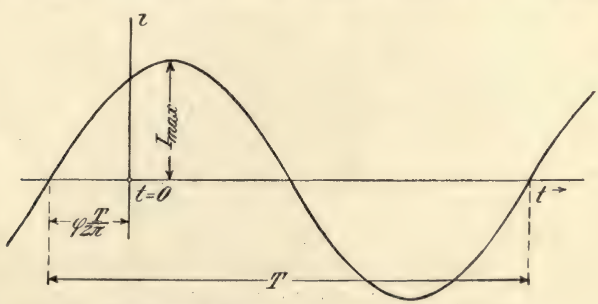


FIG. 28.—Sinusoidal Variation of an Alternating-Current.

is called the *frequency* of the current. Fig. 28 shews such a current, which obeys a sine law, drawn as a function of the time.

With polar co-ordinates, the sine curve is represented by a circle (Fig. 29), whose diameter $\overline{OA}$ equals the amplitude I_{max}. $\overline{OB}$ is the momentary value, whilst ϕ is called the *phase angle* of the current. The point B moves over the circle twice in a cycle; consequently, $\omega = 2\pi c$ represents the *angular velocity* of rotation of the straight line $\overline{OB}$.

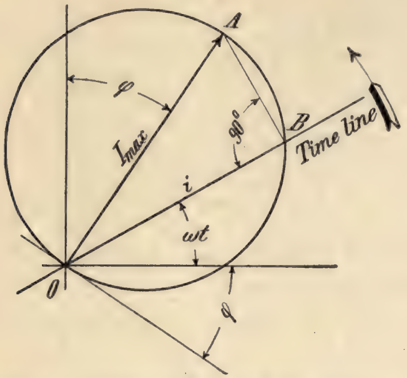


Fig. 29.—Representation of a Sinusoidal Current by Polar Co-ordinates.

The current passes through zero when

$$t = t_0 = -\frac{\phi}{2\pi} T;$$

whence the phase of the current is given by

$$t_0 = \frac{\phi}{2\pi} T.$$

Since the amplitude and phase $\left(\phi \frac{T}{2\pi}\right)$ of the current are given by the magnitude and direction of the vector $\overline{OA}$, the latter represents the current completely. Its momentary value is obtained by projecting the vector $\overline{OA}$ on to a straight line $\overline{OB}$ rotating about O in a counter-clockwise direction with the velocity ω. The rotating line $\overline{OB}$ is therefore called the *time line.*

This method of representation rests on the assumption that the alternating-current is sinusoidal; consequently, the same can also be

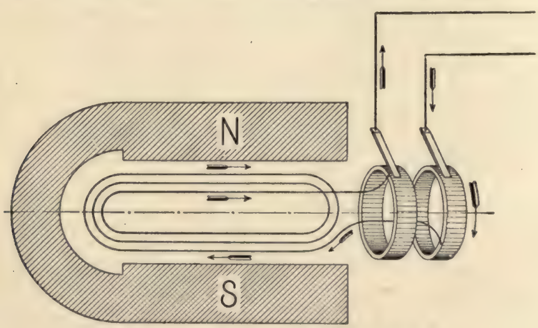


Fig. 30.—Production of a Sinusoidal E.M.F.

applied to an alternating E.M.F. which obeys a sine law. Such an E.M.F. can be produced by the uniform rotation of a rectangular coil about its longitudinal axis between the poles of a magnet, as depicted

in Fig. 30. The poles are assumed to be sufficiently large, so that the field in which the coil rotates is quite uniform.

At the instant considered, the flux passing through the surface F of a turn is (Fig. 31)

$$\Phi = HF \cos \omega t;$$

and since the induced E.M.F. is

$$e = -\frac{d\Phi}{dt},$$

the E.M.F. induced in the turn will be

$$e = -\frac{d(HF\cos\omega t)}{dt} = HF\omega \sin \omega t.$$

Now HF is the maximum flux embraced by the turn during a revolution; denoting this by Φ_{max}, we get

$$e = 2\pi c \Phi_{max} \sin \omega t.$$

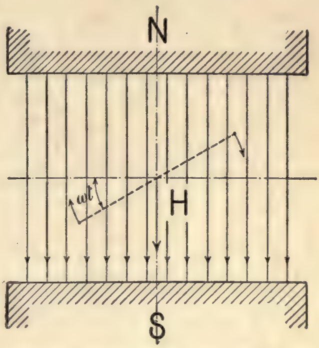


FIG. 31.—Production of a Sinusoidal E.M.F. due to Rotation of a Coil in a Uniform Field.

The embraced flux Φ is a maximum when $\omega t = 0$ and is zero when $\omega t = \frac{\pi}{2}$. The E.M.F. induced by Φ is, on the contrary, zero when $\omega t = 0$, and reaches its maximum when $\omega t = \frac{\pi}{2}$. It is thus apparent that the induced E.M.F. is a minimum when the coil is interlinked with the maximum number of lines of force, i.e. when the coil is perpendicular to the field.

This is also in agreement with the previous statement, that the induced E.M.F. varies directly as the rate of cutting of lines of force,

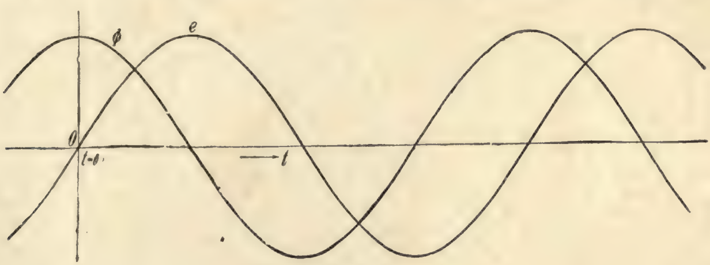


FIG. 32.

for, when the number of interlinkages is zero, the coil is vertical $\left(\text{i.e. } \omega t = \frac{\pi}{2}\right)$ and cuts the lines of force at the maximum rate; consequently, in this position the induced E.M.F. is greatest. In Fig. 32, the flux Φ, and the E.M.F. e induced by it, are drawn as functions of the time. With rising Φ, e is negative; and with falling Φ, e is

positive; in other words, the E.M.F. curve is the differential of the flux curve, with the negative sign prefixed.

If, instead of one turn, there is a coil composed of several turns all in the same plane, the induced E.M.F. will be

$$e = H\Sigma(F)\,\omega \sin \omega t.$$

If all w conductors in a coil-side are so near together that the same flux Φ_{max} is embraced by each turn, then

$$e = 2\pi c w \Phi_{max} \sin \omega t.$$

Since the field-strength H, the sum of all surfaces ΣF of the turns, and the angular velocity ω are constant, we can write

$$e = E_{max} \sin \omega t.$$

If H, F and ω are in C.G.S. units, then e and E will also be in absolute units. To reduce to volts, we must write

$$E_{max} = 2\pi c w \Phi_{max}\, 10^{-8} \text{ volts.} \qquad (14)$$

A cycle in this case corresponds to a revolution of the coil, and the frequency c equals the number of revolutions per second.

The direction of the E.M.F. induced in the coil at any moment can be found from the hand rule on p. 14, and is represented by the arrows (Fig. 30).

8. Summation of Sine Wave Currents. In Fig. 30, all the turns of the rotating coil lie in the same plane, and the E.M.F.'s induced in the several turns all reach their zero together and all attain their maximum together. In this case, the E.M.F.'s are said to be *in phase* with one another.

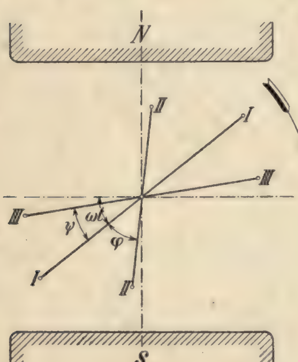


FIG. 33.

If the turns are in different planes, but arranged about a common axis, as in Fig. 33, the E.M.F.'s induced in the several turns will no longer have the same phase, but, in respect to time, they will be displaced in phase. Denoting the E.M.F. induced in coil I. by

$$e_1 = E_{1\,max} \sin \omega t;$$

then the E.M.F. induced in coil II. will have the same frequency as the E.M.F. induced in coil I., since the angular velocity ω is the same in the two cases, but its phase will be different; thus,

$$e_2 = E_{2\,max} \sin (\omega t - \phi),$$

where ϕ is the constant angle by which coil II. *lags* behind coil I. Thus the E.M.F.'s of coils I. and II. are *displaced* from one another by the angle ϕ, which the coils make with one another in space, whence

the angle ϕ is called the *angle of phase-displacement* between e_1 and e_2. The negative sign before ϕ denotes that e_2 *lags* behind, or reaches its maximum after e_1.

Again, the plane of coil III. is displaced from that of coil I. by the angle ψ, in the direction of the sense of rotation. The E.M.F. induced in coil III. can then be written

$$e_3 = E_{3\,\max} \sin(\omega t + \psi),$$

which means that coil III. reaches its maximum E.M.F. (or its zero) before coil I. attains its maximum E.M.F. (or its zero) by an amount corresponding to the time taken for the system to rotate through the angle ψ. Thus e_3 is said to *lead* e_1, and the angle ψ is called the *angle of lead*, in the same way as the angle ϕ above is called the *angle of lag*.

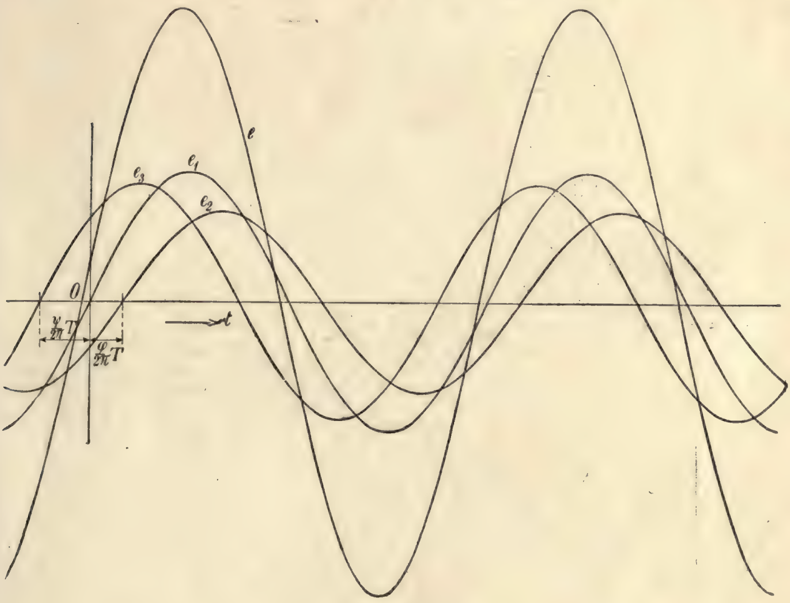


FIG. 34.

In order to obtain the *resultant* E.M.F. induced in the whole coil, the algebraic sum of the *momentary* values of the E.M.F.'s in the several turns must be taken. In Fig. 34, the instantaneous values of the three E.M.F.'s e_1, e_2 and e_3, and their algebraic sum e, are plotted as functions of the time.

We often require the resultant of several E.M.F.'s or currents of different phase. This can be most readily found graphically. The several momentary values e_1, e_2 and e_3 are obtained by projecting the corresponding vectors $E_{1\,\max}$, $E_{2\,\max}$ and $E_{3\,\max}$ on the rotating vector or

time line, in accordance with the well known theorem: *the projection of the resultant* (i.e. *the geometrical sum*) *of several vectors on a straight line equals the sum of the projections of the several vectors on the same line.*

From this it follows that the sum of several sinusoidal E.M.F.'s, which are represented in amplitude and phase by means of vectors, is

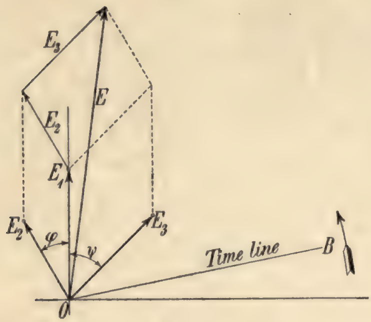


FIG. 35.

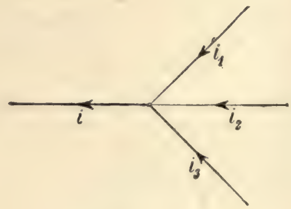


FIG. 36.

given by the resultant of the vectors of the several E.M.F.'s (Fig. 35).* In a similar manner, the sum of several alternating-currents flowing to or from a point (Fig. 36), i.e. the resultant of several parallel currents, can be found by determining the resultant of the vectors of the several currents, as in Fig. 37.* Thus

$$i = I_{\max} \sin(\omega t + \phi) = I_{1\max} \sin(\omega t + \phi_1) + I_{2\max} \sin(\omega t + \phi_2) + I_{3\max} \sin(\omega t + \phi_3).$$

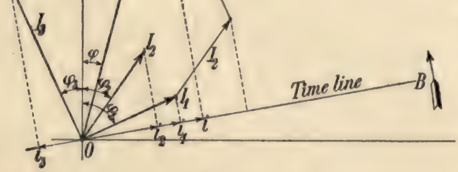


FIG. 37.

From Figs. 35 and 37 it is seen that the amplitude of the resultant E.M.F. or current is not equal to the algebraic sum of the amplitudes of the several components, but depends on the phase displacement of the latter, so that the *geometrical* sum must always be taken.

9. Mean, Effective, and Maximum Values of Sine Wave Currents. Since an alternating-current is continually changing its direction, its *mean value* taken over a whole number of cycles is zero. Thus, such a current

*In Figs. 35 and 37, the vectors denoting the amplitudes of the E.M.F.'s and currents are—for the sake of clearness—denoted by E_1, I_1, etc., instead of by $E_{1\max}$, $I_{1\max}$, etc.

cannot be used directly for charging a battery, nor can it produce any injurious electrolytic effects when flowing as an earth current.

The mean value of an alternating-current is always understood to be the largest mean value which can be obtained during half a period.

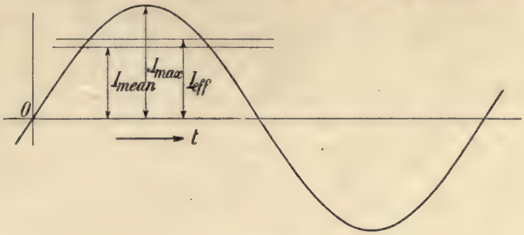


FIG. 38.

Consider the sine curve shewn in Fig. 38, representing

$$i = I_{max} \sin\left(\frac{2\pi}{T} t\right).$$

Then the largest mean value is

$$I_{mean} = \frac{2}{T}\int_0^{\frac{T}{2}} I_{max} \sin\left(\frac{2\pi}{T} t\right) dt$$

$$= \frac{T}{2\pi}\,\frac{2I_{max}}{T}\left[-\cos\frac{2\pi}{T} t\right]_{t=0}^{t=T}$$

$$= \frac{2}{\pi} I_{max} = 0{\cdot}636 I_{max}.$$

Thus the *mean value of a sine curve* is

$$I_{mean} = \frac{2}{\pi} I_{max}. \qquad \text{.............................(15)}$$

The mean value, however, is not of great interest in dealing with alternating-currents or pressures, for the power does not depend on the mean values. From Joule's Law, the work done in overcoming the resistance r of a conductor by a current i in time dt is

$$dA = i^2 r\, dt,$$

whence the mean heating effect is

$$W = \frac{1}{T}\int_0^T dA = \frac{1}{T}\int_0^T i^2 r\, dt = r I_{eff}^2,$$

where I_{eff} is used to denote the current-strength which a continuous-current must have in order to produce the same heating effect as the alternating-current.

Thus $$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2 dt}. \qquad \text{..........................(16)}$$

This is called the *effective* value (or, in accordance with eq. (16), the root-mean-square or R.M.S.-value) of the alternating-current.

Let $$i = I_{\text{max}} \sin\left(\frac{2\pi}{T} t\right).$$

Then $$i^2 = I^2_{\text{max}} \sin^2\left(\frac{2\pi}{T} t\right)$$
$$= \tfrac{1}{2} I^2_{\text{max}} \left\{1 - \cos\left(\frac{4\pi}{T} t\right)\right\}.$$

This is shewn in Fig. 39 as a function of the time.

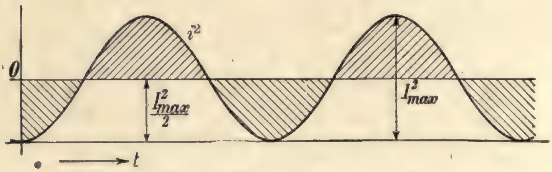


FIG. 39.—Effective Value of Alternating-Current.

The curve i^2 is also a sine wave, but varies with double the frequency of the current i. Further, i^2 does not oscillate about the abscissa-axis, but between zero and I^2_{max}, so that

$$I^2_{\text{eff}} = \frac{1}{T}\int_0^T i^2 dt = \frac{I^2_{\text{max}}}{2},$$

whence $$I_{\text{eff}} = \frac{I_{\text{max}}}{\sqrt{2}} = \frac{I_{\text{max}}}{1{\cdot}414} = 0{\cdot}707 I_{\text{max}}, \quad \text{(17)}$$

or, $$\text{effective value} = \frac{\text{amplitude}}{\sqrt{2}}.$$

From eq. (15) and (17), it follows,

$$I_{\text{eff}} = \frac{\pi}{2} \frac{I_{\text{mean}}}{\sqrt{2}} = 1{\cdot}11 I_{\text{mean}}. \quad \text{(18)}$$

The factor 1·11 is called the *form factor* of a sine curve.

Similarly for the E.M.F.:

$$E_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T e^2 dt} = \frac{E_{\text{max}}}{\sqrt{2}}, \quad \text{(17}a\text{)}$$

and $$E_{\text{eff}} = \frac{\pi}{2\sqrt{2}} E_{\text{mean}} = 1.11 E_{\text{mean}}. \quad \text{(18}a\text{)}$$

On p. 26 it was seen that the maximum E.M.F. induced in a coil of w turns is:

$$E_{\text{max}} = 2\pi c w \Phi_{\text{max}} 10^{-8} \text{ volts.}$$

From eq. (17a) it follows further that the effective E.M.F. will be

$$E_{\text{eff}} = \sqrt{2}\pi c w \Phi_{\text{max}} 10^{-8}$$
$$= 4{\cdot}44 c w \Phi_{\text{max}} 10^{-8} \text{ volts.} \quad \text{(19)}$$

Again, since $$E_{mean} = \frac{2}{\pi} E_{max}, \quad\text{.............................. (15a)}$$

it follows that $$E_{mean} = 4cw\Phi_{max}10^{-8} \text{ volts.} \quad\text{...................... (20)}$$

This last formula can also be simply deduced thus—during one complete cycle, the flux Φ passes from its zero to its positive maximum value Φ_{max} and then sinks again to zero—thus in half a period the flux changes twice—similarly, in the negative half-period, the flux also changes twice, so that in a complete period T the flux Φ_{max} changes 4 times; hence in a second, the flux variation is

$$\frac{4\Phi_{max}}{T} = 4c\Phi_{max},$$

whence formula (20) follows directly.

Since we have made no assumption in deducing this formula as to the way in which the flux varies, it is obvious that the formula (20), i.e. the value of E_{mean}, is independent of the shape of the E.M.F. curve.

In practice the effective value of an alternating-current or pressure plays the most important part. Consequently, in what follows we shall deal almost exclusively with effective values, and in the diagrammatic representation, the vectors will denote such values. If we require the momentary values from such a figure, we have only to multiply the projections of these vectors on to the rotating vector by $\sqrt{2}$. In general, we shall denote instantaneous values by small, and effective values by large letters, whilst maximum or mean values will be denoted by the suffixes *max* and *mean* respectively.

10. Symbolic Representation of Sine Wave Currents. In place of graphical representation of vectors, it is possible to proceed analytically, as in Mechanics, by resolving each vector into two components along axes perpendicular to one another. One axis—the abscissa-axis—coincides with the rotating vector $\overline{OB}$ (Fig. 40) at the instant $t = 0$.

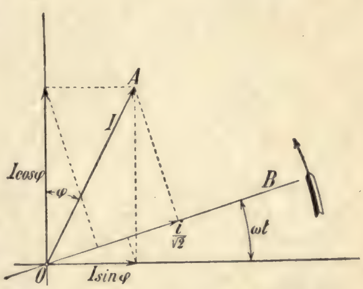


FIG. 40.—Representation of a Sinusoidal Current by two Vector Components.

Now $$i = \sqrt{2}I\sin(\omega t + \phi)$$
$$= \sqrt{2}I(\cos\phi \sin\omega t + \sin\phi \cos\omega t),$$

where I, as above explained, denotes the effective value of the current. Thus the momentary value of a sine function always equals the sum of the momentary values of the two components into which the vector of the sine wave can be resolved.

As seen from Fig. 40, the current i is completely determined by the co-ordinates $I\cos\phi$ and $I\sin\phi$ of the point A.

Just as a complex number can be represented by a point in the plane of the co-ordinates, so a point in the plane of the co-ordinates can be

represented by a complex number. Thus the point A (Fig. 40), and consequently the current I represented by $\overline{OA}$, can be determined from

$$\underset{.}{I} = I\cos\phi - jI\sin\phi,$$

where the vertical co-ordinate is taken as the real axis and the horizontal as the imaginary (Fig. 41). This method was first introduced into electrical theory by *Helmholtz* and *Rayleigh*.

In the expression for the momentary current,

$$i = \sqrt{2}I\sin(\omega t + \phi),$$

ϕ is the phase angle, which shews that the current passes through its zero value at the instant $t_0 = -\frac{\phi}{\omega}$, *i.e.* $\frac{\phi}{\omega}$ before the instant $t = 0$. The greater ϕ is, the earlier the current passes through its zero, *i.e.* the greater the lead. If ϕ is positive, then, as shewn in Fig. 28, the time t_0 must be set off along the *negative* direction of the time axis. In a similar manner, in the vectorial representation of the current in Fig. 40, a positive phase angle ϕ must be set off from the real axis in the *negative* direction of rotation of the time line. In the representation of this current i by means of complex numbers,

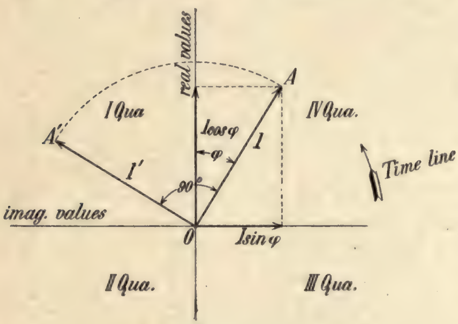


FIG. 41.

$$\underset{.}{I} = I(\cos\phi - j\sin\phi) = I\epsilon^{-j\phi};$$

therefore the phase angle is also $+\phi$; hence, with negative sign, we always obtain a positive phase angle, and *vice versa*.

The system of co-ordinates used in this figure can be regarded as formed from the co-ordinate system in Fig. 29, which is the one generally used in Mathematics, by rotating the latter through 90° in the direction of rotation of the time-line. Hence, in representing sine wave currents symbolically, we set off the real values along the ordinate-axis and the imaginary values along the negative direction of the abscissa-axis.

The current vector can be given either by its magnitude and phase or by the components of the vector along the two axes. The *symbolic expression* $\underset{.}{I}$ implies these two components, so that the vector is completely determined from this symbolic expression.

In what follows, we shall denote effective values by simple capital letters when they merely denote magnitudes, and by capital letters with a dot underneath when the effective value is a vector, representing both magnitude and phase. This method was applied by *Steinmetz*, who has been chiefly instrumental in shewing how technical alternating-current problems can be treated symbolically.

If the vector $\overline{OA}$ is moved through 90°, in the sense of rotation of the time line, to $\overline{OA'}$ (Fig. 41), the co-ordinates of the point A' are

$$I\cos(\phi - 90°) = I\sin\phi$$

and $$-I\sin(\phi - 90°) = I\cos\phi.$$

Thus the complex expression for the vector $\overline{OA'}$ is

$$\begin{aligned} \underset{\cdot}{I}' &= I\sin\phi + jI\cos\phi \\ &= j\{I\cos\phi - jI\sin\phi\} \\ &= j\underset{\cdot}{I}. \end{aligned}$$

We thus see that multiplying a complex or symbolic quantity by j corresponds to moving the vector $\overline{OA}$ through 90° in a counter-clockwise direction. Similarly, multiplying by $-j$ corresponds to rotating the vector 90° in a clockwise direction.

In order to find the components of the resultant of several currents, or E.M.F.'s, we determine the algebraic sum of the several components along the two axes, or, when we proceed symbolically, we can add all real terms together and all the imaginary terms together. Thus, for example, the sum of the currents

$$\underset{\cdot}{I}_1 = a_1 + jb_1 \quad \text{and} \quad \underset{\cdot}{I}_2 = a_2 + jb_2$$

is $$\underset{\cdot}{I} = a + jb = a_1 + a_2 + j(b_1 + b_2).$$

This complex equation can be replaced by two real equations (as shewn in Section 6), namely:

$$a = a_1 + a_2 \quad \text{and} \quad b = b_1 + b_2.$$

Until now we have always spoken of the time-line as revolving; it is possible, however, to suppose this fixed, and let the plane of the co-ordinates rotate about the origin. This must then rotate in a clockwise direction* with the angular velocity ω, and the projection of a vector rotating with the plane on to the fixed vector represents the momentary value of the sinusoidal magnitude represented by the vector revolving with the plane. It is easy to see that *the mutual position of the vectors, also their position with respect to the co-ordinate axes, is the same whether we have a rotating time-line and fixed system of co-ordinates and vectors, or a fixed time-line and a rotating system of co-ordinates and vectors.* Since it is customary to imagine the whole diagram, i.e. the plane of the co-ordinates and the

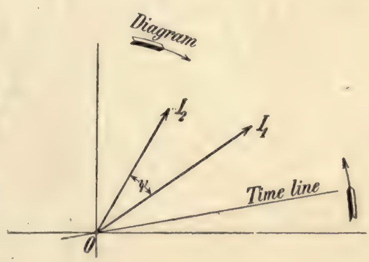


FIG. 42.

* This direction of rotation is opposite to that adopted, since these drawings were prepared, by the International Committee for Electrical Symbols.

vectors fixed in regard to it, as rotating, this method will also be used in what follows, and the arrow will represent the rotation of the diagram—which is always clockwise. Of two vectors, that one always leads which is first in the clockwise direction. Thus, in Fig. 42, I_1 is leading I_2 by the angle ψ.

11. Power given by Sine Wave Currents. It has been shewn on p. 15 that the work done in an electric circuit in time dt is

$$dA = ei\,dt,$$

where e and i denote respectively the E.M.F. and current in the circuit at the moment considered.

Writing $$e = \sqrt{2}E \sin(\omega t + \phi_1)$$
and $$i = \sqrt{2}I \sin(\omega t + \phi_2),$$
where E and I are effective values, the momentary value of the power will be

$$ei = 2EI \sin(\omega t + \phi_1)\sin(\omega t + \phi_2)$$
$$= EI\{\cos(\phi_1 - \phi_2) - \cos(2\omega t + \phi_1 + \phi_2)\}.$$

From this it is seen that the instantaneous value of the power is a function of the time, and varies as a sine function about the mean

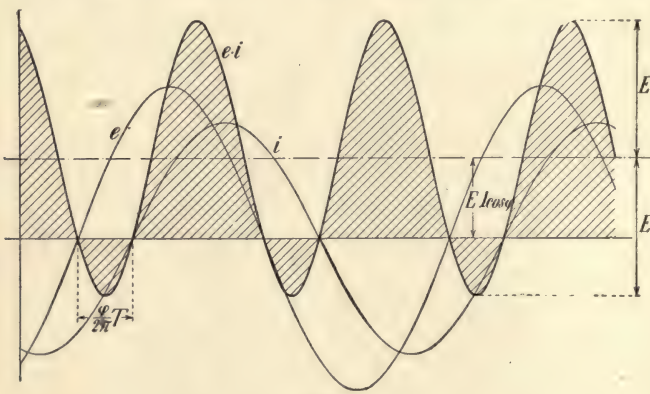


FIG. 43.

value $EI\cos(\phi_1 - \phi_2)$ with double the frequency of the current or pressure (Fig. 43). Hence the mean value of the power during a complete cycle, i.e. the *mean or effective power*, is,

$$W = \frac{1}{T}\int_0^T ei\,dt = EI\cos(\phi_1 - \phi_2)$$
$$= EI\cos\phi, \qquad \qquad (21)$$

where $\phi = \phi_1 - \phi_2$ = phase-angle between the pressure E and current I.

The product EI of E.M.F. and current is called the *apparent power*, and is often referred to as the *volt-amperes*; $\cos\phi$ is equal to the *power-factor*, being the factor by which the volt-amperes EI must be multiplied in order to obtain the *true power* W in watts.

As we have just seen, the power surges to and fro in the circuit—at one instant it is positive, at another negative. This surging will be a minimum when $\phi_1 - \phi_2 = \phi$ is zero, or $\cos\phi$ is unity, i.e. when current and pressure are in phase, for in this case, and in this case only, the momentary value of the power is never negative (Fig. 39). In other words, although the power is transmitted from the generator to the line in the form of pulsations, the line never returns power to the generator. The greatest amount of surging will occur when

$$\phi_1 - \phi_2 = \phi = \frac{\pi}{2},$$

i.e. when $\cos\phi = 0$, for now the mean value of the power is zero, and the power merely surges to and fro between generator and line, but

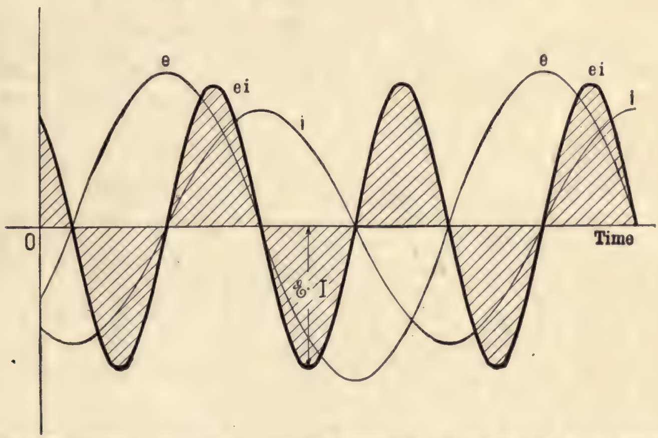


FIG. 44.—Periodic Variation of Pressure, Current and Power when $\phi = \phi_1 - \phi_2 = 90°$.

no actual transmission of power occurs (Fig. 44). In this case, the area of the positive part of the power curve equals that of the negative part.

The momentary power can be shewn diagrammatically by setting off the constant magnitude

$$EI\cos(\phi_1 - \phi_2) = EI\cos\phi$$

on the ordinate axis from O to O_1 (Fig. 45), and describing a circle about O_1 with radius EI. Then, if the radius of this circle rotates with uniform velocity 2ω in a clockwise direction, the momentary power ei will be given by the ordinate drawn from the end A of the radius EI on to the abscissa-axis passing through O.

At the moment $t=0$, the radius EI has the position $\overline{O_1A}$—its component along the ordinate-axis is $-EI\cos(\phi_1+\phi_2)$ and along the abscissa-axis $-EI\sin(\phi_1+\phi_2)$.

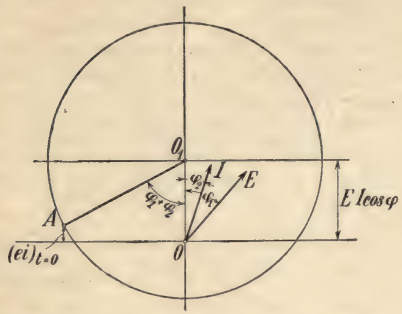


FIG. 45.

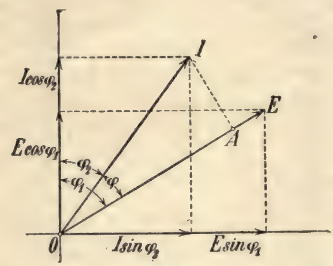


FIG. 46.

Using the graphic representation of Fig. 46 for E.M.F.'s and currents, and resolving the vectors into components along the axes, we get

$$e=\sqrt{2}E\cos\phi_1\sin\omega t+\sqrt{2}E\sin\phi_1\cos\omega t$$

and $$i=\sqrt{2}I\cos\phi_2\sin\omega t+\sqrt{2}I\sin\phi_2\cos\omega t.$$

Since also

$$W=EI\cos(\phi_1-\phi_2)$$
$$=EI\cos\phi_1\cos\phi_2+EI\sin\phi_1\sin\phi_2,$$

we see that *the resultant power equals the sum of the powers of the several components of the vectors.* From Fig. 46, it is also seen that the power equals the E.M.F. multiplied by the projection of the current on to the E.M.F., or equals the current multiplied by the projection of the E.M.F. on the current.

12. Symbolic Representation of Power. If E.M.F. and current are represented symbolically, we get the following expressions for these magnitudes (see Fig. 46):

$$\dot{E}=E\cos\phi_1-jE\sin\phi_1=E\epsilon^{-j\phi_1},$$
$$\dot{I}=I\cos\phi_2-jI\sin\phi_2=I\epsilon^{-j\phi_2},$$

where ϵ denotes the base of natural logarithms. E and I are *absolute magnitudes,* whilst $-\phi_1$ and $-\phi_2$ are called the *arguments* of the complex quantities. To multiply two complex quantities together, we take the product of their absolute magnitudes and the sum of their arguments (see Section 6). Hence the product of the complex expressions for current and pressure is

$$EI\epsilon^{j(-\phi_1-\phi_2)}=EI\{\cos(\phi_1+\phi_2)-j\sin(\phi_1+\phi_2)\}.$$

From this we see that the product of the complex expressions for E and I merely gives the complex expression for that part of the momentary power which varies after a sine law of double frequency (Fig. 45) and has no relation to the actual power.

In practice, however, it is not the momentary power we require, but the mean value $EI\cos\phi$, the apparent power EI and the power factor $\cos\phi$. These are especially important when we come to deal with curves of any desired shape.

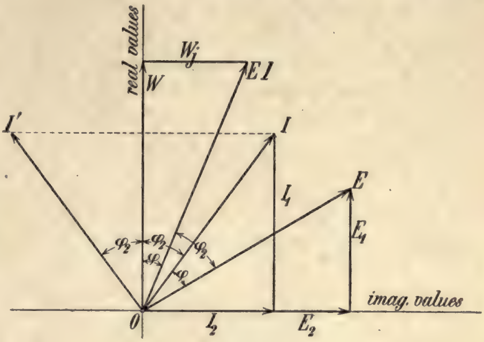


FIG. 47.

For this purpose, it is best to set off the apparent power EI as a vector at angle $\phi = \phi_1 - \phi_2$ to the ordinate-axis (Fig. 47). The projection of this vector EI on to the ordinate-axis then represents the effective power $EI\cos\phi$. Choosing again the ordinate-axis to represent the real and the abscissa-axis the imaginary values, we get the following symbolic expression for the power vector:

$$(\dot{E}\dot{I}) = EI\cos\phi - jEI\sin\phi$$
$$= EI\epsilon^{-j\phi} = W + jW_j.$$

We can suppose the power vector to be formed from the E.M.F. vector, by simultaneously moving the latter through the angle ϕ_2, in the counter-clockwise direction, and multiplying it by the current I. In other words, the power vector is obtained by multiplying the E.M.F. vector by $I\epsilon^{j\phi_2}$. Hence the symbolic expression of the power vector is obtained by multiplying the E.M.F. vector by the conjugate vector $\dot{I}'$ $(= I\epsilon^{j\phi_2})$ of the current vector $\dot{I}$. The vector $\dot{I}' = I\epsilon^{j\phi_2}$ is the image of the current vector $\dot{I} = I\epsilon^{-j\phi_2}$ about the real axis.

Let $$\dot{E} = E\epsilon^{-j\phi_1} = E_1 - jE_2$$

and $$\dot{I} = I\epsilon^{-j\phi_2} = I_1 - jI_2.$$

Then $$(\dot{E}\dot{I}) = W + jW_j = (E_1 - jE_2)(I_1 + jI_2)$$
$$= E_1I_1 + E_2I_2 + j(E_1I_2 - E_2I_1).$$

Hence the effective power W $(= EI\cos\phi)$ is

$$W = E_1I_1 + E_2I_2 \quad \text{...(22)}$$

and the so-called *imaginary power* $(EI\sin\phi)$ is

$$W_j = E_1I_2 - E_2I_1. \quad \text{...(23)}$$

In this method of representation, the imaginary power is positive or negative, according as the current leads or lags in respect to the E.M.F., and is zero when the two are in phase. If we had proceeded otherwise, and called the imaginary power positive, when the current lags, the power vector would have been obtained by multiplying the current vector by the conjugate of the E.M.F. vector.

From the foregoing, we see that *the symbolic expression for the power is obtained by multiplying the symbolic expression for the pressure vector by the symbolic expression for the image of the current vector with respect to the axis of real values.*

The above introduction of the image in the complex expression for the power depends solely on the manner in which the E.M.F., current and power vectors are expressed, and has no physical relation to the expression for the momentary power.

CHAPTER II.

THE PHYSICAL PROPERTIES OF ALTERNATING-CURRENT CIRCUITS.

13. Self-Induction. 14. Capacity. 15. The Pressure Components in a Circuit carrying a Sinusoidal Current. 16. Differential Equation of a Simple Circuit. 17. Graphical Representation of an Alternating-current Circuit. 18. Examples. 19. Resolution of the Current into Watt and Wattless Components.

13. Self-Induction. When a current flows in a conductor, a field is produced encircling the conductor. The flux Φ_x produced by a current i flowing through a conductor of w_x turns is, from equation (9):

$$\Phi_x = \frac{iw_x}{R_x},$$

where R_x is the reluctance of the magnetic path of the flux Φ_x, interlinked with the w_x turns.

If the current changes in strength or direction, the flux Φ_x changes in the same sense, and along with it the stored-up energy $\frac{1}{2}i\Sigma(w_x\Phi_x)$.

Consider any conductor, for example a loop (Fig. 48). If the flux embraced by the loop is varied, an E.M.F. e_s will be induced in the conductor, which, in accordance with the law of induction, is expressed by

$$e_s = -\frac{d\Sigma(w_x\Phi_x)}{dt} = -\frac{d}{dt}\frac{\Sigma(iw_x^2)}{R_x},$$

e_s is called the *counter- or back-E.M.F. of self-induction.*

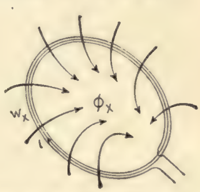


FIG. 48.—Self-induction of a Coil.

Since the same current i flows through each of the turns,

$$e_s = -\frac{d}{dt}i\Sigma\left(\frac{w_x^2}{R_x}\right),$$

where the sum of all fluxes produced by the current i is to be taken.

In general, we write

$$e_s = -\frac{d(Li)}{dt}, \quad \text{.................................(24)}$$

where $$L = \Sigma\left(\frac{w_x^2}{R_x}\right). \qquad (25)$$

The factor L is called the *coefficient of self-induction* of the circuit, and has the same dimension as magnetic permeance, viz. the dimension of a length.

With constant reluctance R_x, the flux Φ_x will be in phase with current i, in accordance with equation (9). If the current varies sinusoidally, the flux and E.M.F. will also follow a sine law, and since the induced E.M.F. e_s lags 90° behind the inducing flux Φ_x, it will also lag 90° behind the current, and we get the curves for Φ_x, i and e_s as shewn in Fig. 49. p_s is the external pressure applied to the coil, and is equal and

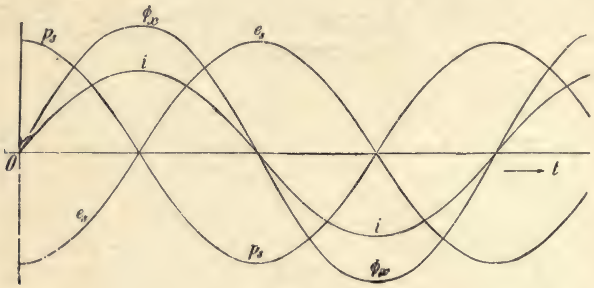


FIG. 49.

opposite to the E.M.F. e_s. The reason e_s has the opposite sign to $d(Li)$, is because the induced E.M.F. always tends to prevent any alteration in the current strength. Thus, in a circuit where the current is rising, the counter-E.M.F. will oppose it, and the current will be *retarded* in its growth. On the other hand, a falling current is always acted on by a counter-E.M.F. which tends to keep the current constant, and so lowers the rate of decrease. Thus, in an electromagnetic circuit, self-induction seeks to prevent any change of current, just as with matter, inertia tends to prevent any change of velocity.

The energy dA supplied to the flux during time dt is:

$$dA = -e_s i\,dt = i\,d\Sigma(w_x\Phi_x)$$
$$= i\,.\,di\Sigma\left(\frac{w_x^2}{R_x}\right) = Li\,di = \frac{L}{2}\,d(i^2).$$

If the coefficient of self-induction L is constant, it follows that the electrical work which must be expended in raising the current from 0 to i (excluding heating losses) is

$$A = \frac{Li^2}{2}. \qquad (26)$$

This work—which is often referred to as the *electromagnetic energy* in the circuit—will be given out again when the current sinks from i to

zero. The coefficient L is measured in absolute units (cm)—the practical unit of self-induction is called the *Henry*, and is chosen equal to 10^9 times the absolute unit.

On page 12, the reluctance of a thin tube of force C was defined as

$$R_x = \int_C \frac{0{\cdot}8\, dl}{\mu f_x}$$

$$= 10 \int_C \frac{dl}{4\pi\mu f_x},$$

so that the flux in the tube can be found directly by dividing the ampere turns interlinked with the tube of force by the reluctance R_x. Thus, R_x is not measured in absolute units, but in units $\frac{1}{10}$ of the absolute; hence

$$L = \Sigma\left(\frac{w_x^2}{R_x}\right) 10^{-8}$$

$$= \Sigma(w_x \Phi_x) 10^{-8}\, henry, \qquad\qquad (27)$$

where Φ_x is the flux due to 1 ampere. In calculating L, we may use the following definition: *The coefficient of self-induction L of a circuit, in absolute units, is measured by the number of interlinkages* $\Sigma(\Phi_x w_x)$ *which the conductor makes with the flux produced by a current of* 10 *amperes* (i.e. by one absolute unit of current).

14. Capacity.* If an E.M.F. is applied to the plates of a condenser, a charge will be taken by the latter. The relation between the acquired charge q and the pressure p_c at the terminals of the condenser is

$$q = Cp_c,$$

where C is called the *capacity* of the condenser. If we make $p_c = 1$, the capacity will be numerically equal to the electric charge which must be supplied to the condenser in order to raise the potential difference between its terminals to unity.

If during the time dt the pressure is increased or decreased by dp_c, the increase or decrease in the charge, i.e. the quantity of electricity passing along the conductor, will be

$$dq = i\, dt,$$

where i is the current in the conductor.

Hence

$$C \,.\, dp_c = i\, dt$$

or

$$i = C\frac{dp_c}{dt}.$$

If the pressure at the terminals of the condenser is altered, the current in the conductor is proportional to the rate of change of the pressure.

* For further information on condensers, see Chap. XIX.

On the other hand, if the rate of change of the current i in the conductor is given, the pressure at the condenser will be

$$p_c=\int\frac{i\,dt}{C}.$$

Hence the energy supplied to the condenser during any time element will be

$$ip_c\,dt=i\,dt\int\frac{i\,dt}{C}.$$

If the current varies periodically, the condenser will be periodically charged and discharged. The energy stored-up in the condenser during charge is given up again during discharge, that is, the charge of the condenser surges to and fro in the circuit.

Assuming that the charging current follows the sine wave

$$i=\sqrt{2}I\sin\omega t,$$

then the pressure taken up by the condenser will be

$$p_c=\int\frac{i\,dt}{C}=\frac{\sqrt{2}I}{\omega C}\sin\left(\omega t-\frac{\pi}{2}\right)=\sqrt{2}P_c\sin\left(\omega t-\frac{\pi}{2}\right).$$

In Fig. 50, the curves of current i and pressure p_c are shewn. The curve p_c—representing the pressure consumed by the condenser—is seen to lag 90° behind the current. This is to be expected when it is

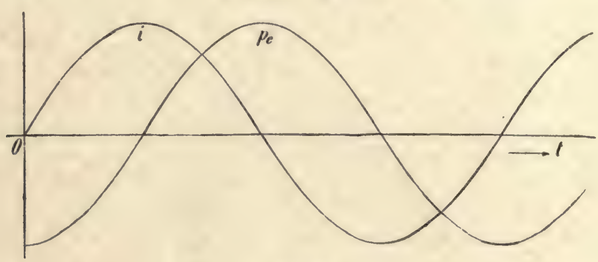


FIG. 50.

remembered that the pressure rises so long as the current is positive and reaches its maximum when the current passes through zero. The pressure curve which coincides with the charging curve q is the integral of the current curve.

As the practical unit of capacity, a condenser may be used whose terminal pressure rises one volt per second when the charging current is one ampere.

The practical unit of capacity equals 10^{-9} absolute units, and is called a *farad*—since this unit is very large, it is usual to use the *microfarad*, which equals one-millionth of one farad or 10^{-15} absolute units.

15. The Pressure Components in a Circuit carrying a Sinusoidal Current. If the current $i=\sqrt{2}I\sin\omega t$ flow along a conductor having the ohmic resistance r, the instantaneous value of the pressure will be

$$p_r=ir=\sqrt{2}Ir\sin\omega t=\sqrt{2}P_r\sin\omega t,$$

where $$P_r=Ir.$$

The pressure curve is thus a sine wave in phase with the current curve.

This is not the case when the circuit possesses self-induction. If the current $$i=\sqrt{2}I\sin\omega t$$

flow in such a conductor whose ohmic resistance is negligible, the pressure at the terminals will be

$$p_s=L\frac{di}{dt}=\sqrt{2}I\omega L\cos\omega t$$

$$=\sqrt{2}P_s\cos\omega t,$$

where $$P_s=I\omega L=Ix_s.$$

Here the terminal pressure p_s leads the current i by 90°. Instead of the resistance, we employ $x_s=\omega L=2\pi cL$ in calculating the effective pressure.

If the conductor possess both resistance and self-induction, the sum of the two respective pressures must be applied to the terminals at any instant. The terminal pressure is then

$$p_{zs}=p_r+p_s=\sqrt{2}Ir\sin\omega t+\sqrt{2}Ix_s\cos\omega t.$$

Substituting, $$\sqrt{r^2+x_s^2}=\sqrt{r^2+(\omega L)^2}=z_s,$$

$$\frac{r}{\sqrt{r^2+(\omega L)^2}}=\frac{r}{z_s}=\cos\phi_s,$$

$$\frac{\omega L}{\sqrt{r^2+(\omega L)^2}}=\frac{x}{z_s}=\sin\phi_s$$

and $$\tan\phi_s=\frac{\omega L}{r}=\frac{x_s}{r},$$

we get $$p_{zs}=\sqrt{2}Iz_s\sin\omega t\cos\phi_s+\sqrt{2}Iz_s\cos\omega t\sin\phi_s$$

$$=\sqrt{2}Iz_s\sin(\omega t+\phi_s)$$

or $$p_{zs}=\sqrt{2}P_{zs}\sin(\omega t+\phi_s).$$

The effective value of the terminal pressure is thus

$$P_{zs}=Iz_s,$$

and the pressure leads the current by ϕ_s.

If a condenser be connected in a circuit, the pressure at its terminals is

$$p_c=\int\frac{i\,dt}{C}.$$

The current is again taken to be

$$i=\sqrt{2}I\sin\omega t.$$

Then
$$p_c=-\frac{\sqrt{2}I}{\omega C}\cos\omega t=-\sqrt{2}P_c\cos\omega t.$$

The effective condenser pressure is therefore

$$P_c=\frac{I}{\omega C},$$

and this pressure lags 90° behind the current.

Lastly, if the current i flow in a circuit in which resistance, self-induction and capacity are all connected in series (as shewn in Fig. 51),

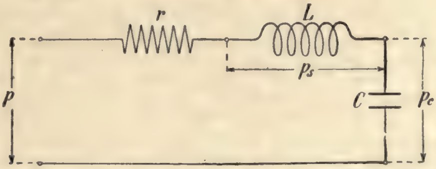


FIG. 51.—Electric Circuit having Resistance, Self-induction and Capacity in Series.

the momentary value of the terminal pressure equals the sum of the several pressures p_r, p_s and p_c. Thus:

$$p=p_r+p_s+p_c=ir+L\frac{di}{dt}+\int\frac{idt}{C}$$

$$=\sqrt{2}I\left[r\sin\omega t+\left(\omega L-\frac{1}{\omega C}\right)\cos\omega t\right].$$

Substituting,
$$\sqrt{r^2+\left(\omega L-\frac{1}{\omega C}\right)^2}=z,$$

$$\frac{r}{\sqrt{r^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}=\cos\phi,$$

$$\frac{\omega L-\frac{1}{\omega C}}{\sqrt{r^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}=\sin\phi,$$

we get
$$p=\sqrt{2}Iz\sin(\omega t+\phi)=\sqrt{2}P\sin(\omega t+\phi).$$

The pressure wave is also sinusoidal in this case and has the effective value

$$P=I\sqrt{r^2+\left(\omega L-\frac{1}{\omega C}\right)^2}=Iz.$$

This pressure leads the current by the amount

$$\phi=\tan^{-1}\frac{\omega L-\frac{1}{\omega C}}{r}.$$

16. Differential Equation of a Simple Circuit. The differential equation of the pressure, developed in the previous section for a circuit possessing resistance, self-induction and capacity (as shewn in Fig. 51), was

$$p = ir + L\frac{di}{dt} + \int\frac{i\,dt}{C}. \quad\text{.............................}(28)$$

This represents Kirchhoff's Second Law in its most generalised form.

Multiplying all through by $i\,dt$, we get the energy equation:

$$pi\,dt = i^2r\,dt + Li\frac{di}{dt}dt + i\,dt\int\frac{i\,dt}{C}. \quad\text{..................}(28a)$$

This tells us that during any time element the energy supplied at the terminals of the circuit equals the sum of the energy consumed in the several parts. Differentiating the pressure equation with respect to dt, we get the differential equation of the current

$$\frac{d^2i}{dt^2} + \frac{r}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{1}{L}\frac{dp}{dt}, \quad\text{........................}(28b)$$

which holds for any pressure p.

In the previous section it was shewn that a sinusoidal current requires a sinusoidal pressure at the terminals of the circuit when r, L and C *are constant.* From this the converse follows, *that a sinusoidal pressure can only produce a sinusoidal current.* Hence, we shall not consider the general solution of this differential equation, but only that for the case when the conditions have become *steady*, a state which is reached soon after switching in. For a sinusoidal pressure at the terminals

$$p = \sqrt{2}P\sin\omega t,$$

we get in eq. (28*b*)

$$\frac{1}{L}\frac{dp}{dt} = \sqrt{2}\frac{\omega}{L}P\cos\omega t.$$

The *special* integral of this equation is then

$$i = \frac{P_{\max}}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\sin\left[\omega t - \tan^{-1}\left(\frac{\omega L}{r} - \frac{1}{\omega Cr}\right)\right]. \quad\text{........}(29)$$

The current is thus a sine wave, but is not in phase with the pressure.

Equation (29) can also be written

$$i = I_{\max}\sin(\omega t - \phi),$$

where

$$I_{\max} = \frac{P_{\max}}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$= \textit{amplitude of the current};$

and

$$\phi = \tan^{-1}\left(\frac{\omega L}{r} - \frac{1}{\omega Cr}\right)$$

$= \textit{angle of phase displacement}.$

The angle of phase displacement ϕ is positive, zero or negative according as

$$\omega L \gtreqless \frac{1}{\omega C} \quad \text{or} \quad \omega \gtreqless \frac{1}{\sqrt{LC}}.$$

When ϕ is positive the current lags behind the pressure, whilst it leads when ϕ is negative.

When $$\omega = \frac{1}{\sqrt{LC}} \qquad \text{.......................(30)}$$

the current and pressure are in phase,

i.e. $$\phi = 0;$$

and the current attains its maximum value

$$I = \frac{P}{r}.$$

When this occurs the self-induction and capacity exactly neutralise one another, and this condition is generally termed "Resonance."

Since in this case the inductance and capacity are in series, we refer to their resonance as *pressure resonance*:* in contradistinction to *current resonance*, which is used for parallel circuits. Using effective values of current and pressure, we get

$$I = \frac{P}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

17. Graphical Representation of an Alternating-current Circuit. In Section 16, it was seen how the solution of the differential equation can be avoided if we start from the current. We shall now see how this method leads to a graphical solution. A sinusoidal current is assumed as given, and we calculate the terminal pressure P. From eq. (28) the momentary value p of the terminal pressure is:

$$p = ir + L\frac{di}{dt} + \int \frac{i\,dt}{C} = ir + p_s + p_c.$$

Thus the applied pressure p can be split up into three components, which are respectively necessary to overcome the ohmic resistance, the counter-E.M.F. of self-induction and the condenser pressure. When the

*This frequency is not the natural period of oscillation of a circuit containing considerable resistance, for in this case

$$c = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{r^2}{4L^2}}.$$

Only when the resistance of the circuit is negligible is the natural period of oscillation equal to the period of supply, when

$$c = \frac{1}{2\pi\sqrt{LC}}.$$

current i is known, each of these three pressures can be calculated. In Fig. 52, the current curve is drawn

$$i = I\sqrt{2}\sin(\omega t - \phi).$$

In phase with the current is the curve ir, which represents the pressure necessary to overcome, or the pressure consumed by, the ohmic resist-

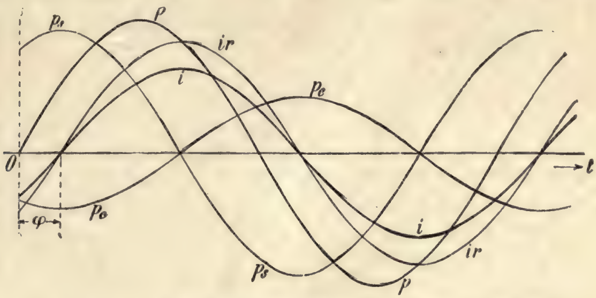


FIG. 52.—Periodic Variation of the E.M.F.'s in a Circuit.

ance of the circuit. This curve ir is also a sine wave, since r is constant.

The pressure p_s required to counter-balance the back-E.M.F. of self-induction e_s is

$$p_s = -e_s = L\frac{di}{dt} = \omega LI\sqrt{2}\sin\left(\omega t - \phi + \frac{\pi}{2}\right).$$

This curve p_s, which must be a sine wave, with sinusoidal current, is shewn in Fig. 52 leading the current by 90°—whilst the counter-E.M.F. e_s (not shewn) lags 90° behind the current.

The pressure p_c required to charge the condenser is

$$p_c = \int\frac{i\,dt}{C} = \frac{I\sqrt{2}}{\omega C}\sin\left(\omega t - \phi - \frac{\pi}{2}\right).$$

Thus the curve p_c is also sinusoidal and lags 90° behind the current.

By summing up the three sine curves ir, p_s and p_c we get the resultant sine curve p, which leads the current curve i by the angle ϕ (Fig. 52). Thus the curve p represents the pressure applied to, or consumed by, the circuit.

Now, since sinusoidal quantities can be represented by vectors, it is possible to represent the phenomena in an alternating-current circuit graphically (see Fig. 53). The current vector I is drawn at an angle ϕ to the ordinate-axis, which is taken to represent the applied pressure P. Since the diagram is taken as rotating right-handedly, and the current is lagging behind the pressure P, the angle ϕ falls to the left of the ordinate-axis. The pressure Ir consumed by r is in phase with I, and must therefore be set off along $\overline{OI}$. The vector representing the

pressure required to overcome the self-induction is given by

$$\omega LI = 2\pi cLI = x_s I$$

and leads the current by 90°. x_s is called the *inductive reactance* of the circuit. It has the dimension of an ohmic resistance, and may therefore be measured in ohms.

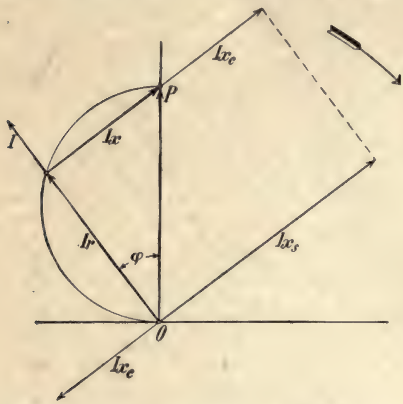


FIG. 53.—Geometric Addition of Pressures in a Circuit.

When L is given in henrys and c in cycles per second, x_s is obtained directly in ohms.

$$x_s = \frac{2\pi c}{10^8}\Sigma\left(\frac{\omega_x^2}{R_x}\right) \text{ ohms.} \quad \ldots\ldots(31)$$

Then Ix_s is set off 90° in advance of I.

The vector representing the pressure P_c used to charge the condenser is $\frac{I}{\omega C}$ and lags 90° behind the current. *Capacity Reactance* x_c is analogous to inductive reactance and is defined as

$$x_c = \frac{1}{\omega C} = \frac{1}{2\pi cC}.$$

This is measured in ohms when c is given in cycles per second and C in farads. The capacity pressure $P_c = Ix_c$ is set off 90° behind I, i.e. in the opposite direction to Ix_s. From this we see that inductance and capacity act directly against one another, and give the resultant component

$$Ix = I(x_s - x_c)$$

or

$$x = x_s - x_c = \omega L - \frac{1}{\omega C}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

x is called the *resultant reactance* or simply the *reactance* of the circuit. When

$$x_s = x_c, \text{ then } x = 0$$

and *resonance* occurs. In this case the current depends only upon the resistance r in the circuit and the angle ϕ is zero, that is, the current i is in phase with the pressure p.

Returning to the general case, we see that the vectors Ir and Ix combine to form the resultant P (see Fig. 53) along the ordinate-axis, at angle ϕ to I.

Thus

$$(Ir)^2 + (Ix)^2 = P^2$$

or

$$I = \frac{P}{\sqrt{r^2 + x^2}} = \frac{P}{z} \quad \ldots\ldots\ldots\ldots\ldots\ldots(29a)$$

where $z = \sqrt{r^2 + x^2}$ is called the *impedance* or apparent resistance of the circuit, whilst

$$\tan\phi = \frac{x}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots(29b)$$

and

$$\cos\phi = \frac{r}{z} = \text{power factor.}$$

When P, r and $x = x_s - x_c$ are given, I can be at once found by drawing a semicircle on $\overline{P}$ as diameter, setting off the angle ϕ and dividing the intercept of $\overline{OI}$ on the circle by r.

As a rule, the applied pressure P is split up into the two components Ir and Ix at right angles to one another. Ir is called the *resistance pressure* and Ix the *reactance pressure.* The effective value of e_s is $-Ix_s$ and is called the *counter-E.M.F. of self-induction*; similarly

$-Ir$ = counter-E.M.F. of resistance,

$-Ix_s$ = „ „ inductance,

$-Ix_c$ = „ „ capacity,

and $-Iz$ = „ „ impedance (or total counter-E.M.F.).

From the diagram in Fig. 54—due to *Bedell and Crehore*—may be seen how the current is affected when the constants r and $x = x_s - x_c$ are altered, whilst the pressure P is kept constant. From the pressure

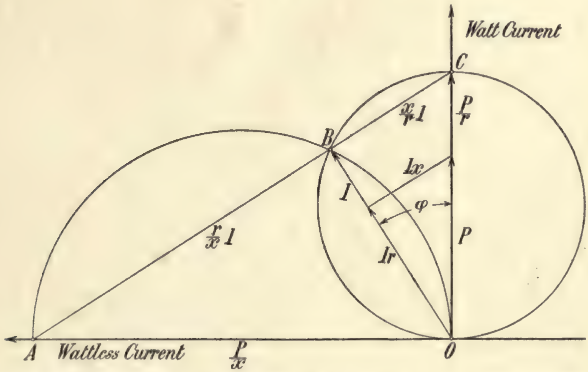


FIG. 54.—Current Diagram of a Circuit with Variation of one of the Constants r or x.

triangle of Fig. 53, the two similar triangles OBC and ABO can be deduced by dividing each side of the pressure triangle by r in the one case and by x in the other. Thus

$$\overline{OA} = \frac{P}{x}; \quad \overline{OB} = I; \quad \overline{OC} = \frac{P}{r},$$

$$\overline{AB} = \frac{Ir}{x} \quad \text{and} \quad \overline{BC} = \frac{Ix}{r}.$$

Hence the current I is represented by the vector $\overline{OB}$. If x is constant and r varied, the point B moves over the semicircle on $\overline{OA}$—from O to A as r decreases from ∞ to 0; that is, on the line ABC the point A is fixed so long as x is constant, whilst the point C moves on the ordinate axis when r is varied; thus the phase displacement ϕ changes from 0 to 90°.

For x positive, $\overline{OA}$ falls to the left, and for x negative to the right of $\overline{OC}$.

If r is kept constant and x varied from zero to $+\infty$ and from $-\infty$ back again to zero, then B moves on the circle on $\overline{OC}$—starting from C, passing through O and coming back to C.

When $x=0$ and r constant, I has its maximum equal to $\overline{OC}$, and the two pressure curves p_s and p_c (Fig. 52) have the same amplitude.

A curve which represents the variation of one magnitude as a function of a second is generally called a *diagram* of the first quantity; thus Fig. 54 is a *current diagram.*

18. Examples.

1. Given the terminal pressure P applied to a circuit possessing resistance, self-induction and capacity of the following values, in series with one another,

$$P=100 \text{ volts}; \qquad r=20 \text{ ohms};$$

$$L=0{\cdot}159 \text{ henry}; \quad C=50 \text{ microfarads}.$$

To determine and to shew graphically the current I, the phase displacement ϕ and the pressures P_c and P_{zs} across the condenser and impedance $z_s=\sqrt{r^2+x_s^2}$ respectively as functions of the frequency c.

At a frequency of 50,

$$x_s=2\pi cL=2\pi 50\times 0{\cdot}159=50 \text{ ohms}$$

and
$$x_c=\frac{1}{2\pi cC}=\frac{1\times 10^6}{2\pi 50\times 50}=63{\cdot}8 \text{ ohms}.$$

Hence $x=x_s-x_c=-13{\cdot}8$ ohms, and the total impedance in this case is

$$z=\sqrt{r^2+x^2}=\sqrt{20^2+13{\cdot}8^2}=24{\cdot}15 \text{ ohms},$$

whence the current I is

$$I=\frac{P}{z}=\frac{100}{24{\cdot}15}=4{\cdot}15 \text{ amperes},$$

$$\tan\phi=\frac{x}{r}=\frac{-13{\cdot}8}{20}=-0{\cdot}64 \quad \text{and} \quad \phi=-32^\circ\ 40',$$

$$P_c=Ix_c=4{\cdot}15\times 63{\cdot}8=264 \text{ volts}.$$

The impedance z_s is

$$z_s=\sqrt{r^2+x_s^2}=\sqrt{20^2+50^2}=53{\cdot}8 \text{ ohms},$$

$$P_{zs}=Iz_s=4{\cdot}15\times 53{\cdot}8=223{\cdot}5 \text{ volts}.$$

In this way, I, ϕ, P_c and P_{zs} are calculated for different frequencies, and are shewn plotted in Fig. 55.

The total reactance of the circuit is zero when the frequency c is

$$c=\frac{1}{2\pi\sqrt{LC}}=\frac{1}{2\pi\sqrt{0{\cdot}159\times 50\times 10^{-6}}}=56{\cdot}5.$$

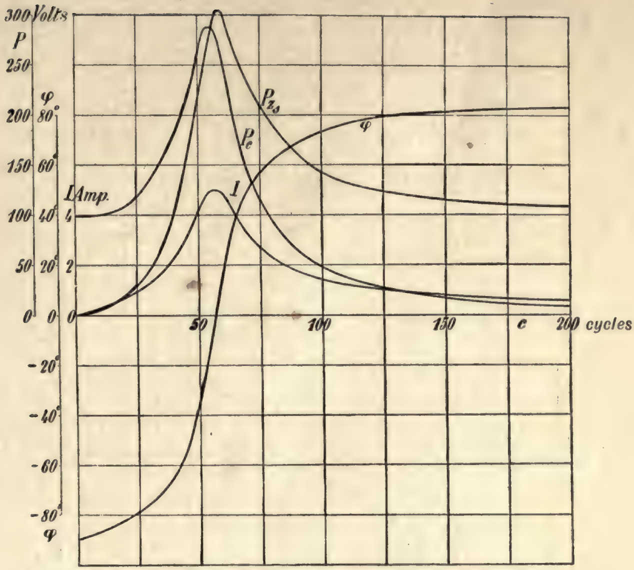

300 Volts
P
250
φ
200 80°
150 60°
I Amp.
100 40° 4
50 20° 2
0 0 0
50
100
150
c
200 cycles
-20°
-40°
-60°
-80°
φ
P_{z_s}
P_c
I
φ

FIG. 55.

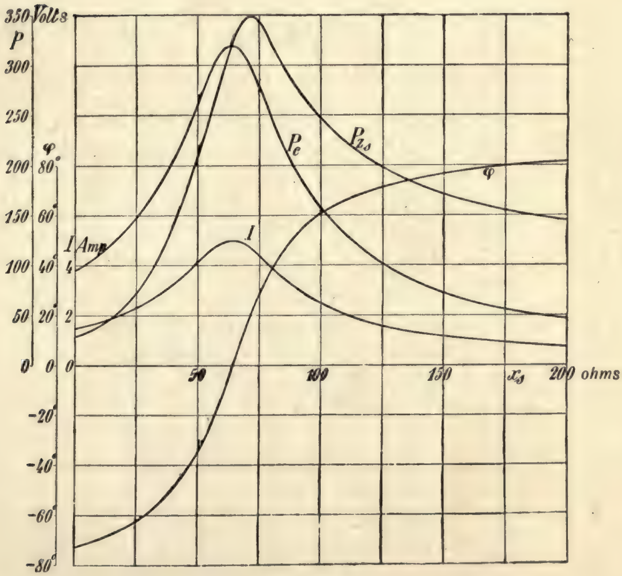

350 Volts
P
300
250
φ
200 80°
150 60°
I Amp
100 40° 4
50 20° 2
0 0 0
50
100
150
x_s
200 ohms
-20°
-40°
-60°
-80°
P_c
P_{z_s}
I
φ

FIG. 56.

At this frequency $\phi=0$ and I is a maximum. The pressures P_c and P_{zs} rise until the frequency has approximately this value, and then fall off. $P_{c\max}$ occurs somewhat before and $P_{zs\max}$ (and $P_{xs\max}$) somewhat after the frequency corresponding to maximum current.

2. Let the terminal pressure, resistance and capacity remain the same as in 1, and the frequency be kept constant at 50 whilst the reactance x_s is varied. Fig. 56 shews the current I, the phase displacement ϕ and the pressures P_c and P_{zs} as functions of x_s.

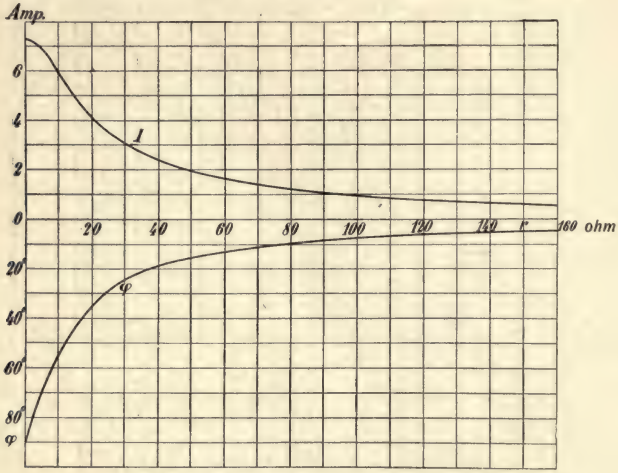


FIG. 57.

When $x_s=x_c=63{\cdot}8$ ohms the current reaches its maximum value, which is the same as in 1, whilst $\phi=0$. In this case $P_{c\max}$ and $P_{s\max}$ both occur at this same value.

3. r is varied whilst P, L and C have the same values as in 1 and $c=50$. By means of Fig. 54, the current I and phase displacement ϕ can be found for different values of r. These are shewn plotted in Fig. 57.

19. Resolution of the Current into Watt and Wattless Components. Instead of resolving the pressure P into two components, the current I may be resolved into two components along co-ordinate axes, one of which is in phase with the pressure P.

Now,
$$i=\frac{\sqrt{2}P}{z}\sin\left\{\omega t-\tan^{-1}\left(\frac{x}{r}\right)\right\}$$
$$=\frac{\sqrt{2}P}{z}\left\{\cos\left(\tan^{-1}\frac{x}{r}\right)\sin\omega t-\sin\left(\tan^{-1}\frac{x}{r}\right)\cos\omega t\right\}$$
$$=\sqrt{2}P\left(\frac{r}{z^2}\sin\omega t-\frac{x}{z^2}\cos\omega t\right).$$

For the sake of simplicity, we write

$$\frac{r}{z^2}=\frac{r}{r^2+x^2}=g=\textit{conductance of the circuit} \quad \ldots\ldots\ldots\ldots(33)$$

and $$\frac{x}{z^2}=\frac{x}{r^2+x^2}=b=\textit{susceptance of the circuit.} \quad \ldots\ldots\ldots\ldots(34)$$

Conductance and susceptance have the reciprocal dimensions of a resistance and are measured in *mhos.*

Thus we can write the current $i=\sqrt{2}P(g\sin\omega t-b\cos\omega t)$, that is, the current vector $\overline{OB}$ in Figs. 53 and 54 is represented by two components Pg and Pb. From Fig. 58, we see:

$$\tan\phi=\frac{b}{g},$$

and further, $$I=P\sqrt{g^2+b^2}=Py,$$

where $$y=\frac{1}{z}=\textit{admittance of the circuit.} \quad \ldots\ldots\ldots\ldots(35)$$

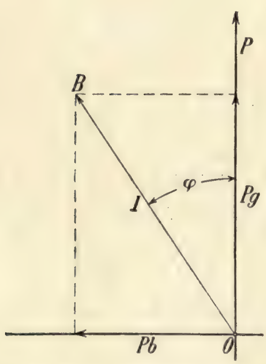


FIG. 58.—Current-triangle.

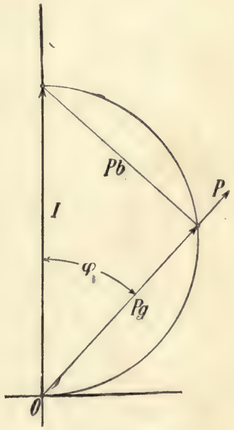


FIG. 59.—Current-triangle.

Rotating Fig. 58 in a clockwise direction through the angle ϕ, we get Fig. 59, which is analogous to Fig. 53.

If it is required that the current in a circuit shall remain constant whilst g and b are varied, the pressure must be correspondingly altered both in phase and magnitude.

From Fig. 60 (analogous to Fig. 54) the pressure $P=\overline{OB}$ can at once be found. If b is constant and g varied, the point B will move over the semi-circle described on $\overline{OA}$, where $\overline{OA}=\frac{I}{b}$. When b is positive, A falls to the right of O, and to the left when b is negative. If g is kept constant and b altered, the circle described on $\overline{OC}$ is the locus of B.

Of the two current components Pg and Pb, only the component Pg, which is in phase with the pressure, does work. Consequently Pg is called the *watt component of the current*, or, simply, the *watt current.*

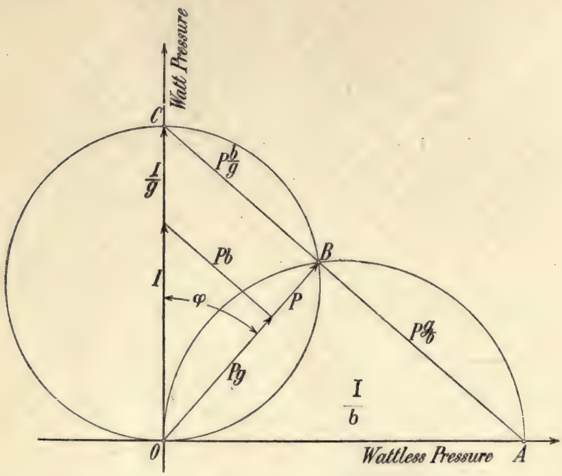


FIG. 60.—Pressure Diagram of a Circuit due to Variation of one of the Constants g or b.

The power is then written

$$W = P \, . \, Pg = P^2 g. \quad \text{.............................(36)}$$

The second component Pb of the current is called the *wattless component of the current*, or, briefly, the *wattless current.*

On p. 52 we saw that the momentary value of the current equals the algebraic sum of the momentary values of the two current components. From the foregoing, we now see that the effective value of the current equals the geometrical sum of the effective values of the watt and wattless components of the current.

Hence, in general, for any circuit containing constant reactances and energy-consuming apparatus (resistances), the impressed sine wave E.M.F. can always be resolved into two components, viz. into the *watt component* Ir which is in phase with the current, and the *wattless component* Ix which leads the current by 90°.

Similarly, the current can be split up into the *watt component* Pg in phase with the pressure, and the *wattless component* Pb which lags 90° behind the pressure.

Thus the constants of a circuit can be written:

$$\frac{\text{watt component of pressure}}{\text{current}} = r = \frac{g}{g^2 + b^2}$$

$$= \textit{effective resistance} \text{ in ohms,} \quad \text{.........................(37)}$$

$$\frac{\text{wattless component of pressure}}{\text{current}} = x = \frac{b}{g^2 + b^2}$$
$$= \textit{effective reactance} \text{ in ohms,} \quad \text{(38)}$$

$$\frac{\text{watt component of current}}{\text{pressure}} = g = \frac{r}{r^2 + x^2}$$
$$= \textit{effective conductance} \text{ in mhos,}$$

$$\frac{\text{wattless component of current}}{\text{pressure}} = b = \frac{x}{r^2 + x^2}$$
$$= \textit{effective susceptance} \text{ in mhos,}$$

$$\frac{\text{pressure}}{\text{current}} = z = \sqrt{r^2 + x^2} = \frac{1}{y}$$
$$= \textit{effective impedance} \text{ in ohms,}$$

$$\frac{\text{current}}{\text{pressure}} = y = \frac{1}{\sqrt{r^2 + x^2}} = \frac{1}{z}$$
$$= \textit{effective admittance} \text{ in mhos.}$$

When several resistances and reactances are in series, it is simplest to use r, x and z, for in this case the corresponding pressure components can be added directly.

On the contrary, when we are dealing with parallel circuits, it is more convenient to use g, b and y, since the current components can then be added in accordance with Kirchhoff's First Law.

CHAPTER III.

ANALYTIC AND GRAPHIC METHODS.

20. The Symbolic Method. Let the instant of time $t=0$ be chosen so that the current vector I coincides with the positive direction of the ordinate axis. We then get the vector diagram shewn in Fig. 61. If all real values are set off along the positive direction of the ordinate axis and all imaginary values along the negative direction of the abscissa axis, we get—as already shewn—a system of co-ordinates similar to that generally used in Mathematics, when the latter system is rotated through 90°.

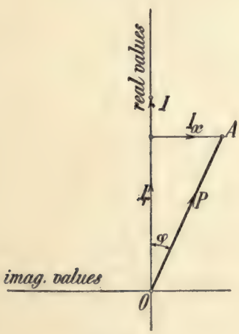


FIG. 61.

The E.M.F. vector P is given by the co-ordinates Ir and Ix of the point A, or, symbolically:

$$P = Ir - jIx = I(r - jx).$$

In order to investigate the meaning of this expression for the general case, where I also is complex, we consider the product of the two complex quantities

$$I = I\cos\phi_2 - jI\sin\phi_2 = I(\cos\phi_2 - j\sin\phi_2) = I\epsilon^{-j\phi_2}$$

and
$$z = r - jx = z(\cos\phi - j\sin\phi) = z\epsilon^{-j\phi}.$$

The product of I and z is

$$Iz = Iz\{\cos(\phi_2+\phi) - j\sin(\phi_2+\phi)\}$$
$$= Iz\epsilon^{-j(\phi_2+\phi)}.$$

This product is represented by a vector which leads the current vector by the angle ϕ, and has an absolute magnitude equal to the

product of the absolute values of the two complex quantities. This vector coincides with the pressure vector; hence we can write for the symbolic expression of the pressure:

$$\underset{\cdot}{P}=\underset{\cdot}{I}z, \quad \text{..............................(39)}$$

where the symbolic expression of the impedance $\underset{\cdot}{z}$ is

$$\underset{\cdot}{z}=r-jx. \quad \text{..............................(40)}$$

Conversely, the symbolic expression of the current $\underset{\cdot}{I}$ is:

$$\underset{\cdot}{I}=\frac{\underset{\cdot}{P}}{\underset{\cdot}{z}}.$$

It is now possible to carry out all the operations of calculation with these symbolic expressions in the same way as with real quantities, and when the calculation is finished, the complex quantities are simply substituted for the symbolic.

The complex expressions can then be changed into the real expressions of the momentary values. We have above:

$$\underset{\cdot}{I}=I\epsilon^{-j\phi_2},$$

$$\underset{\cdot}{z}=z\epsilon^{-j\phi}$$

and

$$\underset{\cdot}{P}=\underset{\cdot}{I}\underset{\cdot}{z}=Iz\epsilon^{-j(\phi_2+\phi)}=P\epsilon^{-j(\phi_2+\phi)}.$$

Then the corresponding momentary values are

$$i=I_{\max}\sin(\omega t+\phi_2)=\sqrt{2}I\sin(\omega t+\phi_2)$$

and

$$p=P_{\max}\sin(\omega t+\phi_2+\phi)=\sqrt{2}P\sin(\omega t+\phi_2+\phi).$$

While the momentary values show directly the amplitude, frequency and phase of a current, the complex quantities only show amplitude and phase, and no more represent the frequency than the graphical method. It is therefore evident that no *direct* mathematical relation can exist between the momentary values and the complex expressions.

The symbolic expression

$$\underset{\cdot}{P}=\underset{\cdot}{I}\underset{\cdot}{z}=\underset{\cdot}{I}(r-jx)$$

shows that the pressure can be analysed into two components, $\underset{\cdot}{I}r$ in phase with the current and $\underset{\cdot}{I}x$ leading it by 90°.

The negative sign in $\underset{\cdot}{z}=r-jx$ is due to the fact that the figure has been rotated in a clockwise direction—if the sense of rotation were reversed, the minus sign would then become plus.

Instead of calculating symbolically, we might also proceed graphically. Like the representation of complex quantities, the graphic representation is also a purely symbolic method, in which the vectors can be added, multiplied, or divided. Up to this stage, we have only used vectors to denote current and pressure. In order, however, to carry out all operations graphically, it is also desirable to represent impedance and admittance by vectors. In Fig. 62, the vector $\overline{OC}$, with the ordinate r, and abscissa x, represents the impedance

$$\underset{\cdot}{z}=r-jx.$$

If now the current is given by the vector $\overline{OB}$, the pressure vector $\overline{OA}$ will be found by turning the current vector through angle ϕ, and at the same time increasing it in the ratio z. If we set off $\overline{OD}$ equal to unity, the two triangles OCD and OAB will be similar, so that the pressure vector can be regarded as formed from the current vector, in the same way as the impedance vector z is formed from unity.

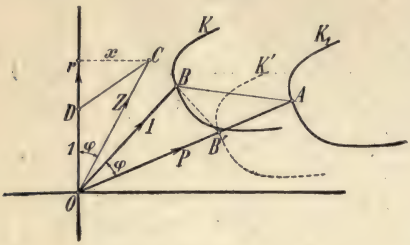


FIG. 62.

Further, assume the current I to vary after some definite law. Graphically this means that the locus of the extremity B of the current vector $\overline{OB}$ is a certain line. For example, let this line be the curve K in Fig. 62. Then the pressure vector $P = Iz$ must also obey some definite law. The locus of the extremity of this vector $\overline{OA}$ will be a curve K_1, which is found by multiplying all vectors of curve K by the constant impedance $z = z\epsilon^{-j\phi}$. The curve K_1 must be similar to the curve K, for this graphical multiplication can be considered as effected by the curve K being moved through the angle ϕ about the origin O, whereby curve K' is obtained, and then all the vectors of K' are increased in the ratio z. By this process, a point B on curve K becomes point A on curve K_1. For any two such corresponding points as A and B, the triangle OAB must have the same shape, since the angle $BOA = \phi$ and $\frac{\overline{OA}}{\overline{OB}} = z$ are constants. Hence the curve K_1 can be regarded as being traced out by the angle at A of the triangle AOB, whilst the latter moves about O, without change of shape, and with its third angle B always on the curve K.

If the curve K is formed from a system of straight lines and circular arcs, its corresponding curve K_1 admits of a very simple geometrical construction.

To multiply a straight line we multiply a point on the same, but keep the angle constant which the vector from this point to the origin makes with the straight line. A circle is multiplied by multiplying its centre and the radius, or its centre and any point on the circumference.

Let the moment of time $t = 0$ be so chosen that the pressure $P = \overline{OA}$ falls on the positive direction of the ordinate axis (Fig. 63). Then we can write symbolically:

$$I = Py = Pg + jPb \qquad \text{(39a)}$$

and

$$y = g + jb, \qquad \text{(41)}$$

in which expression the current is given in terms of two rectangular components, one of which is in phase with, and the other at 90° to,

the pressure. Fig. 64 shews that the current vector is formed from the pressure vector in the same way as the admittance y is formed from unity. Whilst the extremity of the pressure vector moves over the curve K (chosen a circle in this case), the extremity of the current vector describes the circle K_1. In Figs. 62 and 64 it has been tacitly

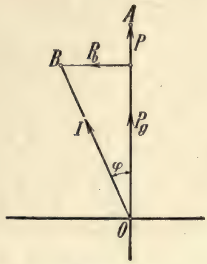


FIG. 63.

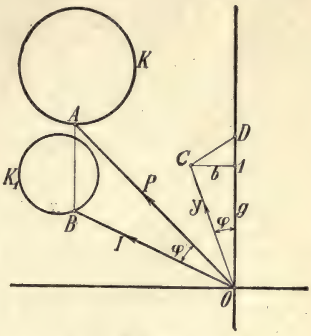


FIG. 64.

assumed, that P, I, z and y are all drawn to the same scale,—that is to say, 1 volt, 1 ampere, 1 ohm and 1 mho are all represented by the same length, e.g. 1 mm—for only in this case are the triangles OCD and OBA similar.

In graphic multiplication, it is to be noted that the rotation of the multiplied vector must be clockwise or counter-clockwise, according as the argument of the second factor is negative or positive.

21. Rotation of the Co-ordinate Axes. It follows directly that in Fig. 62 the curve K_1—which represents the pressure P_1 acting at the terminals of a constant impedance $z_1 = r_1 - jx_1$ and is similar to the curve K of the current vector I—can be obtained by graphic multiplication. At the instant $t=0$, the current and pressure vectors coincide, but are otherwise chosen independently of one another. The scale of the pressure curve depends on that of the current curve K and of the impedance z_1. If the impedance scale is chosen so that 1 cm $= z_1$ ohms, the vectors representing the pressure P_1 will be of the same length as those representing the current. The pressure curve K_1 is then obtained by simply rotating the current curve K through the angle $\phi_1 = \tan^{-1}\frac{x_1}{r_1}$ in a clockwise direction.

Instead of revolving the vectors, the co-ordinate axes can be moved through the angle ϕ_1 in a counter-clockwise direction. If the current curve is drawn so that 1 cm $= m$ amperes, this same curve, with respect to the new axes, will serve as the pressure curve to the scale 1 cm $= z_1 m$ volts.

Rotating the co-ordinate system means that zero time for the

pressures occurs $\frac{\phi_1}{2\pi}T$ seconds earlier than that for the currents. This process is shewn in Fig. 65.

Consider now, the special case of a constant terminal pressure P acting on the circuit, in which the current I is represented by the curve K. Set off the terminal pressure P along the real current axis. The pressure $\dot{P}_1 = \dot{I}\dot{z}_1$ consumed in the line impedance z_1 is represented by the current curve K with respect to the new co-ordinate axes. To obtain the direction and scale of the new real axis (or the pressure axis), we draw the current vector $\dot{I}_K$ for the case when the load is short-circuited, and the only impedance in the circuit is z_1. This is called the *short-circuit current*, and is expressed by

$$\dot{I}_K = \frac{P}{\dot{z}_1} = \frac{P}{z_1}\epsilon^{j\phi_1},$$

and has thus the direction of the real (or pressure) axis in the rotated co-ordinate system.

Now, we have just seen that 1 cm represents z_1 times as many volts in the new co-ordinate system as amperes in the original system. Hence, if $\dot{I}_K$ is set off in the original system, this same vector will represent the terminal pressure P in the new system both in magnitude and direction. This direction coincides with that of the real axis of the new system, since we have taken the terminal pressure as real, i.e. as having no component along the imaginary axis.

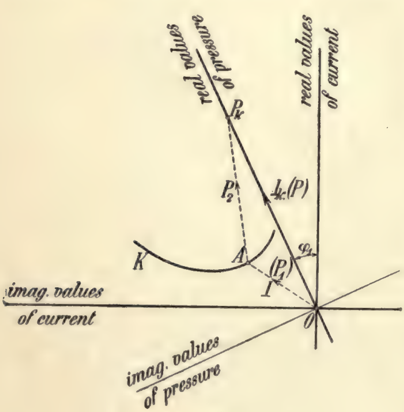


FIG. 65.

The load pressure P_2 which remains after subtracting the pressure $\dot{P}_1$ consumed in the impedance $\dot{z}_1$ from the supplied pressure $\dot{P}$ is

$$\dot{P}_2 = \dot{P} - \dot{P}_1,$$

and is thus given in the new co-ordinate system by the distance of a point A on the curve K from the short-circuit point P_K (see Fig. 65). In other words, the curve K in the new system is the locus of the apex of the pressure triangle, whose two base angles are situated at the origin O and the short-circuit point P_K respectively.

In many cases it is advantageous to take the opposite direction of the vectors as positive in the new system of co-ordinates. This is effected by rotating the co-ordinate system through the angle $\phi_1 + 180°$ in a counter-clockwise direction, and removing the origin to the short-circuit point P_K (Fig. 66). Such a diagram is known

as a *bipolar* diagram—O is the pole for currents and P_K the pole for pressures.

In Fig. 64 it was seen how the current curve K_1—similar to the given pressure curve K—could be formed by multiplying the latter by a constant admittance $y_1 = g_1 + jb_1$. Here also it is not necessary to draw a new curve, if we rotate the co-ordinate system as above. The axes of the new system are moved through the angle $\phi_1 = \tan^{-1}\frac{b_1}{g_1}$ in the direction in which the figure rotates, whilst the current scale in the new system is 1 cm $= y_1 m$ amperes, where 1 cm $= m$ volts in the original system.

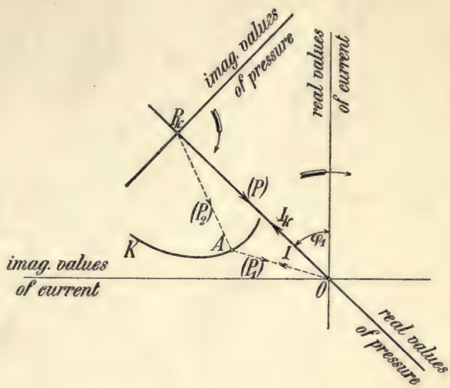


FIG. 66.

An important case is the determination of the pressure curve K of the supplied pressure P when the current I is to be constant at all loads. Let K be such a curve in Fig. 67. Apart from other conductors which may be present, let there be a path of constant admittance y_1. For the time being, suppose all other paths except y_1 to be cut out of circuit. The pressure necessary to produce the constant current I would then be

$$P_0 = \frac{I}{y_1} = \frac{I}{y_1}\epsilon^{-j\phi_1}.$$

P_0 can be called the *no-load* pressure, and coincides with the axis of real values in the new system. Moreover, since a distance represents y_1 times as many amperes in the new system as volts in the original, the no-load pressure vector P_0 in the original system gives the magnitude and direction of the constant current I in the new system.

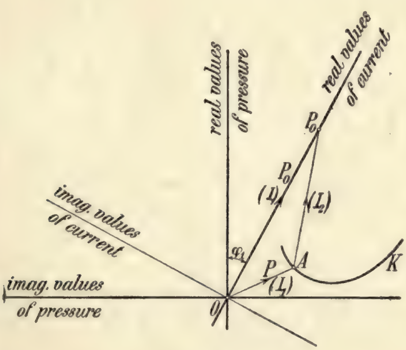


FIG. 67.

When the other branches are in circuit, the current in them is:

$$I_2 = I - I_1,$$

and is represented in the new system by the distance of the point A on the curve K from the no-load point P_0 (see Fig. 67). Hence the

pressure curve is the locus in the new system of the apex of the current triangle, whose two base angles are at the origin O and the no-load point P_0 respectively.

By displacing the origin of the new system to the no-load point P_0 and turning it through 180°, we get the bipolar diagram shewn in Fig. 68, where O is the pole of pressures and P_0 of currents.

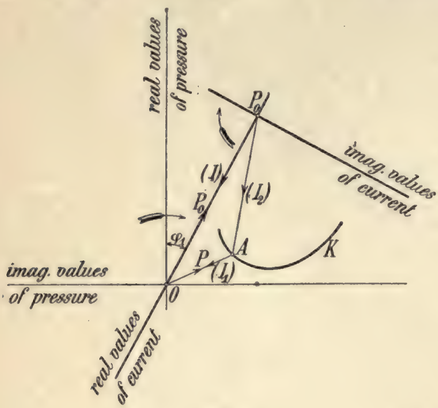


FIG. 68.

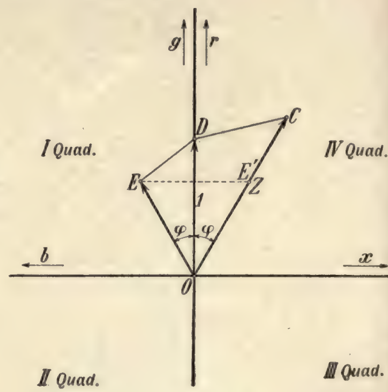


FIG. 69.

22. Inversion. In Fig. 69 the vector $\overline{OC}$ represents the impedance z, and $\overline{OE}$ the corresponding admittance $y=1/z$. They both make the same angle ϕ with the ordinate axis. *In this method of representation, resistance and conductance are set off along the ordinate or real axis, and reactance and susceptance along the abscissa or imaginary axis.* The two triangles ODC and OED are similar when z and y are drawn to the same scale. If we set off $\overline{OE'}=\overline{OE}$ along the impedance vector $\overline{OC}$, then, between the points E', which is the image of E in respect to the ordinate axis, and C there exists the simple relation,

$$\overline{OC}\,.\,\overline{OE'}=z\,.\,y=1.$$

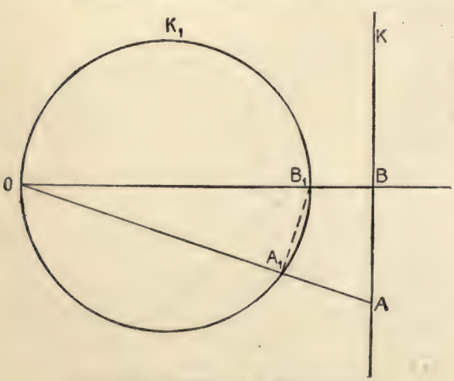


FIG. 70.—Inversion of a Straight Line.

Two such points are called *inverse* points with respect to the origin O, which is called the *centre of inversion.*

In general, if two curves K and K_1 are such that the product of the lengths OA and OA_1 cut from a straight line passing through a fixed

point O is constant, i.e. $\overline{OA} \cdot \overline{OA}_1 = \text{const.} = \mathbf{I}$, the one curve is said to be the *inverse curve* of the other, whilst $\mathbf{I}$ is called the *constant of inversion* and O is the *inversion centre.* A and A_1 are called corresponding points. The inverse curve of a straight line is a circle passing through the centre of inversion (Fig. 70).

Proof. Since triangle OA_1B_1 is similar to triangle OBA,

then $$\overline{OA}_1 : \overline{OB}_1 = \overline{OB} : \overline{OA};$$

thus, for any line $\overline{OA}$,

$$\overline{OA} \cdot \overline{OA}_1 = \overline{OB} \cdot \overline{OB}_1 = \mathbf{I}.$$

Conversely, the inverse curve of a circle which passes through the centre of inversion is a straight line.

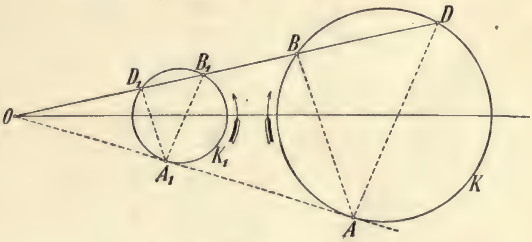


FIG. 71.—Inverse of a Circle.

The inverse curve of a circle, which does not pass through the centre of inversion, is a circle (Fig. 71), and the centre of inversion is similarly situated in respect to each of the circles.

Proof. $$\overline{OD}_1 : \overline{OB} = \overline{OB}_1 : \overline{OD}$$

or $$\overline{OD} \cdot \overline{OD}_1 = \overline{OB} \cdot \overline{OB}_1 = \overline{OA} \cdot \overline{OA}_1 = \mathbf{I}.$$

If both circles coincide, so that the circle is its own inverse curve, then the constant of inversion is $\mathbf{I} = \overline{OA}^2$.

The theorem is equally true when the point O falls within a circle, for the proof is quite independent of the position of O. The point O then falls inside the inverse circle also.

It may be noted that when the point A moves along the curve K in a certain sense, the point A_1 on curve K_1 corresponding to A on curve K will move in the opposite sense.

If the two curves cut or touch at point A, the inverse curves will also cut or touch at the corresponding point A_1.

If the two curves cut one another at A at a certain angle, the inverse curves will also cut at the same angle at A_1. In order to shew how inversion may be applied to the solution of alternating-current problems, consider a circuit along which a constant alternating-current I is flowing; the terminal pressure P must then be varied as the circuit constants are varied, and the end A of the pressure vector $\overline{OA}$ will describe

some curve K (Fig. 72). The abscissa of any point on the curve represents the wattless component and the ordinate the watt component of the corresponding E.M.F.

Since the shape of the curve is independent of the current-strength and also holds for $I=1$, the vector $\overline{OA}$ will also represent the impedance z to another scale. With symbolic representation the impedance

$$\underset{\cdot}{z}=r-jx=z(\cos\phi-j\sin\phi)=z\epsilon^{-j\phi}$$

is given by a radius-vector of length z making angle $-\phi$ with the real axis. Since $yz=1$, the curve K_1, over which the end A_1 of the

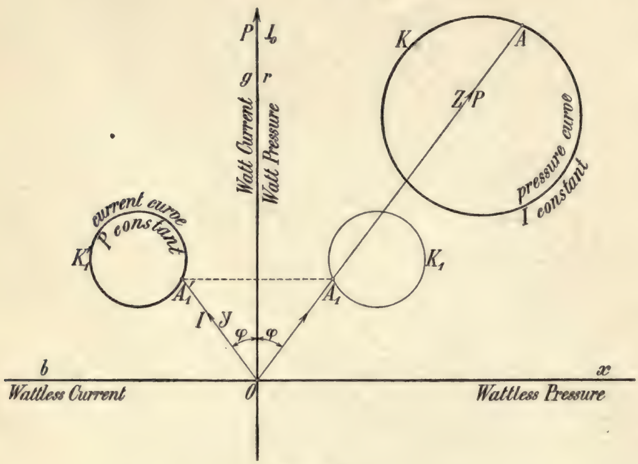


Fig. 72.

admittance vector y moves, is given by inverting the curve K over which the end A of the impedance vector $\underset{\cdot}{z}$ moves.

From the relation $\frac{b}{g}=\frac{x}{r}$ it follows also that the two radii-vectores y and $\underset{\cdot}{z}$ have the same direction, when conductance is set off along the ordinate and susceptance along the abscissa (see Fig. 72). If the radii-vectores of the admittance curve are multiplied by a constant pressure P, the vectors $\overline{OA_1}$ will give the current in the circuit to a certain scale. The ordinates then represent watt currents and the abscissae wattless currents. The admittance y, corresponding to the impedance $\underset{\cdot}{z}=z\epsilon^{-j\phi}$, is

$$\underset{\cdot}{y}=\frac{1}{\underset{\cdot}{z}}=\frac{1}{z\epsilon^{-j\phi}}=y\epsilon^{j\phi}=g+jb.$$

Hence, the admittance vector y will lie in quadrant I. when $\underset{\cdot}{z}$ lies in quadrant IV. and vice versa—or, if $\underset{\cdot}{z}$ lies in quadrant III. then y lies in quadrant II. and vice versa. Thus we see that the vector y cannot coincide with the vector $\underset{\cdot}{z}$ if the same system of co-ordinates is used

for admittance and current vectors as for impedance and pressure vectors.

The direction of the y-vector is the image of the z-vector with respect to the ordinate axis. Hence, if we wish to apply graphical inversion to alternating-currents, *we must in every case substitute the inverse curve* K_1 *(obtained by inversion of the curve* K*) by its image* K'_1 *with respect to the ordinate axis.*

In practice, however, the process of inversion can be simplified as follows. If the admittance or current curve of a circuit is desired, and we wish to derive the same by a *single* inversion of the impedance curve, then, instead of drawing the impedance curve itself, we draw its image with respect to the ordinate axis; the desired admittance or current curve is then obtained directly by inverting this image.

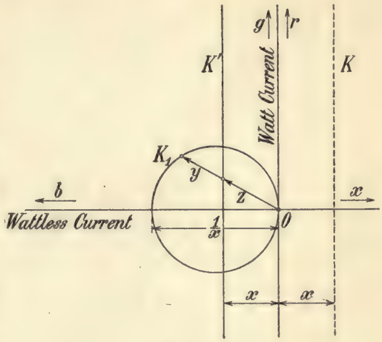


FIG. 73.

The process can be best illustrated by an example. Given a simple circuit with constant reactance x in series with a variable resistance r. The impedance curve is then a straight line K parallel to the ordinate axis and displaced from the same by a distance x (Fig. 73). The image of this straight line about the ordinate axis is K'. The inverse curve of the straight line K' is the circle K_1 of diameter $\frac{1}{x}$. This circle, whose centre lies on the abscissa axis, is then the admittance curve, and when all vectors are multiplied by the pressure P, we get the current curve. This agrees with that in Fig 54, but has been obtained in another way; both circles have the same diameter $\frac{P}{x}$.

Similarly the impedance or E.M.F. curve can be constructed by a single inversion of the image K' of the admittance curve K. For a circuit with constant susceptance b in parallel with a variable conductance g, the curves K and K' are straight lines parallel to the ordinate axis (Fig. 74). The inverse curve K_1, representing the impedance curve, is a circle of diameter $\frac{1}{b}$, whose centre lies on the abscissa axis. By multiplying all the vectors by I, the same pressure diagram is obtained as in Fig. 60, for both circles have the same diameter $\frac{I}{b}$.

It often happens that two inversions must be made in order to obtain a desired diagram.

In this case it is not necessary to draw the image of the inverted

curve, for if the first curve lies in quadrant IV. the inverted curve will lie in I., and the curve obtained after the second inversion will fall again in quadrant IV. Hence, since both the curve from which we start and the curve we obtain lie in the same quadrant, it is more convenient to carry out all the operations in the one quadrant, whereby the figures are also clearer.

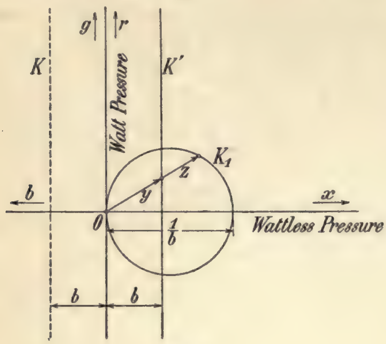


FIG. 74.

In general, therefore we proceed as follows: *According as an even or an odd number of inversions must be carried out in order to obtain a particular diagram, it is desirable to start from the actual diagram, or its image.*

Of the two curves which represent the impedance and admittance of a circuit by polar co-ordinates, the one is always the inverse of the other. The constant of inversion **I** depends on the scales for y and z.

Since the ratio of inversion is a function of the scales, after drawing the first magnitude to a convenient scale it is possible to choose the constant of inversion **I** so that the inverse figure will also be drawn to a suitable scale. This is illustrated by the following example.

In Fig. 72 let the admittance y be set off so that 1 cm $= m$ mhos. Then, if we wish to have the scale of the impedance z such that 1 cm equals n ohms, we get

$$y = m \,.\, \overline{OA}_1 \text{ mhos,}$$

$$z = n \,.\, \overline{OA} \text{ ohms.}$$

Then

$$yz = mn\overline{OA}_1 \,.\, \overline{OA} = 1.$$

Hence, the constant of inversion is

$$\mathbf{I} = \overline{OA} \,.\, \overline{OA}_1 = \frac{1}{mn}. \quad \text{(42)}$$

If Fig. 72 is drawn for currents and pressures to the scales 1 cm $= m$ amps. $= n$ volts, and I_0 and P_0 denote the corresponding constants of the circuit, we have

$$I = m \,.\, \overline{OA}_1 = P_0 y \text{ amperes}$$

and

$$P = n \,.\, \overline{OA} = I_0 z \text{ volts,}$$

whence

$$mn\overline{OA} \,.\, \overline{OA}_1 = I_0 P_0 yz = I_0 P_0,$$

and the constant of inversion is

$$\mathbf{I} = \overline{OA} \,.\, \overline{OA}_1 = \frac{I_0 P_0}{mn}. \quad \text{(42a)}$$

Before leaving inversion, the following theorem may be mentioned: *If we have two figures in which any point and any circle of the one correspond to a point and a circle of the other, then it is always possible, by inversion and multiplication, to convert one system into the other.*

From this it follows that, by means of the foregoing methods, every locus which is a straight line or a circle can be deduced from other loci which are straight lines or circles.

Since the inverse of a circle is a circle, in carrying out the inversion of circles, instead of proceeding point by point, it is simpler to calculate the co-ordinates of the centre and the radius of the new circle.

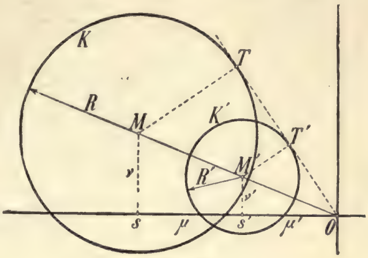


FIG. 75.

Example. Given a circle K of radius R (Fig. 75), the co-ordinates of whose centre M are ν and μ—to calculate the radius R' and the co-ordinates μ' and ν' of the centre M' of the circle K', which is to be the inverse of K for the constant of inversion $\mathbf{I}$. Drawing the common tangent $\overline{OTT'}$ to the two circles, we have

$$\overline{OT'} = \frac{\mathbf{I}}{\overline{OT}}$$

and

$$\overline{OT} = \sqrt{\mu^2 + \nu^2 - R^2}.$$

By aid of the similar triangles $OM'S'$, OMS and OTM, $OT'M'$, it is easy to shew that

$$\nu' = \nu \frac{\mathbf{I}}{\overline{OT}^2} = \mathbf{I} \frac{\nu}{\mu^2 + \nu^2 - R^2},$$

$$\mu' = \mu \frac{\mathbf{I}}{\overline{OT}^2} = \mathbf{I} \frac{\mu}{\mu^2 + \nu^2 - R^2},$$

$$R' = R \frac{\mathbf{I}}{\overline{OT}^2} = \mathbf{I} \frac{R}{\mu^2 + \nu^2 - R^2};$$

so that the new circle is determined both as regards magnitude and position.

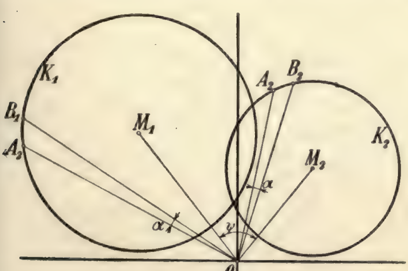


FIG. 76.

Two circles which are formed from one another by multiplication and rotation correspond point for point with respect to the origin of the co-ordinate axes, for we pass from two corresponding points A_1 and A_2 (Fig. 76) of the two circles to two other corresponding points B_1 and B_2 by rotating the vectors about O through the same angle α.

The locus of the sum of the corresponding vectors of two corresponding circles is also a circle. For when the circle K_2 is formed from K_1 by multiplying by a constant k and rotating through a constant angle ψ, we have, for corresponding vectors, a_1 and a_2 of these circles,

$$a_2 = a_1 k\epsilon^{-j\psi}.$$

Hence, the resultant vector,

$$a_1 + a_2 = a_1(1 + k\epsilon^{-j\psi}),$$

is always proportional to a_1 and displaced from it by a constant angle, and consequently moves over a circle.

If two circles correspond in respect of two points on their circumferences, the locus of the sum of the vectors of corresponding points is also a circle.

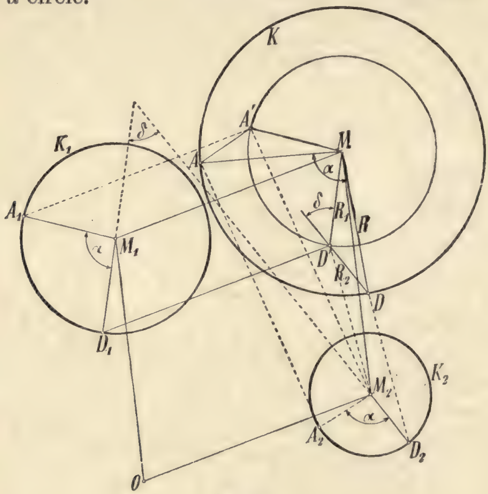


FIG. 77.

In Fig. 77 let K_1 and K_2 be the two circles and D_1, D_2 two corresponding points. If A_1 and A_2 are likewise to be corresponding points, we shall then have

$$\angle D_1M_1A_1 = \angle D_2M_2A_2 = \alpha.$$

The point M is obtained by adding OM_1 and OM_2. Setting off

$\overline{MA'}$ equal and parallel to $\overline{M_1A_1}$,

$\overline{A'A}$ " " " M_2A_2,

then A is the sum of the two points A_1 and A_2. In the same way, D is the sum of the two points D_1 and D_2, where the angle $DMA = \alpha$.

The sum of the two circles K_1 and K_2 is thus the circle K, whose radius is

$$R = \sqrt{R_1^2 + R_2^2 + 2R_1R_2 \cos\delta},$$

where δ is the angle between two corresponding radii of the circles K_1 and K_2.

23. Graphic Representation of the Losses in the Impedance in a Circuit. If a current I is transmitted over a circuit whose line impedance is $z=r-jx$, the energy consumed thereby is $V=I^2r$. We shall now shew how this energy V, which is dissipated in the form of heat, can be represented graphically for the case when the current diagram is a circle. Let μ and ν be the co-ordinates of the centre of a circle whose radius is R, and u and v the co-ordinates of a point on the circle (Fig. 78). The equation of the circle is

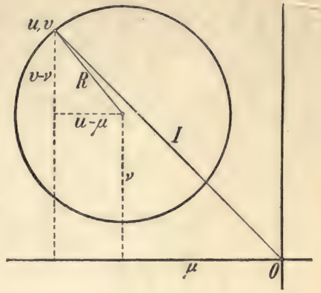


FIG. 78.

$$(u-\mu)^2+(v-\nu)^2=R^2$$

or

$$u^2+v^2-2\mu u-2\nu v=R^2-\mu^2-\nu^2=-\rho^2.$$

The heating losses are

$$V=I^2r=(u^2+v^2)r$$

$$=2r\left(\mu u+\nu v-\frac{\rho^2}{2}\right)=2r\mathsf{V},$$

where, for the sake of brevity,

$$\mu u+\nu v-\frac{\rho^2}{2}=\mathsf{V}.$$

Now $\mathsf{V}=0$ is the equation of a straight line, whilst u and v represent the co-ordinates of points on it.

The *polar* of the current circle, with respect to the origin O, has the equation

$$\mu u+\nu v-\rho^2=0.$$

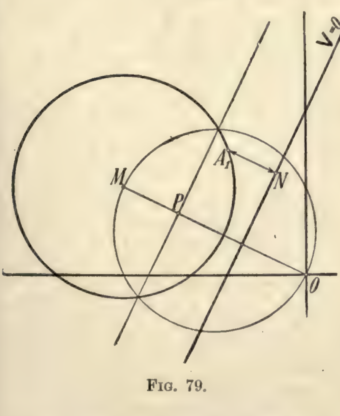


FIG. 79.

From this, we see that the straight line $\mathsf{V}=0$ is parallel to the polar and bisects the distance between it and the origin. Hence the line $\mathsf{V}=\mu u+\nu v-\frac{\rho^2}{2}=0$ is called the *semi-polar* of the circle with respect to the origin O.

To construct the semi-polar $\mathsf{V}=0$, draw a circle on $\overline{OM}$ as diameter, where M is the centre of the current circle (Fig. 79). The circle on $\overline{OM}$ cuts the current circle in two points which lie on the polar, so that the latter can be drawn at once. The semi-polar $\mathsf{V}=0$ is then the line drawn parallel to the polar to bisect the distance $\overline{OP}$.

For any point on the semi-polar, we have:

$$\mathsf{V} = \mu u + \nu v - \frac{\rho^2}{2} = 0,$$

where u, v are the co-ordinates of the point.

For points u, v which do not lie on the semi-polar, the expression for V can be found as follows:

Let the straight line I (Fig. 80) have the equation

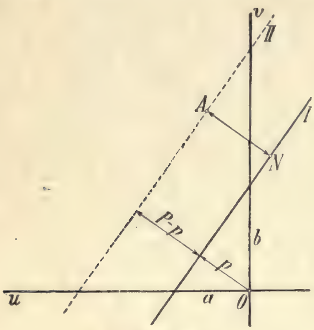


FIG. 80.

$$\frac{u}{a} + \frac{v}{b} - 1 = 0.$$

Further, $p : a = b : \sqrt{a^2 + b^2}$,

$$p = \frac{ab}{\sqrt{a^2 + b^2}}.$$

Hence the equation of the straight line I may also be written

$$bu + av - p\sqrt{a^2 + b^2} = 0.$$

A parallel straight line II at distance P from the origin has the equation

$$bu + av - P\sqrt{a^2 + b^2} = 0.$$

For a point u_1, v_1 on the straight line II,

$$bu_1 + av_1 - P\sqrt{a^2 + b^2} = 0.$$

Hence the equation of the straight line I may also be written:

$$b(u - u_1) + a(v - v_1) + (P - p)\sqrt{a^2 + b^2} = 0.$$

Introducing now into the equation of straight line I, the co-ordinates u_1, v_1, of a point on straight line II, we get

$$(P - p)\sqrt{a^2 + b^2} = \overline{AN}\sqrt{a^2 + b^2}.$$

Returning to the previous case, we see that the linear expression

$$\mu u + \nu v - \frac{\rho^2}{2} = \mathsf{V},$$

for any point u, v in the plane has a value which is proportional to the distance of this point from the straight line whose equation is obtained when we put the linear expression of the co-ordinates equal to zero. The factor of proportionality is $\sqrt{\mu^2 + \nu^2}$, and equals the distance of the centre of the circle M from the origin O.

For any point A_1 corresponding to the current I on the circle (Fig. 79), we thus get the loss V in the impedance,

$$V = I^2 r = 2r\mathsf{V} = 2r \,.\, \overline{OM} \,.\, \overline{A_1N}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(43)$$

where $\overline{A_1N}$ is the distance of A_1 from the semi-polar. In what follows we shall call the semi-polar the *loss line*.

If the circle represents the current due to a constant terminal pressure P, then for the point A_1, the *total supplied power W* is

$$W = PI\cos\phi = P \times \text{ordinate of point } A_1.$$

Until now, it has been assumed that the current scale is unity, so that 1 cm corresponds to 1 ampere. If the current scale is

$$1 \text{ cm} = m \text{ amps.},$$

then the loss V is

$$V = I^2 r = 2m^2 r\overline{OM}\,.\,\overline{A_1N} \text{ watts}, \qquad (43a)$$

and the supplied power W is

$$W = Pm \times \text{ordinate of point } A_1.$$

Hence the ratio of the scales for loss and power is

$$\frac{2mr\overline{OM}}{P}.$$

If the origin O lies on the circle, then the loss line $\mathsf{V}=0$ coincides with the tangent at this point O. For the case when the origin O lies within the circle, as in Fig. 81, the pole P of the origin is found by drawing a perpendicular through O, and where this perpendicular cuts the circle, drawing tangents to meet at P in $\overline{MO}$ produced. The loss line $\mathsf{V}=0$ then bisects $\overline{OP}$ at right angles as previously. Thus, every point has the same loss line as its pole.

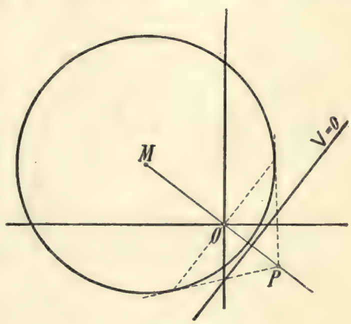


FIG. 81.

If the pressure P between two points in a circuit is represented by a circle diagram and we wish to find the loss consumed in a constant admittance $y = g + jb$ between these two points, we get the same construction as above, for the loss in the admittance is

$$V = P^2 g,$$

where P^2 can be represented by the distances of points on the circle from a loss line $\mathsf{V}=0$, just as I^2 above. Hence, for a point A_1 on the pressure circle whose centre is M, the loss is

$$V = 2g\mathsf{V} = 2g\overline{OM}\,.\,\overline{A_1N}, \qquad (44)$$

where O is the origin and $\overline{A_1N}$ the distance of the point on the circle from the line $\mathsf{V}=0$.

24. Graphic Representation of the Useful Power in the Impedance in a Circuit. With constant terminal pressure P, the power supplied to the circuit is

$$W = P \times \text{watt component of current} = P.v,$$

where v is the ordinate of the current curve. The difference between the supplied power W and the heating losses V, which we shall call the *useful power* W_1, can also be represented graphically. Thus

$$W_1 = W - V = W - I^2r = Pv - 2r\mathsf{V} = 2r\left(\frac{P}{2r}v - \mathsf{V}\right),$$

where
$$\mathsf{V} = \mu u + \nu v - \frac{\rho^2}{2}.$$

Substituting in the same way,

$$\mathsf{W} = \frac{P}{2r}v \quad \text{and} \quad W = 2r\mathsf{W},$$

then $\mathsf{W} = 0$ is the equation of the abscissa axis of the system of co-ordinates. Then

$$W_1 = 2r(\mathsf{W} - \mathsf{V})$$
$$= 2r\mathsf{W}_1,$$

where
$$\mathsf{W}_1 = \mathsf{W} - \mathsf{V} = -\mu u - \left(\nu - \frac{P}{2r}\right)v + \frac{\rho^2}{2} = 0$$

is the equation of a straight line passing through the point of intersection of the loss line with the abscissa axis.

In general, for any point whose co-ordinates are u and v, the expression for W_1 has a value proportional to the distance of the point (u, v) from the straight line $\mathsf{W}_1 = 0$. Denoting this distance by $\overline{A_1N}$, then

$$\mathsf{W}_1 = \sqrt{\mu^2 + \left(\nu - \frac{P}{2r}\right)^2} \cdot \overline{A_1N}.$$

Hence the difference W_1 between the supplied power W and the losses V is given by the distance of the respective point on the current circle from the straight line $\mathsf{W}_1 = 0$. This line $\mathsf{W}_1 = 0$ will be denoted as the *power line* of the diagram.

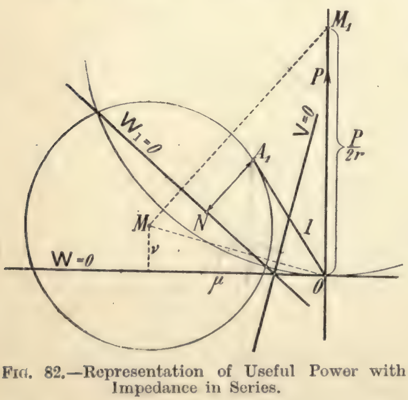


FIG. 82.—Representation of Useful Power with Impedance in Series.

The equation of the power line is obtained by subtracting the equation of the circle

$$u^2 + v^2 - \frac{P}{r}v = 0$$

from the equation of the current curve

$$u^2 + v^2 - 2\mu u - 2\nu v + \rho^2 = 0.$$

Thus the power line passes through the intersection of these two circles and can be constructed, as in Fig. 82, provided the current curve and resistance r are known.

For a point A_1 on the current curve,

$$\mathsf{W}_1 = \sqrt{\mu^2 + \left(\nu - \frac{P}{2r}\right)^2} \,.\, \overline{A_1N} = \overline{MM_1} \,.\, \overline{A_1N},$$

and the power $$W_1 = 2r\mathsf{W}_1 = 2r\overline{MM_1} \,.\, \overline{A_1N}, \quad \text{.....................(45)}$$

or, for the current scale m,

$$W_1 = 2m^2r\overline{MM_1} \,.\, \overline{A_1N} \text{ watts.} \quad \text{..................(45}a\text{)}$$

The points of intersection of the power line with the current circle have a definite physical meaning, which we shall now consider. As already shewn in Ch. II. p. 49, the circle about centre M_1 with radius $\frac{P}{2r}$ would represent the current diagram for the case when only the resistance r and a variable reactance x are in the circuit. This must hold for the points of intersection between power line and current curve, for these points lie on the circle about centre M_1, as proved. At these points, the total supplied power is consumed in the resistance r, and the useful power is therefore zero. One case when this happens is when the applied pressure is short-circuited through the impedance z (short-circuit point), and the other case when the load in series with the impedance z is wattless (no-load point).

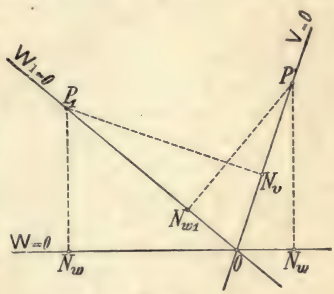


FIG. 83.

The power scales for the diagram can be determined as follows. Let the current curve be drawn to such a scale that 1 cm along the ordinate of a point corresponds to m_w watts. Then for a point P_1 the supplied power (Fig. 83) is

$$W = m_w \,.\, \overline{P_1N_w} = m_w\overline{P_1O}\sin[\mathsf{WW}_1].$$

The power line passes through the points where the supplied pressure equals the losses. If the point P_1 lies on the power line $\mathsf{W}_1 = 0$, we have also

$$W = V = m_v\overline{P_1N_v} = m_v\overline{P_1O}\sin[\mathsf{W}_1\mathsf{V}],$$

where m_v = the scale of the losses. Hence

$$\frac{m_v}{m_w} = \frac{\sin[\mathsf{WW}_1]}{\sin[\mathsf{W}_1\mathsf{V}]}. \quad \text{...........................(46)}$$

Since the loss line $\mathsf{V} = 0$ also passes through the points where the useful power W_1 equals the supplied power W, we get for a point P,

$$m_w\overline{PN_w} = m_{w_1}\overline{PN_{w_1}},$$

$$m_w\sin[\mathsf{WV}] = m_{w_1}\sin[\mathsf{W}_1\mathsf{V}],$$

$$\frac{m_{w_1}}{m_w} = \frac{\sin[\mathsf{WV}]}{\sin[\mathsf{W}_1\mathsf{V}]}, \quad \text{.........................(46}a\text{)}$$

where m_{w_1} is the scale for the useful power. Hence we get the following rule for determining the power scales of the diagram: *If two powers are measured by the perpendicular distances of a point from the corresponding straight lines, the scales are inversely proportional to the sines of the angles, which the respective straight lines make with the line for which the two measured powers are equal.* Since the perpendicular distance of a point from a straight line always remains proportional to the length of the

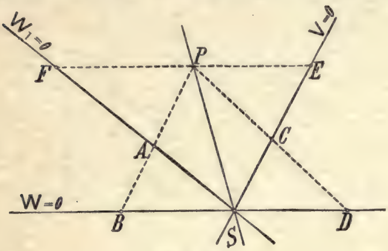


FIG. 84.

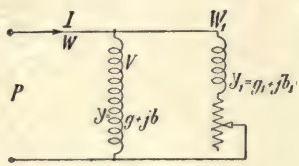


FIG. 85.

line drawn at a constant angle to the straight line, the following Rule will at once be apparent (see Fig. 84). *If two powers are measured by the distances of a point from the corresponding straight lines in the direction parallel to that line for which both the measured powers are equal, then the two powers will have the same scale.*

Thus in Fig. 84, for a point P,

$$\frac{W_1}{W}=\frac{\overline{PA}}{\overline{PB}};\quad \frac{V}{W}=\frac{\overline{PC}}{\overline{PD}};\quad \frac{V}{W_1}=\frac{\overline{PE}}{\overline{PF}}.$$

If the loss does not occur in an impedance in series with the load, but, as is shewn in Fig. 85, in a constant admittance $y=g+jb$ connected in parallel with the load, the useful power is

$$W_1=W-P^2g.$$

Let the pressure vector P move over the circle K in Fig. 86, which has the equation

$$u^2+v^2-2\mu u-2\nu v$$
$$=R^2-\mu^2-\nu^2=-\rho^2,$$

and set off the current $I=I$ along the real axis; we can then write

$$gW_1=2\left(\frac{I}{2g}v-\mathsf{V}\right)=2g(\mathsf{W}-\mathsf{V}),$$

where

$$\mathsf{V}=\mu u+\nu v-\frac{\rho^2}{2},$$

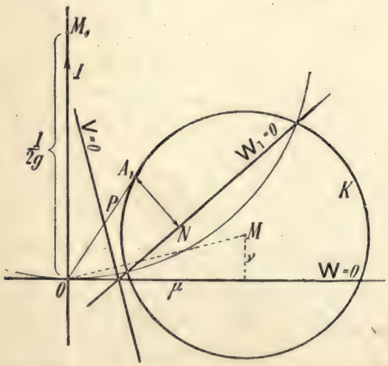


FIG. 86.—Representation of Useful Power with Admittance in Parallel.

whilst $\mathsf{V}=0$ is the equation of the semi-polar of the pressure circle with respect to the origin, or the *loss line*.

Further, $$\frac{I}{2g}v = \mathsf{W},$$

and $\mathsf{W} = 0$ is the equation of the abscissa axis.

We now proceed precisely as above, and put

$$\mathsf{W}_1 = \mathsf{W} - \mathsf{V} = -\mu u - \left(\nu - \frac{I}{2g}\right)v + \frac{\rho^2}{2}.$$

From this we see that with a constant current I the equation

$$\mathsf{W}_1 = 0$$

represents a straight line. This is the *power line* of the circuit.

The power line must pass through the points of the circle K for which the output W_1 is zero. For these points $g_1 = 0$, and consequently the whole conductance between the terminals equals g. Now all pressure vectors for a circuit with a constant current I and conductance g lie on the circle drawn about M_1 with radius $I/2g$ in Fig. 86. Hence the power line $\mathsf{W}_1 = 0$ passes through the point of intersection of this circle with the pressure circle K.

For a point A_1 on the pressure circle K,

$$\mathsf{W}_1 = \sqrt{\mu^2 + \left(\nu - \frac{I}{2g}\right)^2} \cdot \overline{A_1N} = \overline{MM_1} \cdot \overline{A_1N},$$

and the output $$W_1 = 2g\overline{MM_1} \cdot \overline{A_1N}. \qquad (47)$$

If the points M, M_1 and A_1 are set off to the pressure scale 1 cm $= n$ volts, then

$$W_1 = 2n^2g\overline{MM_1} \cdot \overline{A_1N} \text{ watts}. \qquad (47a)$$

25. Graphic Representation of Efficiency. Let a straight line (Fig. 87) from the point P pass through the point of intersection S of the three straight lines $\mathsf{W} = 0$, $\mathsf{W}_1 = 0$ and $\mathsf{V} = 0$, then for all points on this straight line $\overline{SP}$ the ratios between the several powers remain constant, which follows at once from the graphic representation of these ratios.

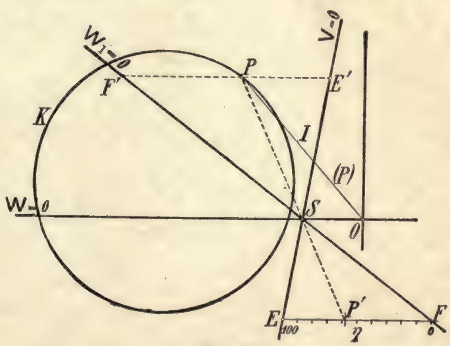


FIG. 87.

From this it is seen that the efficiency of a circuit can be shewn as in Fig. 87. The line $\overline{EF}$ is drawn parallel to the line of supplied power $\mathsf{W} = 0$ between the power line $\mathsf{W}_1 = 0$ and the loss line $\mathsf{V} = 0$. This line $\overline{EF}$ is then divided into 100 equal parts, as shewn.

For a point P on the current (or pressure) circle, the percentage efficiency η is given by the point P', where $\overline{PS}$ produced cuts $\overline{EF}$.

Proof. For every point P on the line $\overline{SP}$, we have (Fig. 87)

$$\frac{V}{W_1}=\frac{\overline{PE'}}{\overline{PF'}},$$

$$\frac{W_1}{W_1}=\frac{\overline{PF'}}{\overline{PF'}},$$

$$\frac{W_1+V}{W_1}=\frac{W}{W_1}=\frac{\overline{PE'}+\overline{PF'}}{\overline{PF'}}=\frac{\overline{E'F'}}{\overline{PF'}}.$$

Thus $$\eta\% = 100\frac{W_1}{W}=\frac{100}{\overline{E'F'}}\cdot\overline{PF'}=\frac{100}{\overline{EF}}\overline{P'F}.$$

When the pressure acting on the two terminals of a circuit is altered, without altering the circuit constants, the current alters in proportion to the pressure, and the ratio between the powers in the several parts remains unaltered.

Hence, the method deduced above for determining the relation between two powers in a current-circle diagram holds even when the pressure changes its value. In like manner the analogous method is applicable for a pressure-circle diagram when the current varies.

CHAPTER IV.

SERIES CIRCUITS.

26. Circuit with two Impedances in Series. As an example of two impedances in series, we have the power transmission line represented in Fig. 88. Since this case is one of the simplest and also of considerable practical importance, we shall investigate it fully.

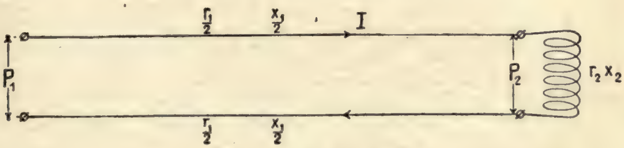


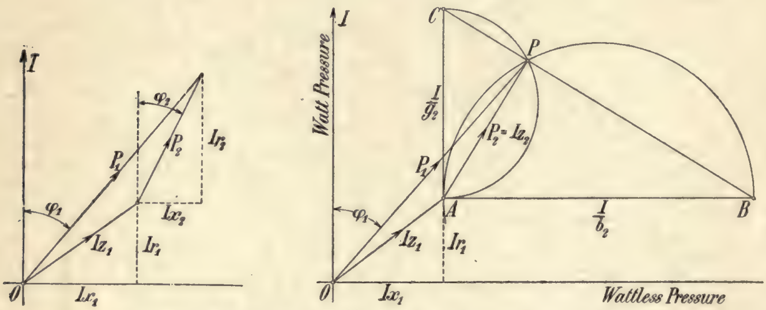
FIG. 88.

The pressure P_1, which is applied at the supply terminals, is consumed by the resistance and reactance of the line and the load at the receiver terminals. Since the pressure required to overcome the E.M.F. of self-induction of the transmission line leads the current by 90°, whilst the pressure consumed by the ohmic resistance of the line is in phase with the current, it is obvious that, with constant supply pressure P_1, the pressure P_2 at the receiver terminals depends to a large extent on the phase displacement of the load. The receiver pressure P_2 can be resolved into two components, the one Ir_2, in phase with the current, and the other Ix_2, leading the current by 90°, where r_2 and x_2 are the constants of the receiver or load circuit. Conversely, the current I can be resolved into two components, one of which P_2g_2 is in phase with the receiver pressure and the other P_2b_2 lags 90° behind it.

Assuming the current I to be given, we can find the receiver pressure P_2 from its components Ir_2 and Ix_2; similarly we can find the pressure drop Iz_1 in the line from its components Ir_1, Ix_1. P_1 is the geometrical sum of these two pressures (see Fig. 89). Fig. 90 follows at once from Fig. 60.

Since r_1 and x_1 are constant, the pressure Iz_1 will be constant so long as the current remains unchanged. Let b_2 be kept constant in the receiver circuit and g_2 varied, then extremity P of the vector P_2 will move over the semi-circle on $\overline{AB}=\frac{I}{b_2}$. If g_2 is maintained constant and b_2 varied, the locus of P is the circle described on $\overline{AC}=\frac{I}{g_2}$.

The assumption of constant current I is of much less practical interest than the case of constant receiver pressure P_2 or constant supply pressure P_1, which we shall now discuss.



FIGS. 89 and 90.—Pressure Diagrams of Two Circuits in Series.

Since all the vectors in Fig. 90 are directly proportional to I, we can suppose the diagrams to be drawn for the case $I=1$; then the vector $\overline{OP}$ will represent the total impedance z in the circuit.

From Fig. 89 it is seen that

$$z=\sqrt{(r_1+r_2)^2+(x_1+x_2)^2},$$

$$\tan\phi_2=\frac{x_2}{r_2}=\frac{b_2}{g_2},$$

$$\cos\phi_2=\frac{g_2}{\sqrt{g_2^2+b_2^2}},$$

$$\tan\phi_1=\frac{x_1+x_2}{r_1+r_2},$$

$$I=\frac{P_1}{z}=\frac{P_1}{\sqrt{(r_1+r_2)^2+(x_1+x_2)^2}},$$

$$P_2=\frac{I}{y_2}=z_2\frac{P_1}{\sqrt{z_2^2+z_1^2+2r_1r_2+2x_1x_2}}$$

$$=\frac{P_1}{\sqrt{1+(r_1^2+x_1^2)(g_2^2+b_2^2)+2r_1g_2+2x_1b_2}};$$

or, by transformation,

$$P_2 = \frac{P_1}{\sqrt{(1+r_1g_2+x_1b_2)^2+(x_1g_2-r_1b_2)^2}} = P_1\alpha, \quad \text{.......(48)}$$

where
$$\alpha = \frac{1}{\sqrt{(1+r_1g_2+x_1b_2)^2+(x_1g_2-r_1b_2)^2}}.$$

The current is then

$$I = Py_2 = P_1\sqrt{\frac{g_2^2+b_2^2}{(1+r_1g_2+x_1b_2)^2+(x_1g_2-r_1b_2)^2}},$$

and the power at the receiver terminals,

$$W_2 = P_2 \times \text{watt component of current}$$
$$= P_2^2 g_2 = P_1^2\alpha^2 g_2.$$

If the susceptance b_2 and the supply pressure P_1 are constant, the power W_2 will have a maximum value. The value of g_2, for which the power W_2 in the receiver circuit is a maximum, is found by differentiation; that is,

$$\frac{dW_2}{dg_2} = \frac{d(P_1^2\alpha^2 g_2)}{dg_2} = 0;$$

or, since the reciprocal of W_2 will then be a minimum, we can put

$$\frac{d}{dg_2}\left(\frac{1}{\alpha^2 g_2}\right) = \frac{d}{dg_2}\left\{\frac{(1+r_1g_2+x_1b_2)^2+(x_1g_2-r_1b_2)^2}{g_2}\right\} = 0.$$

This occurs when
$$g_2 = \sqrt{g_1^2+(b_1+b_2)^2}. \quad \text{.........................(49)}$$

In this case
$$\frac{P_2}{P_1} = \alpha = \frac{1}{\sqrt{2g_2(g_2z_1^2+r_1)}},$$

and the *maximum power* transmitted is

$$W_{2\,\max} = \frac{P_1^2}{2(g_2z_1^2+r_1)}. \quad \text{.........................(50)}$$

Since, in general, the power transmitted to the receiver circuit can be written

$$W_2 = I^2 r_2 \text{ watts},$$

and the total supplied power

$$W_1 = I^2(r_1+r_2),$$

the *efficiency* η is given by
$$\eta\,\% = 100\frac{r_2}{r_1+r_2}\,\%;$$

or, since
$$\eta\,\% = 100\frac{1}{1+\dfrac{r_1}{r_2}}\,\%,$$

it is obvious that the efficiency will be a maximum when $\dfrac{r_1}{r_2}$ is a minimum.

The ratio
$$\frac{r_1}{r_2} = \frac{r_1(b_2^2+g_2^2)}{g_2}$$
has its minimum value (with constant susceptance b_2) when
$$\frac{d}{dg_2}\left(\frac{b_2^2+g_2^2}{g_2}\right)=0,$$
i.e. when
$$g_2 = b_2.$$
Hence the *maximum efficiency* is
$$\eta_{\text{max}}\,\% = \frac{100}{1+r_1\frac{2g_2^2}{g_2}} = \frac{100}{1+2r_1b_2}. \qquad (51)$$

27. Example I. A load, having constant susceptance b_2 and conductance g_2 variable with the load (e.g. asynchronous motors), is fed over a long transmission line, which has both ohmic resistance

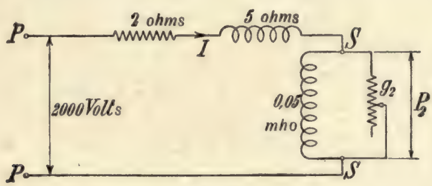


FIG. 91.

and self-induction. In order to better illustrate the effect of these constants on the receiver pressure P_2, they have been chosen larger than would be the case in an efficient installation.

We are given (see Fig. 91):
$$P_1 = 2000 \text{ volts}; \quad r_1 = 2{\cdot}0 \text{ ohms};$$
$$x_1 = 5{\cdot}0 \text{ ohms}; \quad b_2 = 0{\cdot}05 \text{ mho}.$$
Determine first how the receiver (or load) pressure P_2 and the current I depend on the load; and secondly, find the efficiency η and the power-factor $\cos\phi_1$ of the system.

A simple solution of the problem can be obtained by the graphical method of inversion and rotation of the co-ordinate system, whilst at the same time we get a clear insight into the working of the system.

In Fig. 92 the circle K_1 is the image of the impedance of the system for the case when g_2 is varied. The impedance scale is 1 cm = 5 ohms.

Thus
$$\overline{OA}_1 = \frac{z_1}{5} = \frac{5{\cdot}38}{5} = 1{\cdot}076 \text{ cm},$$
$$\overline{A_1B_1} = \frac{1}{5}\,\frac{1}{b_2} = \frac{1}{5}\,\frac{1}{0{\cdot}05} = 4 \text{ cm}.$$
The current curve for a constant pressure P_1 at the supply terminals is, as already explained, the inverse curve K of the image K_1 of the curve representing the total impedance between the supply terminals.

The constant of inversion must now be chosen so that the current curve is drawn to a suitable scale. Let 1 cm = 50 amps., then for two corresponding points P_1 and P, which lie on the impedance and current curves respectively,

$$\overline{OP_1} = \frac{1}{5} z,$$

$$\overline{OP} = \frac{1}{50} I = \frac{1}{50} \frac{P_1}{z} = \frac{40}{z},$$

$$\overline{OP_1} \cdot \overline{OP} = 8 = \text{constant of inversion.}$$

In this way the current curve is obtained as circle K with centre M. Consider the points A and B which lie on this circle. The

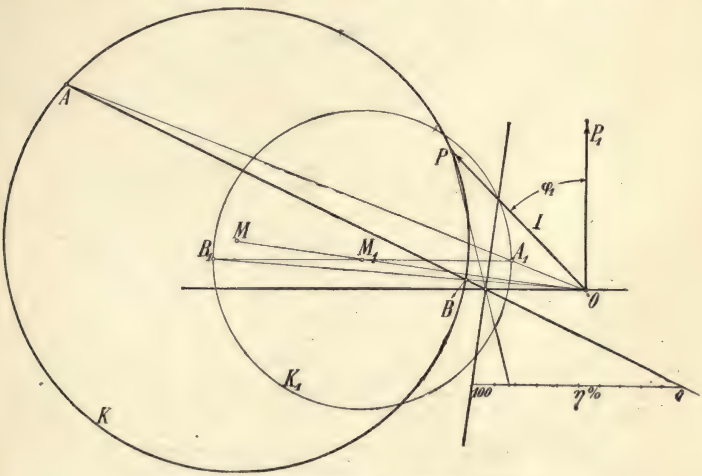


Fig. 92.—Current Diagram of Circuit in Fig. 91.

vector $\overline{OA}$ represents the current when $g_2 = \infty$ or $r_2 = 0$, that is, the current when the receiver terminals are short-circuited (i.e. A is the short-circuit point). The current represented by OA is therefore

$$I_K = \frac{P_1}{z_1} = P_1\left(\frac{r_1}{z_1^2} + j\frac{x_1}{z_1^2}\right).$$

The vector $\overline{OB}$ represents the current when $g_2 = 0$, or $r_2 = \infty$, that is, when the load is purely inductive and possesses only the susceptance b_2 (i.e. B is the no-load point). The no-load current is

$$I_0 = \frac{P_1}{z_1 - j\frac{1}{b_2}} = \frac{P_1}{r_1^2 + \left(x_1 + \frac{1}{b_2}\right)^2}\left\{r_1 + j\left(x_1 + \frac{1}{b_2}\right)\right\}.$$

For any load resistance r_2, the vector $\overline{OP}$ gives the current I both in magnitude and phase displacement ϕ_1.

The pressure P_2 at the receiver terminals is most simply represented by the method given in Section 21, in which we assume a new co-ordinate system with origin at A and real axis passing through O. We then choose the pressure scale so that the distance $\overline{AO}$ represents the supply pressure P_1 (see Fig. 93).

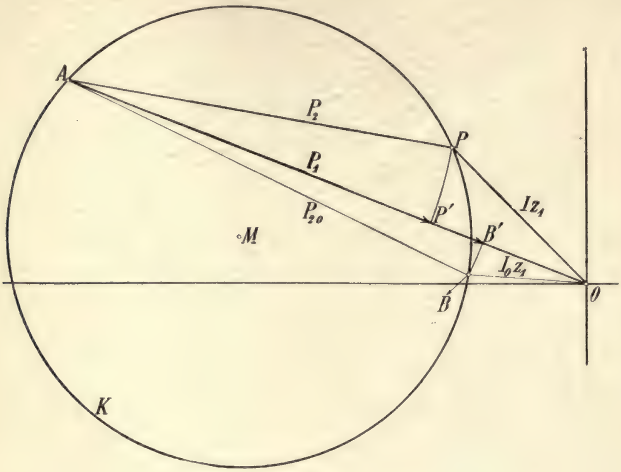


FIG. 93.

Thus
$$\overline{AO} = \frac{1}{50} I_K = \frac{1}{50} \cdot \frac{P_1}{z_1},$$
whence the pressure scale is
$$1 \text{ cm} = 50 z_1 = 50 \,.\, 5{\cdot}38 = 269 \text{ volts}.$$

For any load point P, the vector $\overline{AP}$ gives the receiver pressure P_2 in the new co-ordinate system, whilst the drop of pressure Iz_1 in the line is represented by the vector $\overline{PO}$. The pressure drop is given by the arithmetical difference of the primary and secondary pressures,

i.e. at no-load by $\qquad \overline{AO} - \overline{AB} = \overline{B'O},$

on load by $\qquad \overline{AO} - \overline{AP} = \overline{P'O}.$

The increase of pressure-drop from no-load to load is $\overline{AB} - \overline{AP} = \overline{P'B'}$.

The wattless component of the load current with respect to the receiver pressure is
$$I_{WL} = j P_2 b_2 = j b_2 (P_1 - I z_1)$$
$$= j b_2 z_1 (I_K - I).$$
Hence the watt component of the load current is
$$I_W = I - I_{WL} = I - j b_2 z_1 (I_K - I).$$
At no-load, $\qquad I_W = 0 \quad \text{and} \quad I = I_0,$

whence $$0 = \dot{I}_0 - jb_2 z_1(\dot{I}_K - \dot{I}_0).$$

Subtracting this last equation from the previous one, we get:

$$\dot{I}_W = (1 + jb_2 z_1)(\dot{I} - \dot{I}_0),$$

whilst the wattless component $\dot{I}_{WL}$ of the load current varies in magnitude and direction proportionally to the vector $\overline{PA}$, the watt component $\dot{I}_W$ thus varies proportionally to the vector $\overline{BP}$.

In the previous chapter, the loss line of the diagram was shewn to be given by the semi-polar of the circle K_1 in respect to the origin, whilst the power line passes through the short-circuit point A and the no-load point B. Accordingly, the efficiency η will be represented as shewn in Fig. 92.

The point on the circle K, for which the transmitted power is a maximum, is given by that point on the circumference of K which is at the maximum distance from the power line. At this point the vectors $(\dot{I}_K - \dot{I})$ and $(\dot{I} - \dot{I}_0)$ are equal in magnitude; hence we must have

$$\frac{I_W}{I_{WL}} = \frac{g_2}{b_2} = \frac{\text{value of } (1 + jb_2 z_1)}{\text{value of } jb_2 z_1} = \frac{\sqrt{r_1^2 + \left(x_1 + \frac{1}{b_2}\right)^2}}{z_1}.$$

The condition for maximum power is therefore

$$g_2 = \frac{b_2}{z_1}\sqrt{z_1^2 + 2x_1\frac{1}{b_2} + \frac{1}{b_2^2}}$$

$$= \sqrt{b_2^2 + 2b_1 b_2 + y_1^2} = \sqrt{g_1^2 + (b_1 + b_2)^2}.$$

This is the same condition as that previously deduced (eq. 49, p. 79) in another manner.

From the diagram, we can now measure off the several magnitudes P_2, I, η and $\cos\phi_1$, and plot the same along rectangular co-ordinates as functions of the useful power W_2. This is done in Fig 94.

In the above example, we have

$$g_1 = \tfrac{2}{29} \quad \text{and} \quad b_1 = \tfrac{5}{29};$$

hence for maximum power

$$g_2 = \sqrt{g_1^2 + (b_1 + b_2)^2} = 0{\cdot}232 \text{ mho},$$

whence $$W_{\max} = \frac{P_1^2}{2(g_2 z_1^2 + r_1)} = 229 KW.$$

The maximum efficiency occurs, as shewn above, when

$$g_2 = b_2 = 0{\cdot}05 \text{ mho},$$

so that $$\eta_{\max}\,\% = \frac{100}{1 + 2r_1 b_2} = \frac{100}{1{\cdot}2} = 83{\cdot}3\,\%.$$

As seen from the diagram and curves, for every value of the load W_2 there are two values of P_2, η and $\cos\phi_1$. The curves are drawn

for positive values of the conductance g_2, i.e. for points on the circle which lie above the power line. For this part of the circle the transmitted power is positive. Points of the current diagram

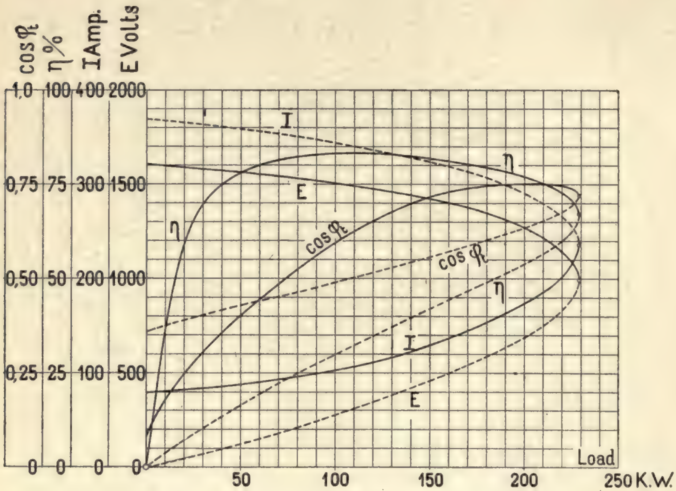


FIG. 94.—Load Curves for Circuit in Fig. 91.

which lie below the power line correspond to negative values of g_2; for this region the supplied power is negative, i.e. the machines in the receiver station act as generators. The curves for negative values of g_2 are not shewn in Fig. 94, since they possess but little interest for us here.

28. Example II. We now consider a power transmission scheme, in which the line has resistance and inductance, while the power-factor $\cos\phi_2$ of the receiver circuit remains constant at all loads. For such a system, the formulae deduced on p. 79 can be applied although in this case the variable is not g_2 but y_2.

We can write, therefore,

$$P_2 = aP_1 = \frac{P_1}{\sqrt{(1 + r_1 g_2 + x_1 b_2)^2 + (x_1 g_2 - r_1 b_2)^2}}$$

$$= \frac{P_1}{\sqrt{(1 + r_1 y_2 \cos\phi_2 + x_1 y_2 \sin\phi_2)^2 + (x_1 y_2 \cos\phi_2 - r_1 y_2 \sin\phi_2)^2}},$$

and since $I = P_2 y_2$,

$$P_1 = \sqrt{\{P_2 + I(r_1 \cos\phi_2 + x_1 \sin\phi_2)\}^2 + I^2(x_1 \cos\phi_2 - r_1 \sin\phi_2)^2}.$$

From this we get

$$P_2 = \sqrt{P_1^2 - I^2(x_1 \cos\phi_2 - r_1 \sin\phi_2)^2} - I(r_1 \cos\phi_2 + x_1 \sin\phi_2).$$

It follows from that equation that the curve of the receiver terminal pressure P_2 as function of the current is part of an ellipse.

The power in the receiver circuit is

$$\begin{aligned} W_2 &= P_2 I \cos\phi_2 \\ &= I\cos\phi_2\{\sqrt{P_1^2 - I^2(x_1\cos\phi_2 - r_1\sin\phi_2)^2} - I(r_1\cos\phi_2 + x_1\sin\phi_2)\} \\ &= I^2 r_2 \\ &= \frac{P_1^2 z_2 \cos\phi_2}{(r_1 + z_2\cos\phi_2)^2 + (x_1 + z_2\sin\phi_2)^2} \\ &= \frac{P_1^2 z_2\cos\phi_2}{z_1^2 + z_2^2 + 2z_2(r_1\cos\phi_2 + x_1\sin\phi_2)}. \end{aligned}$$

The *maximum power* transmitted occurs when $\frac{dW_2}{dz_2} = 0$.

This is the case when

$$z_1^2 + z_2^2 + 2z_2(r_1\cos\phi_2 + x_1\sin\phi_2) - z_2\{2z_2 + 2(r_1\cos\phi_2 + x_1\sin\phi_2)\} = 0;$$

thus when $z_2 = z_1$, ...(52)

that is, whence the impedance of the receiver circuit equals the impedance of the transmission line.

Substituting this value for z_2, we get the following expression for the maximum useful power:

$$W_{2\max} = \frac{P_1^2\cos\phi_2}{2(z_1 - r_1\cos\phi_2 + x_1\sin\phi_2)}. \quad \text{.................(53)}$$

Assume the same line constants as in previous example:

$$P_1 = 2000 \text{ volts}; \quad r_1 = 2 \text{ ohms}; \quad x_1 = 5 \text{ ohms}.$$

For the sake of comparison, we shall develop the diagram for the following three cases:

$\cos\phi_2'' = 0{\cdot}9$, current lagging.
$\cos\phi_2' = 1$, current in phase.
$\cos\phi_2''' = 0{\cdot}9$, current leading.

In Fig. 95 $\overline{OA_1}$ is the image of the impedance z_1, drawn to the scale 1 cm $= 2$ ohms.

Draw the three straight lines K_1', K_1'' and K_1''' through the point A_1 at angles ϕ_2', ϕ_2'' and ϕ_2''' respectively to the vertical. These straight lines are the images of the sum of the impedances $z_1 + z_2$ for the three cases under consideration. By the inversion of the impedance curves K_1', K_1'' and K_1''' we get the three circles K', K'' and K''', which are the current curves of the system. The current scale is chosen so that 1 cm $= 75$ amps. If P_1 and P are two corresponding points on the impedance curve and current curve respectively, then

$$\overline{OP_1} = \tfrac{1}{2}z,$$

$$OP = \frac{1}{75}I = \frac{1}{75}\cdot\frac{2000}{z} = \frac{2000}{75\,.\,2\,.\,\overline{OP_1}};$$

consequently the constant of inversion is $\overline{OP_1}\,.\,\overline{OP} = 13{\cdot}3$.

The circles can be still more simply determined if we remember that they must all pass through one common short-circuit point A and through the origin O of the co-ordinate axes. Consequently the centres M', M'' and M''' of the circles must all lie on the line which bisects $\overline{OA}$ at right angles. Further, the lines $\overline{OM'}$, $\overline{OM''}$ and $\overline{OM'''}$ make angles ϕ_2', ϕ_2'' and ϕ_2''' respectively with the abscissa axis. For

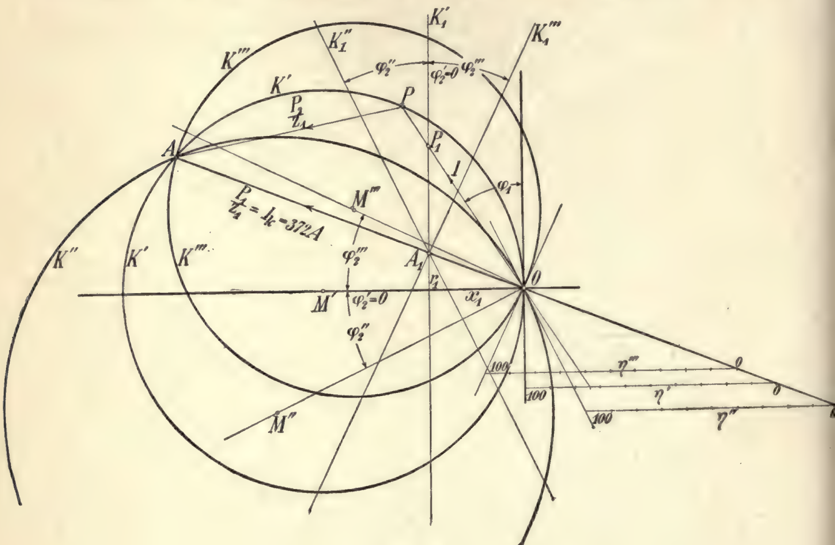


FIG. 95.

the case $\phi_2' = 0$ (non-inductive load), the centre M' falls on the abscissa axis. The receiver pressure P_2 is

$$P_2 = P_1 - Iz_1 = z_1\left(\frac{P_1}{z_1} - I\right) = z_1(I_K - I),$$

whilst the short-circuit current I_K is given by the vector $\overline{OA}$. For a point P on the current curve, the current $(I_K - I)$ is represented by the vector $\overline{PA}$. When we choose the pressure scale so that the length $\overline{AO}$ represents the supply pressure, $P_1 = 2000$ volts, then the distance $\overline{AP}$ from the short-circuit point A to the respective load point P on the current curve gives the receiver pressure P_2. It is seen that the drop of pressure is greatest for inductive loads. For non-inductive loads the pressure drop is not so large, whilst for capacity loads there is a pressure rise at small loads, provided $\phi_2 > \phi_1$.

At no-load, the power-factor $\cos\phi_1$ of the system approaches the value $\cos\phi_2$, for in this case the effect of the line is negligible. As the load increases, the effect of the line reactance begins to make itself felt, and the power-factor $\cos\phi_1$ falls as the inductive or non-inductive

load rises. On the other hand, with capacity load, $\cos\phi_1$ rises until

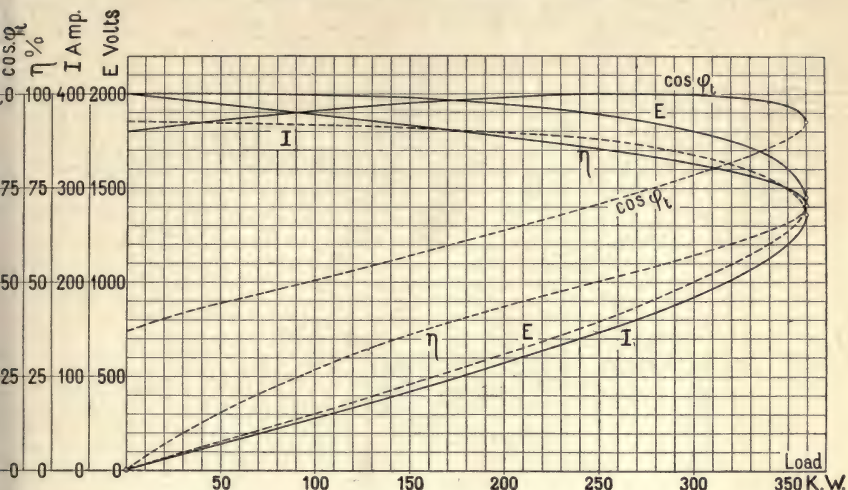


FIG. 96.—Load Curves for Leading Power Factor at Secondary Terminals.

it reaches unity, as the load increases, but falls again when the load is further increased.

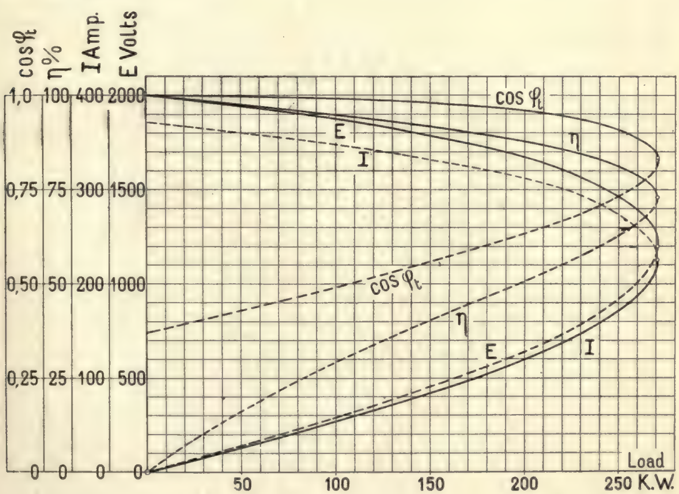


FIG. 97a.—Load Curves for Unity Power Factor at Secondary Terminals.

All the circles have the same power line $\overline{OA}$ with different scales. The maximum power is obtained when the extremity of the current

vector lies midway between O and A on the current curve. At this point the vector of the pressure drop in the line has the same length as the vector of the pressure in the receiver circuit; thus,

$$Iz_1 = Iz_2, \quad \text{i.e. } z_1 = z_2,$$

as shewn previously by another method.

Each current curve has its own loss line, which is the tangent to the circle at the origin. The efficiency for each kind of load is found in the usual way (see Fig. 95).

In Figs. 96, 97*a*, 97*b*, the curves for P_2, I, η and $\cos\phi_1$, as taken from the diagram, are plotted as functions of the load. It is seen that

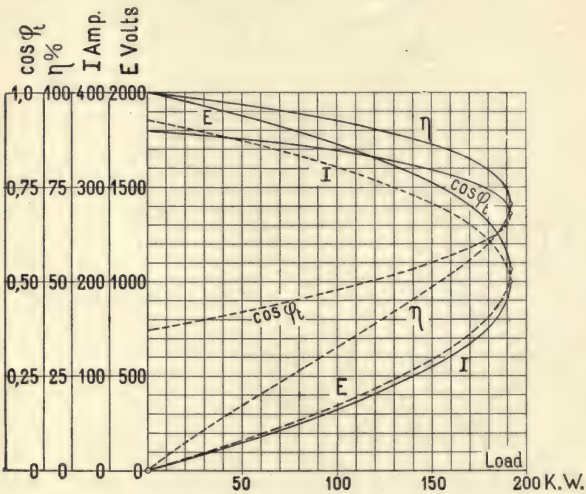


FIG. 97*b*.—Load Curves for Lagging Power Factor at Secondary Terminals.

the maximum power is greatest for the capacity load and least for the inductive. Here also, as in Example I., for every load there are two corresponding values of each of the respective magnitudes. Of these two values, that which lies on the full-line curve is the usual one—it corresponds to the point on the current curve which lies between the origin and the point of maximum power.

Points on the current curve lying below the power line correspond to the case when the receiver circuit works as generator. This part of the diagram has not been plotted in the rectangular co-ordinates.

29. Several Impedances in Series. If several impedances, with the constants r_1, x_1; r_2, x_2; r_3, x_3; and so on, are connected in series, the resistance of each impedance will require an E.M.F. component in phase with the current, and the reactance an E.M.F. component which leads the current vector by 90°.

To drive the current I through the circuit, a terminal pressure P is required.

$$P = I(r_1 - jx_1) + I(r_2 - jx_2) + I(r_3 - jx_3) + \dots$$
$$= I(r_t - jx_t) = Iz_t,$$

where $$r_t = r_1 + r_2 + r_3 + \dots = \Sigma(r)$$

and $$x_t = x_1 + x_2 + x_3 + \dots = \Sigma(x).$$

The total impedance of a circuit consisting of several impedances in series is equal to the geometric sum of these impedances; or, expressed symbolically,

$$z_t = z_1 + z_2 + z_3 + \dots \qquad (54)$$

Fig. 98 shews the graphical addition of the E.M.F.'s necessary to drive the current I through the several impedances. Since the current is the same throughout the whole circuit, the same result would have been obtained by summing up the impedances of the circuit.

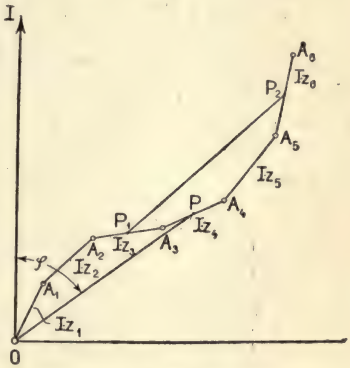


FIG. 98.

Assuming that each part of the circuit is uniform, i.e. r and x are uniformly distributed over the respective portions of the circuit, and also that one terminal of the circuit has zero potential, then the polygon $OA_1A_2A_3$, and so on, illustrates the distribution of potential in the circuit. The potential at any point in the circuit is given by the distance of the corresponding point P in the polygon from the origin, and the phase displacement of this potential from the current I equals the angle ϕ which the vector $\overline{OP}$ makes with the ordinate axis. The difference of potential between two points P_1 and P_2 in the circuit equals the distance between the two corresponding points on the polygon. The straight line $\overline{P_1P_2}$ gives this potential difference both in magnitude and direction.

CHAPTER V.

PARALLEL CIRCUITS.

30. Circuit with Admittances in Parallel. 31. Current Resonance. 32. Equivalent Impedance of Two Parallel Impedances.

30. Circuit with Admittances in Parallel. We shall now consider the case in which a pressure $p=\sqrt{2}P\sin\omega t$ acts at the terminals A and B of a compound circuit having two parallel branches (Fig. 99). We denote the currents in these two branches by I_0 and I_2. These can be resolved, as shewn above, into the components Pg_0, Pb_0 and Pg_2, Pb_2. By setting off these components, as in Fig. 100, we get the currents I_0 and I_2, and hence their geometric sum, the resultant current I_1. Let P, g_0 and b_0 be constant; we can then represent what takes place in the circuit when g_2 or b_2 is varied by the diagram in Fig. 101 (cp. Fig. 54). If x_2 is kept constant, whilst r_2 is varied, the locus of I_1 will be the semi-circle O_1BA.

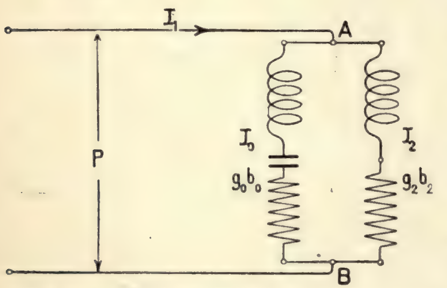


FIG. 99.—Circuit with Two Admittances in Parallel.

Conversely, when r_2 is constant and x_2 varied, the current vector will move over circle O_1BC. The semi-circle lying to the right of $\overline{O_1C}$ applies to the case when x_2 is a capacity-reactance.

If several admittances having the constants g_1, b_1; g_2, b_2; g_3, b_3; and so on, are connected in parallel, the pressure P applied at the terminals will send a current

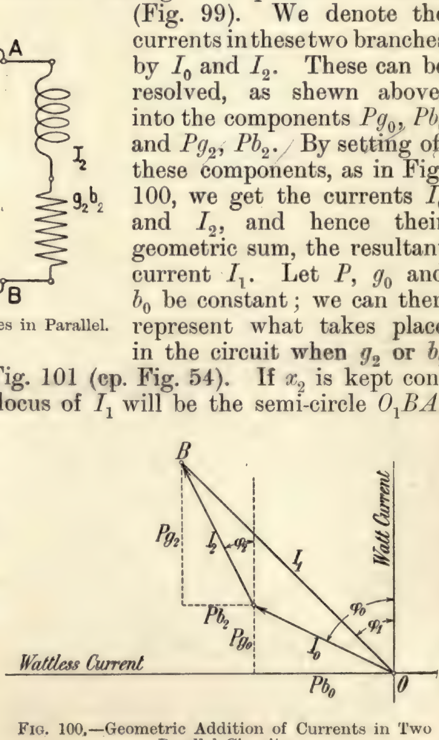


FIG. 100.—Geometric Addition of Currents in Two Parallel Circuits.

through each admittance, which can be resolved into a watt component Pg in phase with the pressure and a wattless component Pb

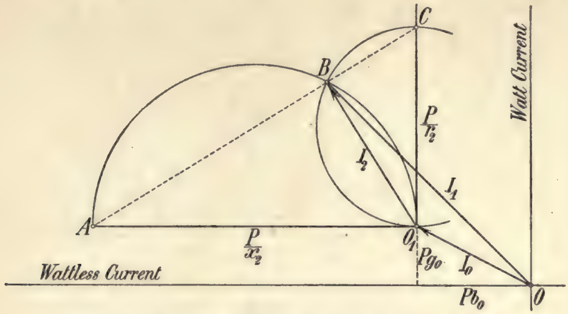


FIG. 101.—Current Diagram for Two Parallel Circuits.

lagging 90° behind the pressure. Hence the current flowing in the whole circuit is

$$I = P(g_1 + jb_1) + P(g_2 + jb_2) + P(g_3 + jb_3) + \dots$$
$$= P(g_t + jb_t) = Py_t,$$

where

$$g_t = g_1 + g_2 + g_3 + \dots = \Sigma(g),$$
$$b_t = b_1 + b_2 + b_3 + \dots = \Sigma(b);$$

whence it follows, that *the total admittance y_t of a circuit with several admittances connected in parallel equals the geometric sum of these admittances*; or, expressed symbolically,

$$y_t = y_1 + y_2 + y_3 + \dots.$$

31. Zero Susceptance. If two circuits are connected in parallel, one of which contains capacity and the other inductance, the current in the former will lead and in the latter lag in respect of the applied pressure. Consequently the wattless component of the resultant current will be less than the wattless components of the currents in the branches. If the wattless currents in the two branches are equal but of opposite sign, the resultant current will be in phase with the pressure, and the total susceptance will therefore be zero. In such a case, *resonance* is said to prevail in the circuit, and, in distinction to pressure resonance—which we have seen (p. 46) takes place in series circuits—resonance in parallel circuits is called *current resonance.**

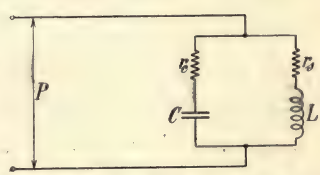


FIG. 102.—Circuit for Current Resonance.

We can write the reactance of the two circuits in Fig. 102 thus:

$$x_c = \frac{1}{\omega C}, \quad x_s = \omega L.$$

* As explained for series circuits, this condition can only truly be termed "Resonance" when the resistance of the oscillatory circuit is negligible.

The condition necessary to give equal and opposite wattless currents in the two circuits is

$$b_c = b_s,$$

or

$$\frac{x_c}{r_c^2 + \frac{1}{\omega^2 C^2}} = \frac{x_s}{r_s^2 + \omega^2 L^2}.$$

If we draw $\overline{OA_c} = \overline{OA_s} = b_c = b_s$ in Fig. 103, the above condition for resonance is fulfilled as soon as the extremity B_c of vector y_c falls on the vertical through A_c, and the extremity B_s of vector y_s on the vertical through A_s; for then the resultant admittance $y = \overline{OD}$ coincides with the ordinate axis. The circles on $\overline{OA'_c}$ and $\overline{OA'_s}$ are the loci of the images of the impedances z_c and z_s.

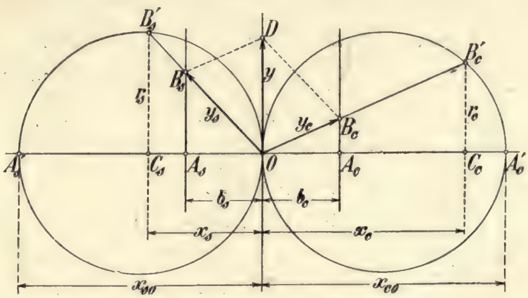


FIG. 103.—Diagram for Zero Susceptance.

When $r_c = r_s = 0$, we have the same condition for zero susceptance in the parallel circuits as for zero impedance in the series circuit (see Sect. 16). We then get

$$\omega L_0 = \frac{1}{\omega C_0} = x_{s_0} = x_{c_0} = x_0,$$

$$L_0 C_0 = \frac{1}{\omega^2}. \quad \text{.................................} (55)$$

When $r_c = r_s = r_1 \lesseqgtr 0$, the total susceptance becomes zero in two cases:

CASE 1. When $x_s = x_c = x_1$,

$$\omega L = \frac{1}{\omega C},$$

$$LC = \frac{1}{\omega^2}. \quad \text{.................................} (56)$$

In this case the resultant conductance of the two branches is

$$g = g_s + g_c,$$

or, since

$$g_s = g_c = g_1,$$

$$g = 2g_1$$

equals double the conductance in one branch.

CASE 2. When $$r_1^2 = x_s(x_{s_0} - x_s) = x_s x_c = \frac{L}{C}. \quad \text{(56a)}$$

This case is shewn in Fig. 104. The resultant conductance of the two paths is here

$$g = g_s + g_c = b_s \frac{r_1}{x_s} + b_c \frac{r_1}{x_c}.$$

Now, since $$b_s = b_c = b_1,$$

we get also

$$g = r_1 b_1 \left(\frac{1}{x_s} + \frac{1}{x_c}\right) = r_1 b_1 \left(\frac{1}{x_s} + \frac{x_s}{r_1^2}\right)$$

$$= r_1 b_1 \frac{r_1^2 + x_s^2}{x_s r_1^2} = r_1 b_1 \frac{1}{r_1^2} \frac{1}{b_1} = \frac{1}{r_1}.$$

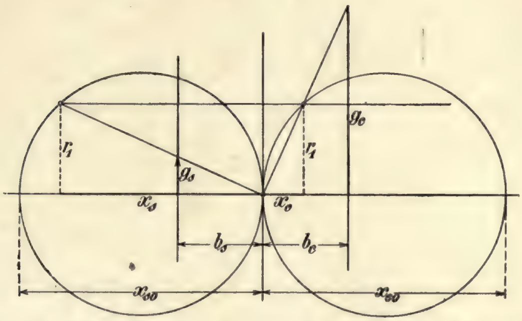


FIG. 104.—Diagram for Zero Susceptance independent of Frequency.

The resultant resistance between the terminals is therefore

$$r = \frac{1}{g} = r_1,$$

equal to the resistance in one branch.

This latter example of a circuit with zero susceptance is of special interest as the effect is independent of the frequency.

32. Equivalent Impedance of Two Parallel Impedances. If the two impedances z_1 and z_2 are connected in parallel, and we write symbolically:

$$z_1 = \frac{1}{y_1}, \quad z_2 = \frac{1}{y_2},$$

then the impedance of the parallel circuit is

$$z = \frac{1}{y},$$

where $$y = y_1 + y_2 = \frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2},$$

or $$z = \frac{1}{y_1 + y_2} = \frac{z_1 z_2}{z_1 + z_2}. \quad \text{(57)}$$

This expression is similar to that for the resultant resistance of two ohmic resistances joined in parallel.

The impedance z can be determined graphically in a simple manner. In Fig. 105, let

$$\overline{OA} = z_1 \quad \text{and} \quad \overline{OB} = z_2.$$

Then
$$\overline{OC} = z_1 + z_2 = z'.$$

Make $\triangle ODB$ similar to $\triangle OAC$.

Then
$$\overline{OD} = \overline{OB}\,\frac{\overline{OA}}{\overline{OC}} = \frac{z_1 z_2}{z'} = z.$$

Hence the required impedance z is given by the vector $\overline{OD}$.

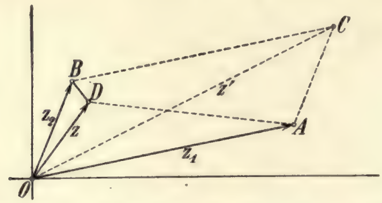


FIG. 105.—Graphical Construction of Equivalent Impedance for Two Parallel Impedances.

This can also be proved as follows: If we write equation (57) in the form

$$\frac{z}{z_2} = \frac{z_1}{z_1 + z_2} = \frac{z_1}{z'} = \frac{z\epsilon^{-j\phi}}{z_2\epsilon^{-j\phi_2}} = \frac{z_1\epsilon^{-j\phi_1}}{z'\epsilon^{-j\phi'}},$$

then, for the absolute values, we have

$$\frac{z}{z_2} = \frac{z_1}{z'},$$

and for the angles
$$\phi - \phi_2 = \phi_1 - \phi',$$

or
$$\angle BOD = \angle COA.$$

From this we see that the construction of Fig. 105 is correct.

The point D can also be found from the following construction (Fig. 106). Draw $\overline{OM_2}$ and $\overline{OM_1}$ perpendicular to the impedances z_2 and z_1. Determine the points A' and B', which are respectively the images of points A and B with respect to these perpendiculars $\overline{OM_2}$ and $\overline{OM_1}$. Then we have

$$\triangle OAB' \text{ similar to } \triangle OA'B \text{ similar to } \triangle BCO,$$

whence
$$\angle OAB' = \angle BCO = \angle DAO,$$
$$\angle OBA' = \angle BOC = \angle DBO.$$

The desired point D therefore lies on the two lines $\overline{AB'}$ and $\overline{BA'}$, i.e. D is the point where these two lines cut.

For the case when the impedance z_2 is altered in amount but not in phase—i.e. its direction remains unchanged—the point B moves on a straight line through O and B. Thus $\angle OBC = \angle ODA$ remains

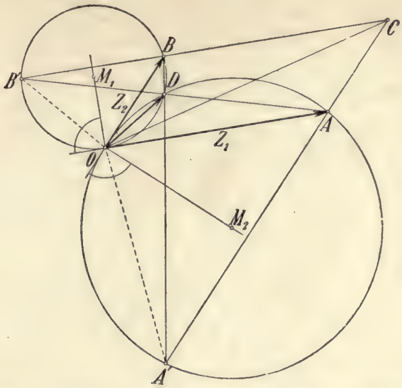


FIG. 106.—Impedance Diagram for Two Parallel Impedances.

constant. The point D then moves over a circle described about M_2 as centre and passing through the points O, A and A'. Conversely, if z_2 is constant and z_1 alters in value but not in direction, the points B and B' remain fixed, whilst the point D moves over the circle described about M_1 which passes through O, B and B'.

CHAPTER VI.

THE GENERAL ELECTRIC CIRCUIT.

33. Impedance in Series with Two Parallel Circuits. 34. Pressure Regulation in a Power Transmission Scheme. 35. Compounding of a Power Transmission Scheme. 36. Losses and Efficiency in a Compounded Transmission Scheme.

33. Impedance in Series with Two Parallel Circuits. Having now dealt with compound circuits consisting of a number of impedances connected respectively in series and in parallel, we may proceed to the more complex case, in which one impedance is in series with two others connected in parallel.

Almost all the circuits met with in practice may be reduced to such a circuit, provided the constants of the circuit are in fact *constant.* The case is so generally applicable that it may be termed the *General Electric Circuit.*

Such a case is met with, for example, when power is transmitted over an inductive line to a receiver station, where two admittances are joined in parallel. Fig. 107 shows a circuit of this kind, in which we have the line impedance r_1, x_1, in series with two parallel branches. We may take the case, in which the admittance y_a of the first branch and also the reactance x_2 of the second branch remain constant, whilst the load resistance r_2 is varied at will.

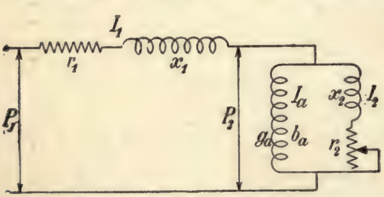


FIG. 107.—Circuit with Impedance in Series with Two Parallel Circuits.

The graphical process by which the current curve is obtained for this circuit, with constant supply pressure P_1, may be summarised as follows.

The combined admittance curve of the two parallel branches is first obtained (as in Fig. 101, p. 91) by graphical addition of the constant admittance y_a and the variable admittance corresponding to x_2 and r_2.

The total impedance of the circuit is now obtained by adding the impedance corresponding to $x_1 r_1$ to the combined impedance of the two parallel branches, obtained by inversion of the curve of their combined admittance.

The third and final step is the inversion of the total impedance curve in order to obtain the admittance of the circuit, which multiplied by the constant pressure P_1 gives the current curve. This final inversion would naturally be unnecessary, if it were required to

determine the voltage required to maintain a constant current in the circuit.

This graphical process is analogous to the algebraic calculation with complex quantities, in which we get the combined admittance of the parallel circuits

$$y = y_a + \frac{1}{z_2},$$

and the total impedance of the complete circuit

$$z_t = z_1 + \frac{1}{y} = z_1 + \frac{1}{y_a + \dfrac{1}{z_2}}.$$

Hence the total current $I = \dfrac{P_1}{z_t} = \dfrac{P_1}{z_1 + \dfrac{1}{y_a + \dfrac{1}{z_2}}}.$

The current diagram (Fig. 108) has been drawn for the following values:

$$z_1 = 2 - j5 \text{ ohms}; \quad z_2 = r_2 - j4 \text{ ohms},$$
$$y_a = 0{\cdot}0033 + j0{\cdot}02 \text{ mho},$$
$$P_1 = 1000 \text{ volts}.$$

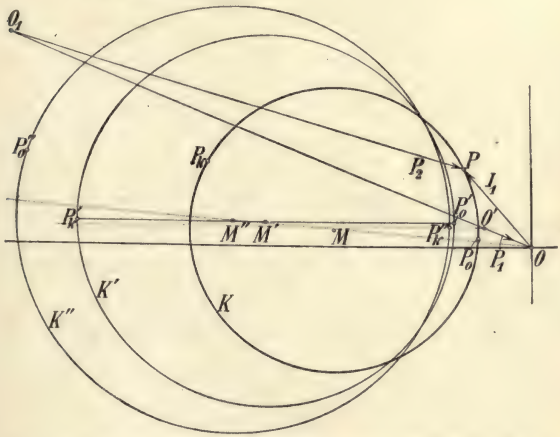


FIG. 108.—Construction of Current Diagram for Circuit in Fig. 107.

Take 1 cm $= 0{\cdot}05$ mho, and mark off P_0' at a distance $\dfrac{b_a}{0{\cdot}05} = 0{\cdot}4$ cm to the left of O' and $\dfrac{g_a}{0{\cdot}05} = 0{\cdot}066$ cm above it. The vector $\overline{O'P_0'}$ represents the admittance y_a. Draw $\overline{P_0'P_K'} = \dfrac{1}{0{\cdot}05} \cdot \dfrac{1}{x_2} = 5$ cm parallel to the abscissa axis, and on it as diameter describe the circle K' to represent the admittance $y_a + \dfrac{1}{z_2}$. By the inversion of the circle K' with

respect to O', we get the impedance

$$\frac{1}{y_a + \frac{1}{z_2}} = \frac{z_2}{1 + z_2 y_a}.$$

The impedance scale is 1 cm = 8 ohms, hence the constant of inversion is

$$\mathbf{I} = \frac{1}{0{\cdot}05 \,.\, 8} = 2{\cdot}5.$$

The inverse circle of K' is K''.

Starting again from point O' and setting off $\frac{x_1}{8} = 0{\cdot}625$ cm to the right and $\frac{r_1}{8} = 0{\cdot}25$ cm downwards, we get the point O. As OO' represents the line impedance z_1, the circle K'', in respect to point O, then represents the impedance between the supply terminals. If we now wish to have the admittance between the supply terminals to the scale 1 cm = 0·025 mho, we must take the inverse of circle K'' with respect to O with the constant of inversion

$$\mathbf{I} = \frac{1}{8 \,.\, 0{\cdot}025} = 5.$$

The inverse of K'' is the circle K. Since the supply pressure $P_1 = 1000$ volts, the circle K represents the current I_1 to the scale 1 cm = 0·025 × 1000 = 25 amps.

The point P_0 corresponds to the load $r_2 = \infty$, and is called the *no-load* point of the system. The *no-load current* I_{10} is given by the vector $\overline{OP_0}$. The point P_K is the *short-circuit* point, and corresponds to the load $r_2 = 0$. The *short-circuit current* I_{1K} is given by the vector $\overline{OP_K}$.

If O_1 is the inverse point of the origin O' to the ratio of inversion 5, then $\overline{OO_1}$ corresponds to the current $\frac{P_1}{z_1}$. O_1 is thus the short-circuit point for the case when the receiver terminals are short-circuited. Let P be any point on the circle K, then the vector $\overline{PO_1}$ represents a current

$$\frac{P_1}{z_1} - I_1 = \frac{P_1 - I_1 z_1}{z_1} = \frac{P_2}{z_1}.$$

Hence if we construct a new co-ordinate system with the origin O_1 and with the real axis passing through O, and further choose the pressure scale so that $\overline{O_1O} = P_1$ volts, then in this new system the vector $\overline{O_1P}$ represents the receiver pressure P_2 (see Chap. III. Sect. 21). In this system of co-ordinates, therefore, the triangle O_1PO is the pressure triangle of the installation. The pressure drop in the transmission line equals the algebraic difference $\overline{O_1O} - \overline{O_1P}$. At no-load, the drop of pressure is $\overline{O_1O} - \overline{O_1P_0}$. From no-load to load, therefore, the pressure falls $\overline{O_1P_0} - \overline{O_1P}$.

The current I_a in the constant admittance y_a is proportional to the pressure P_2. Whence

$$I_a = \frac{P_2}{P_{2_0}} I_{a_0} = \frac{P_2}{P_{2_0}} I_0,$$

where $I_{a_0}' = I_0$ is the no-load current and P_{2_0} is the no-load receiver pressure. From the diagram, we get

$$\frac{P_2}{P_{2_0}} = \frac{\overline{O_1P}}{\overline{O_1P_0}}, \quad I_0 = \overline{OP}_0.$$

Hence, in the original current scale,

$$I_a = \frac{\overline{OP}_0}{\overline{O_1P_0}} \overline{O_1P}.$$

To complete the diagram, we draw in the loss and power lines. For the loss in the impedance z_1, we put

$$V_1 = I_1^2 r_1 = B_2 \mathsf{V}_1,$$

where $\mathsf{V}_1 = 0$ is the shortened form of the equation of the loss line (see Section 23). This line $\mathsf{V}_1 = 0$ is the semi-polar of the origin O, with respect to the circle K (Fig. 108), and is constructed as previously shewn.

The loss in the parallel connected admittance y_a is

$$V_a = P_2^2 g_a.$$

Since P_2 can be here represented by the vector $\overline{O_1P}$, the line for the loss V_a is the semi-polar of the point O_1 in respect to the circle K. Writing $\mathsf{V}_a = 0$ for the equation of this straight line, we get

$$V_a = P_2^2 g_a = B_a \mathsf{V}_a,$$

where B_a is a constant, and the co-ordinates of the point P are inserted in the linear expression V_a. Similarly, writing the equation of the abscissa axis $\mathsf{W}_1 = 0$, the equation of the supplied power can be written in the form : $W_1 = P_1 I_1 \cos \phi_1 = A_1 \mathsf{W}_1,$

where A_1 is a constant. In this particular case, A_1 is simply equal to the supply pressure and W_1 is the watt current, or the ordinate of the point P.

The power received by branch 2 of the parallel circuits is :

$$W_2 = W_1 - V_1 - V_a = W_1 - V_{1a}$$
$$= A_2 \mathsf{W}_2 = A_1 \mathsf{W}_1 - B_1 \mathsf{V}_1 - B_a \mathsf{V}_a = A_1 \mathsf{W}_1 - B_{1a} \mathsf{V}_{1a}.$$

Since, on the one hand,

$$B_{1a} \mathsf{V}_{1a} = B_1 \mathsf{V}_1 + B_a \mathsf{V}_a,$$

$\mathsf{V}_{1a} = 0$ is the equation of a straight line passing through the point of intersection of $\mathsf{V}_1 = 0$ and $\mathsf{V}_a = 0$. Thus $\mathsf{V}_{1a} = 0$ is the resultant loss line of the current diagram.

Since, however, on the other hand,

$$A_2 \mathsf{W}_2 = A_1 \mathsf{W}_1 - B_{1a} \mathsf{V}_{1a},$$

then $\mathsf{W}_2 = 0$ is the equation of the useful power line of the circuit.

This line $W_2 = 0$ passes through the point where the resultant loss line $V_{1a} = 0$ cuts the abscissa axis $W_1 = 0$. Again, since the power line $W_2 = 0$ passes through the points for which the power in the load r_2 is zero, it is obvious that it passes through the no-load point P_0 and the short-circuit point P_K, and can thus be drawn at once. To find the resultant loss line $V_{1a} = 0$, on the one hand, we have the point of intersection of the two loss lines $V_1 = 0$ and $V_a = 0$, and, on the other hand, the point of intersection of the power line and the abscissa axis, and from this follows the construction for the determination of the efficiency as shewn in Fig. 109. This figure is drawn for the same constants and to the same scale as Fig. 108.

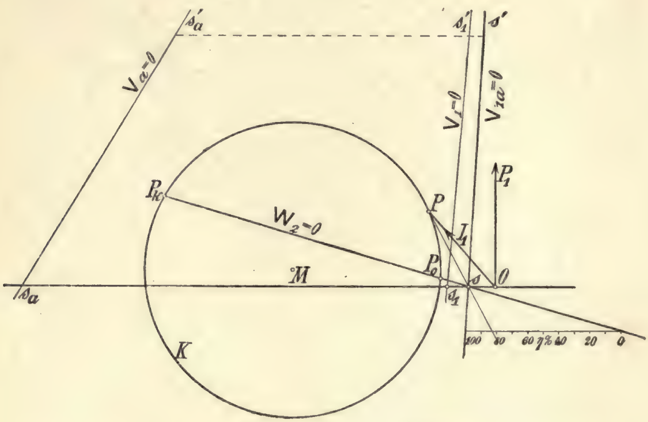


FIG. 109.—Complete Current Diagram.

Since the straight lines $V_1 = 0$, $V_a = 0$ and $V_{1a} = 0$ must all cut at a point, the direction of the straight line $V_{1a} = 0$ can be found from

$$\overline{s_1's'} : \overline{s_1 s} = \overline{s_a's_1'} : \overline{s_a s_1},$$

since the ratio of the intercepts of the three lines on any horizontal straight line is the same as that of the intercepts on the abscissa axis.

34. Pressure Regulation in a Power Transmission Scheme. Until now we have always assumed that the pressure at the supply terminals was maintained constant, and have determined the pressure at the receiver terminals for various loads. In practice, it is often required to maintain a constant receiver pressure. This can be accomplished by suitable regulation of the supply pressure. If, by way of example, it is required to maintain a constant receiver pressure P_2 at the end of a transmission line of impedance $\underset{.}{z}_1$, then the pressure at the supply terminals must be

$$\underset{.}{P}_1 = \underset{.}{P}_2 + \underset{.}{I}_1 \underset{.}{z}_1.$$

We may take, by way of example, the case in which the load current $\underset{.}{I}_1 = \underset{.}{P}_2 \underset{.}{y} = \underset{.}{P}_2(g + jb)$ is given by the curve K in Fig. 110.

This curve K to another scale also represents the admittance curve of the load due to the constant receiver pressure P_2 which is set off along the ordinate axis.

Since
$$\dot{P}_1 = \dot{z}_1\left(\frac{\dot{P}_2}{\dot{z}_1} + \dot{I}_1\right)$$
we get
$$\frac{\dot{P}_2}{\dot{z}_1} = \dot{P}_2\left(\frac{r_1}{z_1^2} + j\frac{x_1}{z_1^2}\right).$$

This $\left(\frac{\dot{P}_2}{\dot{z}_1}\right)$ is the short-circuit current in the line under pressure P_2. If we displace the origin to O_1 by making
$$\overline{OA} = P_2\frac{x_1}{z_1^2} \quad \text{and} \quad \overline{AO_1} = P_2\frac{r_1}{z_1^2}$$
then the current $\frac{\dot{P}_2}{\dot{z}_1} + \dot{I}_1$ is given by the vector $\overline{O_1P}$. Hence, if we

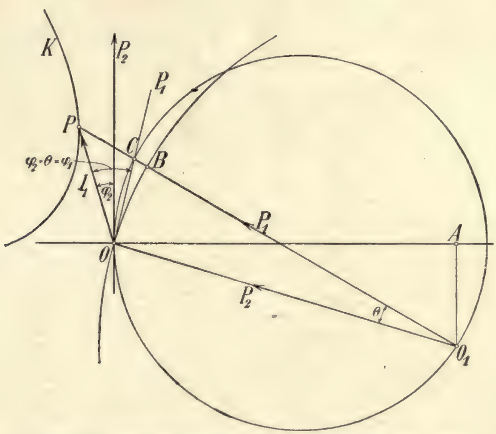


Fig. 110.—Pressure Regulation of a Transmission Line.

choose the pressure scale so that the line $\overline{O_1O}$ equals the constant receiver pressure P_2, the line $\overline{O_1P}$ will give the supply pressure P_1 corresponding to the current vector $I_1 = \overline{OP}$. The rise of pressure is thus $\overline{BP}$.

The supply pressure P_1 leads the receiver pressure P_2 by the angle θ, whilst the current I_1 lags behind the receiver pressure P_2 by the angle ϕ_2. Hence the phase displacement at the supply terminals is $\phi_1 = \phi_2 + \theta$. If we draw a circle to pass through O and O_1, and with its centre on the abscissa axis, then
$$\angle P_2OC = \theta \quad \text{and} \quad \angle POC = (\phi_2 + \theta) = \phi_1.$$

We will now determine graphically the loss and efficiency of the transmission line for the usual case, in which the current curve is represented by the circle K as in Fig. 111, with the receiver pressure P_2.

The loss in the line is $V_1 = I^2 r_1,$

and is represented as before by the loss line $\mathsf{V}_1 = 0$, the semi-polar of the circle K with respect to the origin O.

The power given to the receiver circuit is

$$P_2 I_W = P_2 v,$$

where u and v are the co-ordinates of a point P on the circle K. The power line $\mathsf{W}_2 = 0$ is therefore the abscissa-axis in this case.

The supplied power is

$$W_1 = W_2 + V_1 = P_2 v + I^2 r_1 = P_2 v + (u^2 + v^2) r_1.$$

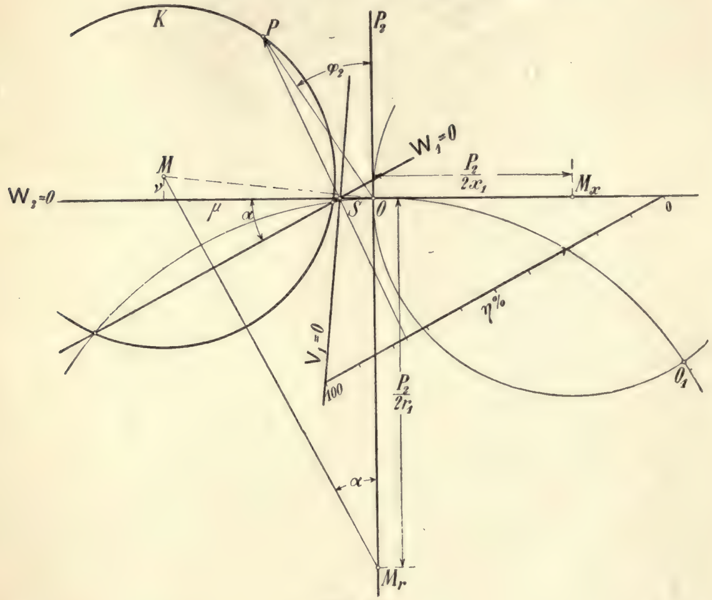


FIG. 111.

Since the equation of the circle is

$$(u - \mu)^2 + (v - \nu)^2 = R^2$$

or

$$u^2 + v^2 = 2\mu u + 2\nu v - \rho^2,$$

we get for the supplied power

$$W_1 = P_2 v + 2r_1 \mu u + 2r_1 \nu v - r_1 \rho^2 = A_1 \mathsf{W}_1,$$

where $\mathsf{W}_1 = 0$ is the shortened equation of the power line.

If now α is the angle this line makes with the abscissa axis, we have

$$\tan \alpha = -\frac{2r_1 \mu}{P_2 + 2r_1 \nu} = \frac{-\mu}{\frac{P_2}{2r_1} + \nu}.$$

Since the line $W_1 = 0$ must further pass through the intersection of the loss line with the abscissa axis, it can at once be constructed (Fig. 111). The power line is perpendicular to the line $\overline{M_r M}$. A circle described about M_r as centre with radius $\frac{P_2}{2r_1}$ must pass through the short-circuit point O_1; and the power line $W_1 = 0$ and this new circle cut the circle K in the same points.

To obtain the efficiency of the system at any point P on the current curve, we now proceed as follows:

Draw a line 0–100 parallel to the power line $W_1 = 0$ between the loss line and the power line $W_2 = 0$; join PS and produce to cut this line. Marking off the line 0–100 into ten parts to represent 10 %, 20 %, up to 100 % efficiency, the efficiency at the point P may be read off directly at the point where this efficiency line is cut by PS produced.

35. Compounding of a Power Transmission Scheme. From Fig. 110 it is seen that the pressure-rise $P_1 - P_2$ only depends on the magnitude and direction of the current vector I_1, and that P_1, and consequently

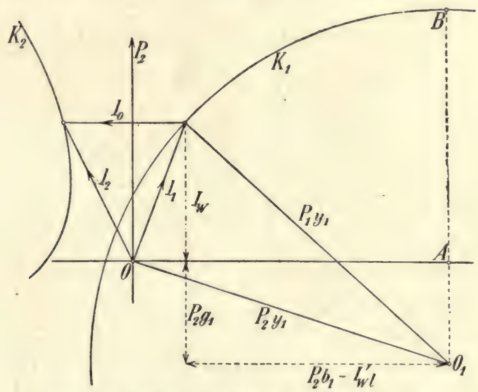


FIG. 112.—Compounding of a Transmission Line.

$P_1 - P_2$, will be constant so long as the extremity P of the current vector I_1 moves over a circle described about O_1 as centre. But the current curve K of the load is not a circle as a rule. It is possible, however, to connect a machine to the receiver terminals—i.e. in parallel with the load—whose current I_0 can be so regulated that the line current vector $I_1 = I_2 + I_0$ describes a circle whose centre is at O_1. A transmission scheme in which this is the case is said to be *compounded.* The current I_0 can be a pure wattless current. Such a machine joined to the receiver terminals for the purpose of giving or taking a wattless current is called a *phase regulator.*

In Fig. 112, curve K_2 represents the load (current) diagram for the constant receiver pressure P_2. The current is represented by

$$I_2 = I_2(\cos\phi_2 + j\sin\phi_2) = P_2(g_2 + jb_2) = I_W + jI_{WL},$$

O_1 is the short-circuit point for the line of impedance z_1.

$$\overline{OA} = P_2 \frac{x_1}{z_1^2} = P_2 b_1, \qquad \overline{AO}_1 = P_2 \frac{r_1}{z_1^2} = P_2 g_1.$$

Let K_1 be the circle about O_1 whose radius equals the constant supply pressure P_1, then I_1 is the line current, and consequently I_0 is the lagging wattless current given by the phase regulator.

The line current I_1 possesses the same watt current I_W as the load current I_2, and in addition it has a *leading* wattless component I'_{WL}, which can be determined from Fig. 112 as follows:

$$(I_W + P_2 g_1)^2 + (P_2 b_1 - I'_{WL})^2 = P_1^2 y_1^2,$$

$$I'_{WL} = P_2 b_1 - \sqrt{P_1^2 y_1^2 - (P_2 g_1 + I_W)^2}.$$

The lagging wattless current supplied by the phase regulator will be, therefore,

$$I_0 = I_{WL} + P_2 b_1 - \sqrt{P_1^2 y_1^2 - (P_2 g_1 + I_W)^2}. \quad \text{..............(58)}$$

Dividing all through by P_2 and putting, as before, $\frac{P_2}{P_1} = a$, we get

$$-b_0 = b_2 + b_1 - \sqrt{\frac{y_1^2}{a^2} - (g_1 + g_2)^2}, \quad \text{..............(58a)}$$

where b_0 is the susceptance of the phase regulator. We write $-b_0$ because I_0 is not the lagging wattless current consumed by, but produced by the phase regulator. Hence, so long as the right-hand side of the equation is positive, the phase regulator acts as a capacity.

The wattless current produced by the phase regulator consists of two parts. The one part I_{WL} is the wattless current of the load and is given by the current curve K_2 as a function of the watt current of the load. The other part I'_{WL} is the leading wattless current which is necessary for the line. The latter is likewise given as a function of the watt current by the circle K_1, and depends therefore on the value of the supply pressure P_1. If $P_1 > P_2$, I'_{WL} will be zero for a certain watt current, and will lag at small loads. A part of the load wattless current can then be supplied by the line current, and the current of the phase regulator will be correspondingly smaller. The wattless current of the phase regulator is always given by the horizontal distance between the two curves K_1 and K_2. If these two curves cut, then $I_0 = 0$ at the point of intersection. Passing beyond this point, I_0 becomes negative, i.e. the current *given out* of the phase regulator is leading, or that *taken in* by it is lagging—the same then acts as an inductance and b_0 becomes positive.

With a given transmission line and given pressures P_1 and P_2, the transmitted power has a maximum which is given by the highest point B on the circle K_1. At this point

$$\left.\begin{aligned} I_W &= P_1 y_1 - P_2 g_1 = P_2\left(\frac{y_1}{a} - g_1\right), \\ g_2 &= \frac{y_1}{a} - g_1. \end{aligned}\right\} \quad \text{..............(59)}$$

The same condition for maximum power is given by equation (58a), since for larger values of g_2 the root becomes imaginary. In this case, the wattless current of the phase regulator is

$$I_0 = I_{WL} + P_2 b_1 - b_0 \doteq b_2 + b_1.$$

The maximum power is

$$W_{\max} = P_2^2 g_2 = P_1^2 a^2 g_2 = P_1^2 a^2 \left(\frac{y_1}{a} - g_1\right). \quad \text{.................(60)}$$

If the supply pressure P_1 is maintained constant, whilst the receiver pressure P_2 is varied, we get different circles K_1, all of which have the same radius, and whose centres lie on the straight line $\overline{OO_1}$ at distances from O proportional to P_2.

The highest points B of these circles, and accordingly the watt currents at maximum load, are represented by a parallel to OO_1. Thus, whilst P_2 increases as a straight line function, the watt current I_W decreases as a straight line function. Hence there is a certain ratio $a = \frac{P_2}{P_1}$ for which the maximum power, which can be transmitted over a line of given constants r_1 and x_1, attains its highest value. This value of a can be found from the condition:

$$\frac{dW_{\max}}{da} = P_1^2(y_1 - 2ag_1) = 0 \quad \text{or} \quad a = \frac{y_1}{2g_1} = \frac{z_1}{2r_1}.$$

For this maximum, therefore,

$$g_2 = \frac{y_1}{a} - g_1 = g_1, \quad -b_0 = b_1 + b_2. \quad \text{.................(61)}$$

The maximum power itself is

$$W_{\max} = P_1^2 \frac{z_1^2}{4r_1^2} g_1 = \frac{P_1^2}{4r_1}, \quad \text{.......................(62)}$$

and represents the maximum power which can be transmitted over the given line at the given supply pressure P_1.

It is also of interest to determine the phase displacement at the supply terminals of a compounded power transmission scheme. Fig. 113 represents the same diagram as Fig. 112, except that the current curve K_2 of the load and the load current I_2 have been omitted. The extremity C of the vector of the line current I_1 moves over the circle K_1 described about O_1. This circle is thus the current diagram of the line current. The receiver pressure P_2 coincides with the ordinate axis.

The angle P_2OC is thus the angle of lead of the line current with respect to the receiver pressure. On the other hand, if we consider $\overline{O_1O}$ as the real axis of a new system of coordinates in respect to the origin O_1, then, as shewn, P_2 is represented by the vector $\overline{O_1O}$ and P_1 by $\overline{O_1C}$. The angle by which the supply pressure P_1 leads the receiver pressure P_2 is thus $\angle OO_1C = \theta$. If we draw a circle K to pass through O and O_1 with its centre on the abscissa axis, it will then be seen that

$$\angle OO_1C = \angle P_2OD = \angle \theta,$$

and the angle COD gives the phase displacement ϕ_1 at the supply terminals.

The radius of the circle K is, as already shewn above, $\frac{P_2}{2x_1}$. Since $\overline{O_1C} = P_1 y_1$, the phase displacement at the supply terminals cannot become zero unless

$$P_1 y_1 \leqq \frac{P_2}{x_1} \quad \text{or} \quad \frac{P_2}{P_1} = a \geqq x_1 y_1 = \frac{x_1}{z_1} = \sin\phi_1.$$

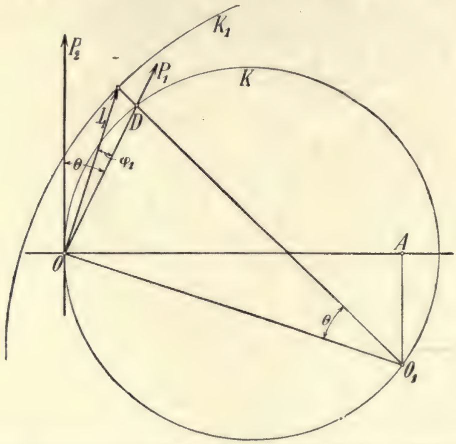


FIG. 113.—Phase Displacement between Current and Pressure at Primary Terminals.

When the equality sign holds, the phase displacement only disappears for the one load where $g_2 = g_1$. If the inequality sign holds, the phase displacement in the supply circuit disappears at two loads, which are graphically determined by the points of intersection of the two circles K_1 and K. Between these two load points the current in the supply station leads—otherwise it lags.

If it is required to *over-compound* the transmission scheme, then P_1, r_1 and x_1 are constants, whilst the receiver pressure P_2 increases with the load.

If we put, for example, $P_2 = P_{2,0} + I_W r_W$,

where $P_{2,0}$ is the receiver pressure at no-load and r_W is a resistance, in this case, the wattless current I_0 is obtained by an equation similar to equation (58).

$$I_0 = I_{wL} + (P_{2,0} + I_W r_W) b_1 - \sqrt{P_1^2 y_1^2 - \{P_{2,0} g_1 + (r_W g_1 + 1) I_W\}^2}. \quad \ldots (63)$$

Hence in an over-compounded system, when

$$I_W = \frac{P_1 y_1 - P_{2,0} g_1}{r_W g_1 + 1} = \frac{P_1 z_1 - P_{2,0} r_1}{r_W r_1 + z_1^2},$$

we get maximum power $W_{\max} = P_2 I_W = (P_{2,0} + I_W r_W) I_W$

or

$$W_{\max} = P_{2,0} \frac{P_1 z_1 - P_{2,0} r_1}{r_W r_1 + z_1^2} + \left(\frac{P_1 z_1 - P_{2,0} r_1}{r_W r_1 + z_1^2}\right)^2 r_W. \quad \ldots\ldots\ldots\ldots (64)$$

36. Losses and Efficiency in a Compounded Transmission Scheme. Since the diagram of the line current in a compounded transmission scheme is a circle (see above), we can represent the powers and losses by straight lines, as shewn in Sections 23 to 25. Since, however, in that case we started with the diagram for the supply circuit, we obtained the abscissa axis for the line of supplied power. In this case, on the contrary, we start from the diagram for the receiver circuit, and consequently get the abscissa axis as the line of the power given out.

The loss in the line is $V_1 = I_1^2 r_1$,

and is represented by a loss line which is the semi-polar of the origin with respect to the circle. Denoting the co-ordinates of the current curve K_1 in Fig. 112 by (u, v), and taking abscissae to the right as positive, we then get the equation of the circle K_1:

$$(u - P_2 b_1)^2 + (v + P_2 g_1)^2 = P_1^2 y_1^2 = P_2^2 y_1^2 \frac{1}{a^2}$$

or

$$u^2 + v^2 - 2P_2 b_1 u + 2P_2 g_1 v = P_2^2 y_1^2 \left(\frac{1}{a^2} - 1\right).$$

The heating losses in the line are therefore

$$(u^2 + v^2) r_1 = 2P_2 r_1 y_1^2 \left[x_1 u - r_1 v + \frac{P_2}{2}\left(\frac{1}{a^2} - 1\right)\right] = B_1 \mathsf{V}_1,$$

where

$$B_1 = 2P_2 g_1$$

and

$$\mathsf{V}_1 = x_1 u - r_1 v + \frac{P_2}{2}\left(\frac{1}{a^2} - 1\right) = 0.$$

This latter is the equation of the loss line. The power given out is

$$W_2 = P_2 v,$$

and the power supplied,

$$\begin{aligned} W_1 &= W_2 + V_1 = P_2 v + B_1 \mathsf{V}_1 \\ &= 2P_2 g_1 x_1 u - 2P_2 g_1 r_1 v + P_2 v + P_2^2\left(\frac{1}{a^2} - 1\right) g_1 = A_1 \mathsf{W}_1. \end{aligned}$$

The straight line $\mathsf{W}_1 = 0$ is thus the line for the supplied power. As seen from the form of its equation, this line passes through the point where the loss line $\mathsf{V}_1 = 0$ cuts the abscissa axis $v = 0$. In order to be able to draw this line $\mathsf{W}_1 = 0$, we further determine the tangent of the angle α which it makes with the ordinate axis. This is

$$\tan \alpha = \frac{g_1 r_1 - \frac{1}{2}}{g_1 x_1} = -\frac{\frac{1}{2r_1} - g_1}{b_1}.$$

As already shewn, the point O_1 is the point of intersection of two circles, one of which has the radius $\frac{P_2}{2r_1}$ and the other $\frac{P_2}{2x_1}$. These two circles cut one another rectangularly in the origin O and at

the point O_1. In Fig. 114 the centres of the two circles are denoted by M_r and M_x. As seen from this figure, the power line $W_1=0$ is perpendicular to the line $\overline{M_rO_1}$, and is therefore parallel to the line $\overline{M_xO_1}$. The efficiency of the scheme can now be determined from the loss line and the two power lines (see the construction in Fig. 114).

The efficiency of the line depends on the line constants r_1 and x_1 on the ratio $\alpha=\frac{P_2}{P_1}$ and also on the watt current of the load; but is independent of the wattless current of the load. In practice, synchronous machines are used as phase regulators.* As is well known, such machines yield a leading or lagging wattless current according

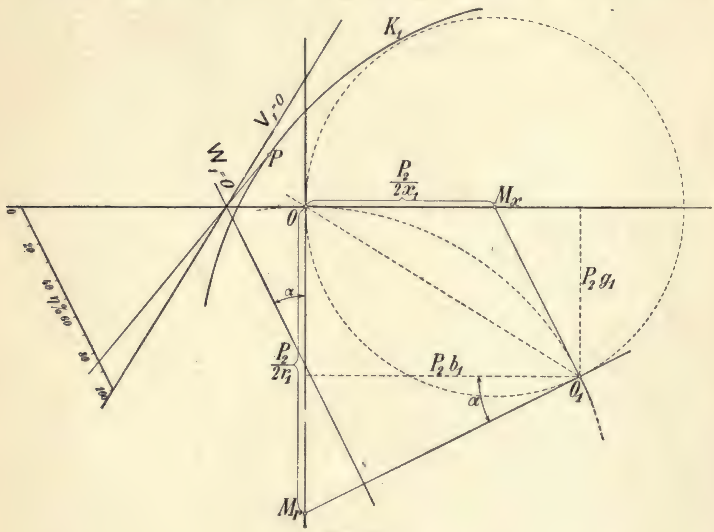


FIG. 114.

as they are over- or under-excited. In the former case they act as a capacity, in the latter as self-induction. In addition to the wattless current, the phase-regulator on no-load also requires a watt current to cover its losses, which form an additional load in the system. The phase regulator can also be used for other purposes at the same time, e.g. as a motor giving out mechanical work or as generator for the production of a watt current.

By means of the above diagrams, a whole series of problems on compounding of transmission schemes can be solved. A comparison of these diagrams with the load diagram of a synchronous motor with constant excitation shows the great similarity between the two.†

* For details of the use of synchronous machines as phase regulators, see Arnold-la Cour, *Wechselstromtechnik*, vol. iv. p. 447.

† Arnold-la Cour, *Wechselstromtechnik*, vol. iv. p. 418.

CHAPTER VII.

MAGNETICALLY INTERLINKED ELECTRIC CIRCUITS.

37. Magnetic Interlinkage between Two Circuits. Until now we have investigated only the phenomena which occur in a single closed circuit. Since, however, the E.M.F.'s in a circuit are generally due to induction, as is the case, for example, in all electromagnetic machines and transformers, it is of the greatest importance to study exactly the relation between two electric circuits. The simplest of all electrical apparatus met with in practice is the single-phase transformer, which consists of two electric circuits—a primary and a secondary—magnetically linked to one another. In Fig. 115 the principle of a transformer of this type, viz. a *mantle* or *shell* transformer, is represented diagrammatically, whilst Figs. 116*a* and *b* shew photographs of such transformers. Both primary and secondary, which are insulated from one another, are wound on the core in the centre, whilst the two outer cores or *mantle* serve as a return path for the flux. The single-phase current is supplied to the transformer on the primary side, and is withdrawn, transformed, from the secondary side. Fig. 116*b* is a view with part of the stampings removed, to shew the windings more clearly.

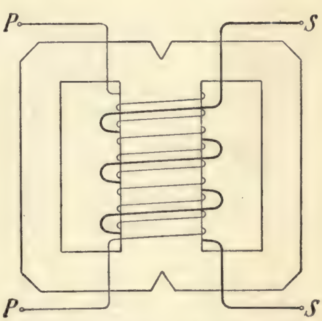


FIG. 115.—Diagram of a Shell Transformer.

Fig. 117*a* shews how the field is distributed in such a transformer. *I* is the primary winding and *II* the secondary. As a rule, the number of turns w_1 on the primary is not the same as the number w_2 on the secondary, although these may be equal. The chief part of the

flux passes through the laminated iron core, and thus embraces the total turns of each winding. Another part of the flux is interlinked with some of the primary turns or with some of the secondary, but not with both; whilst still another part may be interlinked with many turns of the one winding, but only with few of the other.

The magnetic force in the air gap for the section *aa* is represented by the curve *C* in Fig. 117*b*.

In developing the theory of the transformer, it is best to split up the field into tubes of force. Considering a single tube of force inter-

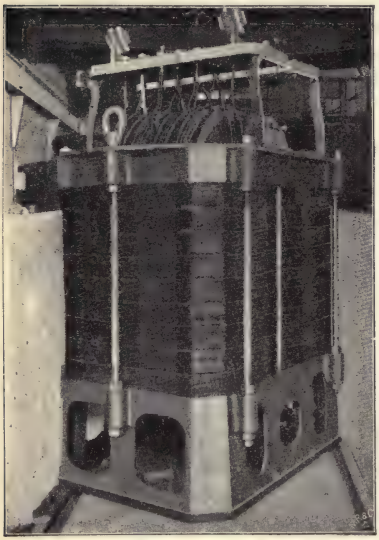


FIG. 116*a*.

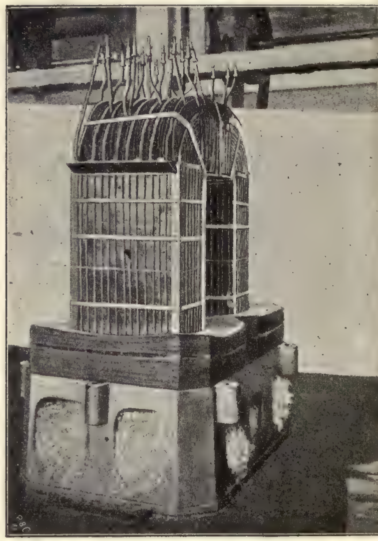


FIG. 116*b*.

linked with w_{1x} primary turns and w_{2x} secondary turns, then the flux in this tube is proportional to $i_1 w_{1x} + i_2 w_{2x}$, where i_1 and i_2 denote the currents in the primary and secondary windings respectively. If the number of turns on primary and secondary is the same, the currents i_1 and i_2 will be very nearly equal to one another, but will flow in opposite directions.

Now, since

$$i_1 w_{1x} + i_2 w_{2x} = (i_1 + i_2) w_{2x} + i_1 (w_{1x} - w_{2x})$$

or

$$= (i_1 + i_2) w_{1x} + i_2 (w_{2x} - w_{1x}),$$

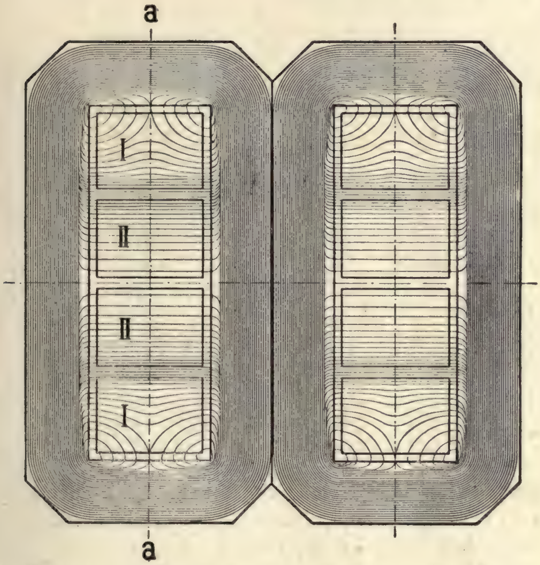
the flux can be split up into two parts, one of which is proportional to the magnetising current $(i_1 + i_2)$ and the other either to the primary or the secondary current. The first part of this flux is called the *main flux* and the second the *stray flux*.

The flux of this tube induces an E.M.F. in the primary winding proportional to

$$\frac{dw_{1x}(i_1 w_{1x} + i_2 w_{2x})}{dt} = \frac{d}{dt}\left\{(i_1 + i_2) w_{1x} w_{2x} + i_1 w_{1x}(w_{1x} - w_{2x})\right\},$$

and an E.M.F. in the secondary proportional to

$$\frac{dw_{2x}(i_1 w_{1x} + i_2 w_{2x})}{dt} = \frac{d}{dt}\left\{(i_1 + i_2) w_{1x} w_{2x} + i_2 w_{2x}(w_{2x} - w_{1x})\right\}.$$



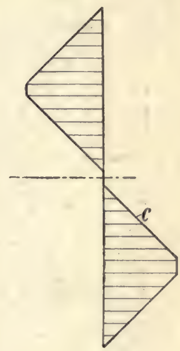


FIG. 117a.—Diagram of Tubes of Force in a Shell Transformer. **FIG. 117b.**

From this, we see that the main flux of every tube always induces the same E.M.F. in both primary and secondary windings, whilst the E.M.F.'s induced by the stray flux are proportional to the currents in the respective windings. The stray flux has a large part of its path in air, and is therefore in phase with the current which produces it. Most of the main flux, however, has an iron path, the hysteresis of which will cause this flux to lag (by an amount equal to the hysteretic angle of advance) behind the magnetising current $(i_1 + i_2)$.

Summing up the E.M.F.'s induced in each winding, we get, for the primary circuit, the differential equation:

$$P_1\sqrt{2}\sin(\omega t + \theta_1) = i_1 r_1 + S_1\frac{di_1}{dt} + w\frac{d\Phi_h}{dt}, \quad \ldots\ldots\ldots\ldots(65a)$$

and for the secondary circuit:

$$0 = P_2\sqrt{2}\sin(\omega t + \theta_2) + i_2 r_2 + S_2\frac{di_2}{dt} + w\frac{d\Phi_h}{dt}, \quad \ldots\ldots\ldots\ldots(65b)$$

where P_1 and P_2 are the respective primary and secondary terminal pressures; S_1 denotes the sum of the interlinkages with the primary stray flux (that is, that part of the primary flux which is not interlinked with the secondary) produced by unit current in the primary. Similarly for S_2. S_1 and S_2 are called the *coefficients of stray induction*, and are

$$\left.\begin{aligned} S_1 &= \Sigma\frac{w_{1x}(w_{1x}-w_{2x})}{R_x}, \\ S_2 &= \Sigma\frac{w_{2x}(w_{2x}-w_{1x})}{R_x}, \end{aligned}\right\} \quad \text{.......................(66)}$$

where R_x is the reluctance offered to the tube of force which is interlinked with w_{1x} primary and w_{2x} secondary turns. Φ_h is the ideal main flux, which is completely interlinked with both primary and secondary windings and induces an E.M.F. in both which is proportional to the sum of all the interlinkages $\Sigma(\Phi_h w_x)$ of the main flux.

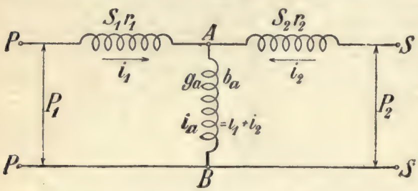


FIG. 118.—Equivalent Circuit of a Transformer.

The above two differential equations (65a and b) apply both to the transformer (Fig. 115) and to the circuit shewn in Fig. 118.

In the branch AB, the current $i_1 + i_2 = i_a$ flows, and requires between the terminals A and B the pressure

$$e = w\frac{d\Phi_h}{dt},$$

which is equal but opposite in direction to the E.M.F. $-e$ induced by the main flux in the two circuits. This E.M.F., of course, has the same frequency c in both circuits, since they are embraced by the same flux and fixed with respect to one another. Since Φ_h lags behind the magnetising current i_a by the angle $\frac{\pi}{2} - \psi_a$, the pressure e leads the magnetising current i_a by the angle ψ_a. Hence we can thus write:

$$\dot{I}_a = \dot{E}\dot{y}_a = \dot{E}(g_a + jb_a),$$

where
$$\tan\psi_a = \frac{b_a}{g_a}.$$

In this way we may replace the transformer by the circuit represented in Fig. 118, and can treat the same analytically just as any other circuit having an impedance in series with two parallel branches.

Denoting $2\pi cS_1$ by x_1 and $2\pi cS_2$ by x_2, we may then write the above differentials as follows:

$$\dot{P}_1 - \dot{E} = \dot{I}_1 r_1 - j\dot{I}_1 x_1 = \dot{I}_1 \dot{z}_1, \quad -\dot{E} - \dot{P}_2 = \dot{I}_2 r_2 - j\dot{I}_2 x_2 = \dot{I}_2 \dot{z}_2, \quad \text{......(67)}$$

where
$$\dot{I}_1 = \dot{I}_a - \dot{I}_2 = \dot{E}\dot{y}_a - \dot{I}_2$$

and
$$\dot{z}_1 = r_1 - jx_1, \quad \dot{z}_2 = r_2 - jx_2.$$

When the secondary circuit is open, i.e. when the transformer is on *no-load*, $I_2 = 0$, and the primary current I_1 equals the magnetising current I_a. Since the resistances and reactances, and also the magnetising current, of a normal transformer are usually very small, it follows that at no-load the secondary pressure $P_2 = E$ will be nearly equal to the primary pressure P_1, assuming of course that the number of turns on the primary is the same as that on the secondary, i.e.

$$w_1 = w_2 = w.$$

The currents and pressure of a transformer can be best shewn graphically, as in Fig. 119. Set off the main flux Φ_h along the negative direction of the abscissa axis, then the E.M.F. $-E$ induced by Φ_h falls along the negative direction of the ordinate axis (since the induced E.M.F. lags 90° behind the inducing flux). The flux itself, however, is not in phase with the M.M.F. (or magnetising current), but follows the same at the angle $\frac{\pi}{2} - \psi_a$. This lagging of the flux behind the magnetising current is due to the hysteresis and eddy currents, caused by the continuous reversal of the magnetisation in the core, which is treated more fully in Chapter XVIII.

The magnetising current I_a can be calculated from the circuit constants and set off in the diagram. Further, if the secondary current I_2 is known, the secondary pressure P_2 can be found by geometrically subtracting the secondary impedance pressure $I_2 z_2$ from the induced E.M.F. $-E$.

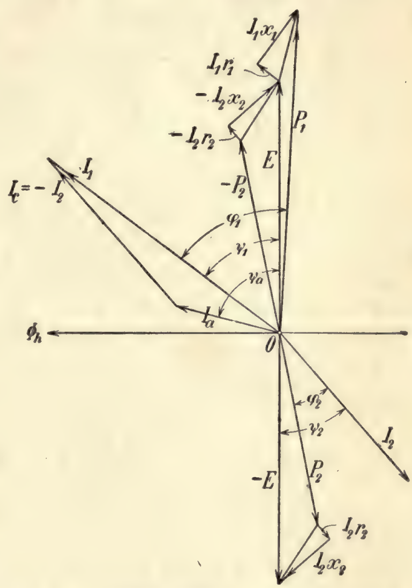


FIG. 119.—Vector Diagram of the Currents and Pressures in a Transformer.

Since the current I_2—induced in the secondary winding by the flux Φ_h—is always directed so that it tends to weaken the inducing field, it is obvious that a primary current $-I_2$ must be supplied to overcome the reaction of the secondary current I_2 on the field, if Φ_h is to be kept constant. Consequently, the current supplied to the primary has two components. The one component is the magnetising current I_a, necessary for producing the field, while the other component is the compensating current $-I_2$ required to neutralise the reaction of the secondary current I_2 on the main field. Hence the primary current I_1

is simply the resultant of the currents I_a and $-I_2$. Again, if we add the impedance pressure Iz_1 to the pressure E, which is equal and opposite to the E.M.F. $-E$ induced by the main flux Φ_h, the primary pressure P_1 will be obtained. If we now turn the pressure-triangle $-E$, P_2 through 180° to the position E, $-P_2$, we get a clear view of the pressure drop from the primary terminal pressure P_1 to the secondary terminal pressure $-P_2$. The pressure E often termed the E.M.F. consumed by the counter-electromotive force $-E$, is required for driving the magnetising current I_a through the circuit, and therefore leads the latter by the angle ψ_a, as shewn in the figure.

The power

$$EI_a \cos\psi_a = E^2 g_a$$

is consumed by the iron losses in the magnetic circuit, and is dissipated in the form of heat.

The phenomena which occur in a transformer occur in every other form of electromagnetic apparatus, although in a somewhat modified form. *In every case, however, we have the secondary current induced by the main flux, and the corresponding compensating current which combines with the magnetising current necessary to produce the flux to form the primary current. The main flux serves to transmit the power from the primary side to the secondary, just as a belt transmits the power from one pulley to another.*

In the stationary transformer the power $EI_1 \cos\psi_1$ is transmitted from the primary circuit to the main flux. Here, in the main flux, the iron losses $EI_a \cos\psi_a$ are consumed, so that the power transmitted to the secondary circuit is

$$EI_2 \cos\psi_2 = EI_1 \cos\psi_1 - EI_a \cos\psi_a;$$

but since $EI_a \cos\psi_a$ is usually very small, nearly the whole power is conveyed from the primary to the secondary.

The frequency of both primary and secondary is the same. The only reason therefore for using a stationary transformer is to effect a change of pressure as the power is transmitted from primary to secondary. This is achieved by choosing different numbers of turns for the primary and secondary windings. If there are w_1 turns on the primary side and w_2 on the secondary, then the E.M.F. induced in the latter will be

$$E_2 = \frac{w_2}{w_1} E_1 = \frac{E_1}{u},$$

since the flux Φ_h induces the same E.M.F. in every turn. The secondary current is

$$I_2 = \frac{w_1}{w_2} I_c = uI_c,$$

where I_c is the compensating current in the primary winding. This follows at once from the fact that the ampere turns of the two currents I_2 and I_c must be equal and opposite. u is the ratio of transformation, which in a stationary transformer is the same for currents as pressures. In the equivalent circuit, where the primary

and secondary circuits are electrically connected, all secondary pressures must be reduced to the primary by multiplying by u, and secondary currents by dividing by u. The powers remain unaltered, since

$$(E_2 I_2) = \left(\frac{E_1}{u} \cdot u I_c\right) = (E_1 I_c).$$

On the other hand, the impedances must be converted in the ratio u^2 since

$$\frac{E_2}{I_2} = \frac{E_1}{u} \cdot \frac{1}{u I_c} = \frac{E_1}{u^2 I_c}.$$

By these reductions the equivalent circuit and all the calculations may be made independent of the ratio of conversion of the transformer.

38. Self-, Stray and Mutual Induction of Two Circuits. Neglecting the iron losses in a transformer, the main flux at no-load can be written,

$$\Phi_h = \frac{i_{10} w_1}{R},$$

where w_1 = number of primary turns and R = magnetic reluctance offered to the ideal flux completely interlinked with both primary and secondary windings. The E.M.F. induced in the secondary winding is then

$$e_2 = -w_2 \frac{d\Phi_h}{dt} = -\frac{w_1 w_2}{R} \frac{di_{10}}{dt} = -M \frac{di_{10}}{dt}.$$

$M = \frac{w_1 w_2}{R}$ is called the *coefficient of mutual induction* between the primary and secondary windings. Introducing this coefficient into equation (65a) we get at no-load:

$$P_1 \sqrt{2} \sin(\omega t + \theta_{10}) = i_{10} r_1 + \left(S_1 + M \frac{w_1}{w_2}\right) \frac{di_{10}}{dt} = i_{10} r_1 + L_1 \frac{di_{10}}{dt},$$

where L_1 denotes the total interlinkages of the primary winding with the flux produced by unit current in this winding. This is called the *coefficient of self-induction* of the primary winding. Between the coefficients of self-, stray and mutual induction, there exists therefore the following relation,

$$L_1 = S_1 + M \frac{w_1}{w_2}, \quad \text{..............................(68}a\text{)}$$

for the primary winding, and similarly

$$L_2 = S_2 + M \frac{w_2}{w_1} \quad \text{..............................(68}b\text{)}$$

for the secondary winding.

By multiplying these two expressions, we get

$$M^2 = (L_1 - S_1)(L_2 - S_2). \quad \text{......................(68}c\text{)}$$

Of the flux produced by and interlinked with the primary, the part corresponding to $M \frac{w_1}{w_2}$ is interlinked with the secondary, whilst the part corresponding to S_1 is interlinked only with the primary.

In practice, the ratio $$\frac{L_1}{M\frac{w_1}{w_2}} = \frac{L_1}{L_1 - S_1} = \sigma$$

is known as the *leakage coefficient*, a name given by J. Hopkinson. σ is always greater than unity, and represents the ratio between the total flux and the flux Φ_h which is interlinked with the secondary; or, in other words, the ratio between the total and the useful flux. The flux which is only interlinked with one winding is called *stray flux*. Both the primary and the secondary have their own stray fluxes.

In electromagnetic machinery, we have nearly always to deal with a main flux and a stray flux, or with corresponding magnitudes, viz. the coefficients of mutual and of stray induction. This is due to the fact that these fluxes are actually present in the machine, whilst the fluxes corresponding to the coefficients of self-induction do not as a rule exist, and consequently are not easy to calculate. Moreover, the former method of calculation has the advantage that all machines can be analytically replaced by equivalent electric circuits, since in the equivalent circuits the only constants which occur are:

$$b_a = \frac{1}{2\pi c M \frac{w_1}{w_2}}, \quad x_1 = 2\pi c S_1 \quad \text{and} \quad x_2 = 2\pi c S_2.$$

On the contrary, the reactance $2\pi c L_1$ is not at all confined to one electric circuit, but is distributed over two circuits in which different currents flow. Consequently, with machines, it is not convenient to work with the reactance due to self-induction.

In the case of mains or other similar circuits however, where little or no iron at all is present, the conditions are different. Here the reaction of the currents in neighbouring conductors is often so small that the stray flux is larger than the main flux. In such cases it is best to use the coefficients of self-induction, and estimate as nearly as possible, by approximate calculations and experiments, the damping effect of secondary currents in the neighbourhood or in the conductors themselves.

When a circuit is influenced by a closed secondary circuit in its neighbourhood, the differential equations ($65a$ and b) appear in the following form:

$$p_1 = i_1 r_1 + S_1 \frac{di_1}{dt} + M\frac{w_1}{w_2}\frac{di_1}{dt} + M\frac{di_2}{dt} = i_1 r_1 + L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} \quad \ldots\ldots(65c)$$

and $$0 = i_2 r_2 + S_2 \frac{di_2}{dt} + M\frac{w_2}{w_1}\frac{di_2}{dt} + M\frac{di_1}{dt} = i_2 r_2 + L_2 \frac{di_2}{dt} + M\frac{di_1}{dt}. \quad \ldots(65d)$$

Instead of solving these two equations with the unknowns i_1 and i_2, each of which would bring us to a differential equation of the second degree for i_1 alone or i_2 alone, we may demonstrate the damping effect of secondary circuits by the following simpler considerations.

For the sake of simplicity, assume that the resistances r_1 and r_2 in the equivalent circuits are negligibly small compared with the

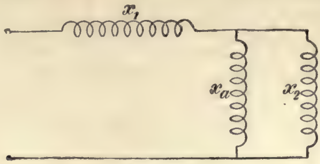


FIG. 120.

reactances. We then get the circuit shewn in Fig. 120. The total reactance of this circuit is

$$x_t = x_1 + \frac{1}{\frac{1}{x_a} + \frac{1}{x_2}} = x_1 + \frac{x_a x_2}{x_a + x_2}$$

$$= 2\pi c \left\{ S_1 + \frac{\frac{w_1}{w_2} \left(\frac{w_1}{w_2}\right)^2 S_2}{\frac{w_1}{w_2}\left(M + \frac{w_1}{w_2} S_2\right)} \right\} = 2\pi c \left(L_1 - \frac{M^2}{L_2} \right).$$

Thus the secondary currents reduce the self-induction of the main conductor, and the greater the ratio of the mutual induction to the self-induction of the secondary conductor, the greater is this reduction. When $w_1 = w_2$, M is always smaller than L_2; and if we take, for example, $M = \frac{1}{2} L_2 = \frac{1}{8} L_1$, then the total reactance of the main circuit will be

$$x_t = 2\pi c L_1 \left(1 - \frac{\left(\frac{1}{8}\right)^2}{\frac{1}{4}} \right) = 2\pi c L_1 \tfrac{15}{16},$$

i.e. some 6 % less than when the secondary circuit is not present.

Taking into account the resistances r_1 and r_2, and also denoting

$$x_{L_1} = 2\pi c L_1 \quad \text{and} \quad x_{L_2} = 2\pi c L_2,$$

we obtain the total impedance,

$$z_t = z_1 + \frac{1}{y_a + \frac{1}{z_2}} = z_1 + \frac{z_2}{1 + y_a z_2},$$

or, neglecting g_a, $\qquad z_t = r_t - j x_t,$

in which expression the resistance and reactance are given by the values

$$\left.\begin{aligned} r_t &= r_1 + \frac{x_a^2 r_2}{r_2^2 + x_{L_2}^2} \\ x_t &= x_{L_1} - \frac{x_a^2 x_{L_2}}{r_2^2 + x_{L_2}^2}. \end{aligned}\right\} \qquad \text{..............................(69)}$$

Thus the secondary currents in neighbouring conductors and the eddy currents in the conductor itself cause an apparent increase in the

resistance and a decrease in the self-induction of the main circuit. This is also what one would expect, for example, in a round conductor; the eddy currents are so directed that at the centre of the conductor they flow against the main current, and at the surface with the main current. Owing to this unsymmetrical distribution of the current over the section of the conductor, the losses are of course increased; and since at the centre of the conductor—where the self-induction is greatest—the current density is least, the total self-induction of the conductor will be less than that calculated on the assumption that no eddies are present. We shall shew in Chap. XXI. how the effect of the eddy currents on the circuit constants can be calculated.

Lastly, it may be pointed out that formula (69) shews clearly that the disturbing influences of the secondary and eddy currents increase with the frequency of the main current and with the dimensions of the conductor.

39. Conversion of Energy in the General Transformer. In the above section we have considered two magnetically-interlinked electric circuits and have seen that the magnetic flux serves to transmit the energy from one to the other. If the primary and secondary circuits are fixed relatively to one another, the total energy given out by the primary will be taken in by the secondary, neglecting iron losses. In many cases, however, the two windings may be capable of motion relatively to one another.

For example, the primary winding may be fixed and the secondary arranged on a rotating axis in such a way that the magnetic field is still linked with both windings. This condition is obtained by placing the secondary winding in slots on the periphery of a laminated cylinder and the primary in slots on the inner surface of a coaxial ring, inside which the cylinder rotates. In such a machine the inter-linkages of the two windings with the rotating field pulsate with different frequencies c_1 and c_2.

For the fixed primary winding, the frequency c_1 is proportional to the speed of the rotating field, while for the moving secondary, c_2 is proportional to the speed of the flux relative to the rotating winding. In this case the total power given out by the primary will not be taken in by the secondary.

If, for example, the main flux Φ_h induce in the primary an E.M.F.

$$E_1 = 4{\cdot}44 c_1 w_1 \Phi_h 10^{-8},$$

having the frequency c_1; and in the secondary an E.M.F.

$$E_2 = 4{\cdot}44 c_2 w_2 \Phi_h 10^{-8},$$

having the frequency c_2.

Then the two E.M.F.'s have the ratio

$$\frac{E_2}{E_1} = \frac{c_2 w_2}{c_1 w_1}.$$

Since in this case also the compensating ampere-turns of the primary

circuit must equal the ampere-turns of the secondary circuit, we must have $m_1 I_c w_1 = m_2 I_2 w_2$, where m_1 denotes the number of similar primary circuits having the turns w_1, and m_2 the number of similar secondary circuits having the turns w_2. On transposing, this becomes

$$\frac{w_1}{w_2} = \frac{m_2 I_2}{m_1 I_c},$$

and combining this with the above ratio of the E.M.F.'s, we get

$$\frac{m_2 E_2 I_2}{m_1 E_1 I_c} = \frac{c_2}{c_1}. \quad \text{.............................(70)}$$

We have here $\angle(E_1 I_c) = \angle(E_2 I_2) = \psi_2$, as in the transformer diagram (Fig. 119).

The power taken in by the secondary circuit is therefore less than that given out by the primary, in the same ratio as the frequency of the secondary current is less than that of the primary. The difference

$$(m_1 E_1 I_c - m_2 E_2 I_2) \cos \psi_2 = \frac{c_1 - c_2}{c_1} m_1 E_1 I_c \cos \psi_2$$

between the power given out by the primary and that taken in by the secondary must therefore appear in some other form, since energy cannot be lost. This difference does not appear in the form of electrical but mechanical energy, and it is thus possible for the general transformer to work also as a motor. The power transmitted from the primary circuit to the magnetic circuit therefore appears partly as electrical power in the secondary circuit and partly as mechanical power. The former (the electrical) part is proportional to

$$\frac{c_1 - c_2}{c_1} = 1 - \frac{c_2}{c_1},$$

i.e. proportional to the velocity with which the secondary circuit lags behind the primary, whilst the latter (the mechanical) part is proportional to the velocity $\frac{c_2}{c_1}$ with which the secondary circuit is cut by the main flux.

Putting
$$c_2 = s c_1,$$
then
$$E_2 = s \frac{w_2}{w_1} E_1, \quad \text{.............................(71)}$$

or, with the same number of turns on the primary as on the secondary, i.e. $w_1 = w_2$,

$$E_2 = s E_1. \quad \text{.............................(71}a\text{)}$$

Assuming further that the number of primary and secondary circuits is the same, i.e. $m_1 = m_2$, then

$$I_2 = I_c$$

and
$$z_2 = \frac{E_2}{I_2} = \frac{s E_1}{I_c} = s z_2', \quad \text{.............................(72)}$$

where z_2' denotes the impedance of the secondary circuit reduced to the primary. Further, let x_2 denote the reactance of the secondary circuit at frequency c_1, then

$$z_2 = r_2 - j\frac{c_2}{c_1}x_2 = r_2 - jsx_2.$$

Hence
$$z_2' = \frac{z_2}{s} = \frac{r_2 - jsx_2}{s} = \frac{r_2}{s} - jx_2.$$

We may therefore replace the general transformer with relatively movable primary and secondary circuits by an equivalent electric circuit (Fig. 121); for, by reducing the secondary frequency to that

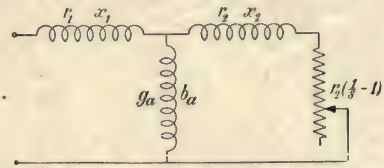


Fig. 121.—Equivalent Circuit of the General Transformer.

of the primary, the continuity of the transmission of energy remains. In the equivalent scheme, the power given out by the primary is:

$$I_2^2\frac{r_2}{s} = I_2\frac{E_2}{s}\cos\psi_2 = I_cE_1\cos\psi_2.$$

Since, however, only the power $V_2 = I_2^2 r^2$
appears in the secondary circuit as electric energy, the difference

$$W_2 = I_2^2\frac{r_2}{s} - I_2^2 r_2 = I_2^2 r_2\left(\frac{1}{s} - 1\right) \quad \text{.....................(73)}$$

must appear in the form of mechanical energy. To absorb an amount of electrical power corresponding to the motor effect of the general transformer, we may therefore employ a resistance in the secondary circuit of the equivalent diagram, having the value

$$r_2\left(\frac{1}{s} - 1\right) \text{ ohms.} \quad \text{..............................(73}a\text{)}$$

This is, of course, a completely non-inductive load. Hence, in spite of the mutual displacement of the primary and secondary windings, we can represent the general transformer by a simple equivalent electric circuit, whose frequency and pressure are those of the primary; whilst all the formulae deduced for the equivalent circuit hold also for the general transformer. The ratio of conversion of the pressures, assuming the same frequency in both secondary and primary, is

$$u_e = \frac{E_1}{E_2} = \frac{w_1}{w_2},$$

whilst the ratio of conversion of the currents is

$$u_i = \frac{I_2}{I_c} = \frac{m_1 w_1}{m_2 w_2};$$

and since
$$\left(\frac{E_2}{I_2}\right)=\frac{E_1}{u_e}\cdot\frac{1}{u_i I_c}=\frac{E_1}{u_i u_e I_c},$$

the ratio of conversion of the impedances is

$$u_e u_i=\frac{m_1 w_1^2}{m_2 w_2^2}, \qquad \text{.............................(74)}$$

where w_1 and w_2 denote the number of effective primary and secondary turns.

FIG. 122.—Induction Motor.

The ordinary form of the general transformer is the asynchronous motor, which consists of a stationary laminated core, or stator, on which the primary is wound, and a rotating laminated core, or rotor, which carries the secondary windings. The two windings are embedded in slots in their cores and lie directly opposite to one another, and as near to the surface as possible so as to reduce the stray flux to a minimum. Fig. 122 shews the photograph of a modern induction motor, with the bearing shield removed.

Example. For

$$P_1 = 500 \text{ volts},$$
$$r_1 = r_2 = 1 \text{ ohm},$$
$$x_1 = 5 \text{ ohms},$$
$$x_2 = 2{\cdot}5 \text{ ohms},$$
$$g_a = 0{\cdot}002 \text{ mho},$$
$$b_a = 0{\cdot}01 \text{ mho},$$

the curves in Figs. 123*a* and *b* have been plotted for the following powers as functions of the slip s:

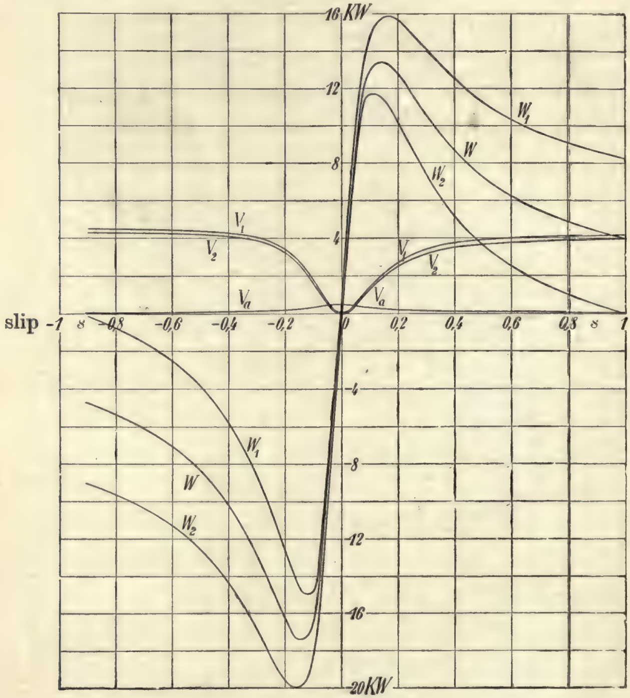


FIG. 123*a*.

1. The power supplied to the primary $W_1 = P_1 I_1 \cos \phi_1$.
2. The primary copper loss $V_1 = I_1^2 r_1$.
3. The iron losses $V_a = E^2 g_a$.
4. The power transferred to the secondary
$$W = W_1 - V_1 - V_a = EI_2 \cos \psi_2.$$
5. The secondary copper loss $V_2 = I_2^2 r_2$.
6. The mechanical power $W_2 = I_2^2 r_2 \left(\frac{1}{s} - 1\right) = W(1 - s)$.

In Fig. 123*b* the scale of the abscissa axis has been increased, in order to shew the curves more clearly in the neighbourhood of synchronism.

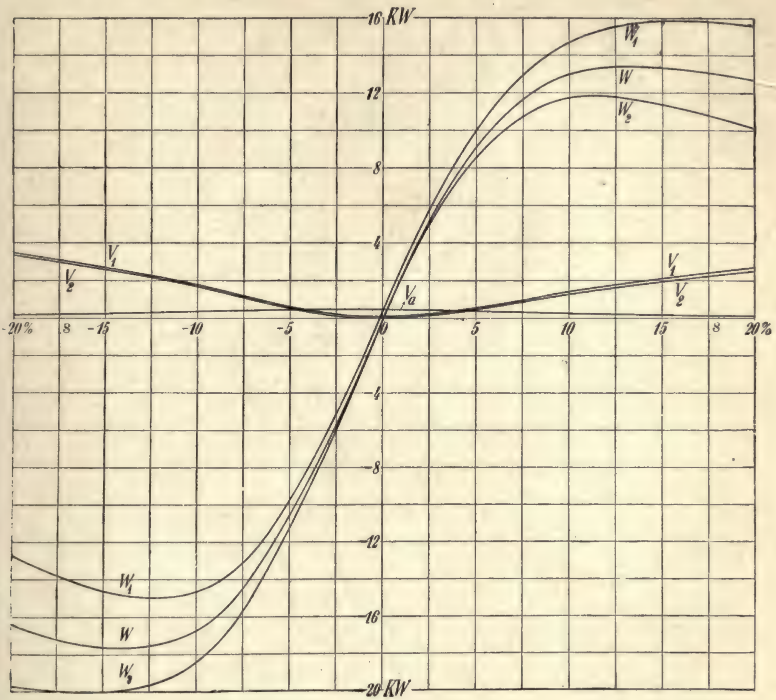


FIG. 123*b*.

As seen from these figures, the general transformer works as motor between $s=0$ and $s=1$, i.e. between rest and the speed at which no E.M.F.'s are induced in the rotor circuits. This speed (i.e. $s=0$) is called the synchronous speed, being that speed at which the secondary circuit is at rest relatively to the main flux, as the rotor rotates *synchronously* with the main flux. s is called the *slip*, since this ratio shews how much the secondary slips relatively to the main flux.

From $s=0$ in the negative direction the general transformer works as a generator and supplies electrical energy to the mains; and from $s=1$ in the positive direction it works as an electric brake, receiving both electrical and mechanical power, both of which are dissipated in the transformer.

CHAPTER VIII.

CAPACITY IN CIRCUITS.

40. Transmission of Power over Lines containing Capacity. For transmitting alternating-current over long distances, overhead lines are usually employed. The capacity effects of such lines are comparatively small, except at very high pressures. Often, however, the current must be taken along cables laid in the earth over parts where overhead wires cannot be used, and the capacity of these sections has to be considered. A simple and approximate calculation of the capacity effects in all such cases can be obtained by assuming the total capacity of the conductors and cables to be concentrated at the centre of gravity of the distributed capacity. We thus get the equivalent circuit shewn in Fig. 124, which can be treated in the same way as the circuit described in Chap. VII. By way of example, we shall here consider the case where the load current at the receiving end of the line is chiefly used for driving induction motors. The current vector will then move over a curve which will be approximately a circle when all the motors are uniformly loaded. Let this circle be represented by K_b in Fig. 125 and the power line by $\overline{P_0^b P_K^b}$.

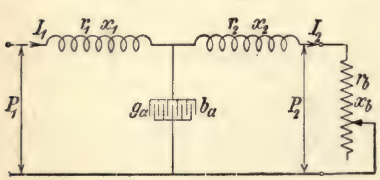


FIG. 124.

By inversion of this circle, we get the load impedance z_b. To this add the impedance z_2, and then a second inversion gives the admittance y'', which is in parallel with y_a. After adding y_a and a further

inversion, we get the impedance z', which is in series with z_1. Lastly, by adding z_1 to z' and once more inverting, we get the load current in the supply circuit, which is represented by the circle K. All the loss and power lines can be now drawn, but it will here suffice if

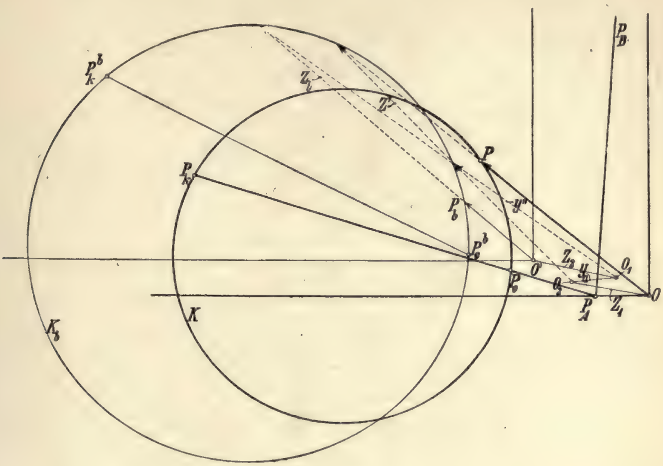


FIG. 125.

we merely shew the line $\overline{P_A P_B}$ for the total losses and the resultant power line $\overline{P_0 P_K}$, from which the efficiency and maximum power of the system can be obtained.

41. Condenser Transformers. In 1891 *Boucherot* proposed to use condensers to transform from a constant pressure to a constant current or vice versa. Such transformers—known as *condenser transformers*—were employed by Boucherot for series circuits—for example, for tunnels or gardens lighted by arc-lamps or incandescent lamps in series, in which cases this system can be used with advantage.

Three systems proposed by Boucherot are shewn in Figs. 126*a-c*. They are all for the same purpose, viz. for obtaining a constant current in the load circuit AB independently of the load, when the supply pressure P_1 is constant. Considering first the scheme shewn in Fig. 126*a*; we have

$$I_2 = \frac{E}{r_2 - jx_2}, \quad I_a = \frac{E}{jx_a},$$

and the total current

$$I_1 = I_a + I_2 = E\left(\frac{1}{r_2 - jx_2} + \frac{1}{jx_a}\right).$$

The supply pressure is therefore

$$P_1 = E - I_1 jx_1 = E\left(1 - \frac{jx_1}{r_2 - jx_2} - \frac{jx_1}{jx_a}\right)$$

or

$$P_1 = E\left(1 - \frac{x_1}{x_a}\right) - jx_1 I_2.$$

Choosing the reactances x_1 and x_a equal, then the current in the receiver circuit will be

$$I_2 = j\frac{P_1}{x_1} \quad \text{or} \quad I_2 = \frac{P_1}{x_1}.$$

That is, with constant supply pressure P_1, the current I_2 in the load circuit is constant, and is thus independent of the load resistance.

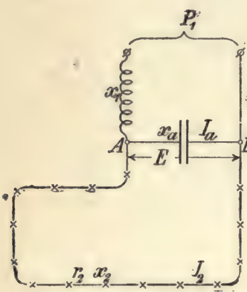


FIG. 126*a*.

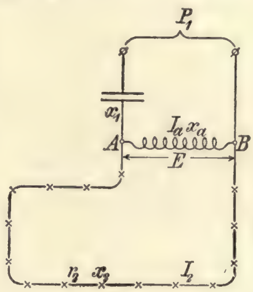


FIG. 126*b*.

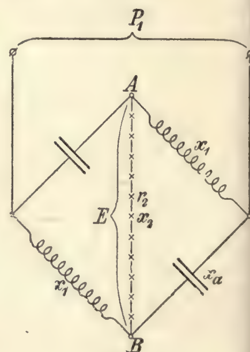


FIG. 126*c*.

The total current I_1 is

$$I_1 = E\left(\frac{1}{r_2 - jx_2} + \frac{1}{jx_1}\right),$$

but

$$E = \frac{r_2 - jx_2}{-jx_1} P_1;$$

hence

$$I_1 = \frac{r_2 - jx_2}{x_1^2} P_1 + j\frac{P_1}{x_1} = \frac{P_1}{x_1^2}\{r_2 - j(x_2 - x_1)\}$$

or

$$I_1 = \frac{P_1}{x_1^2}\sqrt{r_2^2 + (x_2 - x_1)^2}.$$

Thus the total current is a minimum when $x_2 = x_1$. In this case

$$I_{1\,\min} = \frac{P_1 r_2}{x_1^2} = \frac{P_1 r_2}{x_2^2}.$$

When the load circuit is open (i.e. no-load) $r_2 = \infty$, and I_1 will therefore be infinite; whilst when the load resistance is short-circuited, (i.e. $r_2 = 0$), $I_1 = 0$. In other words, no-load in the load circuit acts as a short-circuit to the supply terminals, and conversely, short-circuit in

the load circuit acts as no-load to the supply circuit. For this reason, care must be taken that the circuit is not broken when a lamp is extinguished. This is effected by connecting choking coils in parallel with the lamps, or by using a small transformer for each lamp. In the latter case all danger of short-circuit is removed.

Of the different schemes given above, that in Fig. 126*c* is the best, since here the current I_1 is zero when the load circuit is short-circuited ($z_2=0$), instead of $I_1=\frac{P_1}{x_1}$, as in the other two cases.

Recently the condenser transformer has also been used for producing pulsations of high pressure and frequency. If, for example, a path containing inductance, resistance and a spark-gap is placed in parallel with the condenser (Fig. 127), electric pulsations will be set up, provided the self-induction L_2 is made large enough compared with the resistance r_2. When an alternating pressure P_1 is applied at the supply terminals, a large pressure will be set up across the gap and will give rise to a spark. The pressure then falls immediately, and the spark is extinguished by the rising air warmed by itself. This, however, is scarcely completed when the pressure again rises and produces a further spark. In this manner, sparking will continue, and the frequency c_{ei}—which only depends on the constants of the receiver circuit—will be found to be the natural frequency of the circuit, viz.:

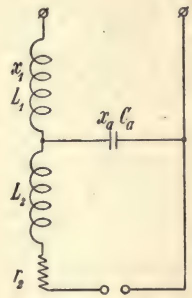


FIG. 127.

$$c_{ei}=\frac{1}{2\pi}\sqrt{\frac{1}{L_2C_a}-\left(\frac{r_2}{2L_2}\right)^2}.$$

This frequency is almost invariably much greater than that of the applied pressure. The oscillations in the receiver circuit produce similar oscillations in the supply circuit also. When the natural frequency is much greater than that of the supply pressure, the oscillations disappear during the time the condenser is discharged.

42. Transmission of Power over Lines containing Distributed Capacity. We now come to the most general case of the transmission of power by alternating-currents. We commence by considering the physical occurrences in the conductors and in neighbouring bodies.

Let a constant alternating E.M.F. act at the supply terminals of a long two-wire system used for the transmission of a single-phase alternating-current to the receiver circuit which contains the load. At any instant, every point in the line will have its own definite potential. Regard the earth as having zero potential. In order to give the line its potential, a certain charging current is necessary, and, since there are both conductors and dielectrics in the electrostatic field due to the line-potential, this charging current will be

dependent on the constants of these bodies, and may be quite considerable. Moreover, every conductor has imperfections in its insulation, through which a quantity of electricity proportional to the potential difference passes. To this latter, we must also add the escape of electricity into the air—known as "silent or glow discharge" (corona).

This potential—which varies from point to point along the line—requires a current due to which an electromagnetic field is formed around the conductor. Moreover, this current is not constant at every section of the conductor, but varies according to the quantity of electricity required for charging, for insulation leakage and for discharge into the air.

The above, however, applies only to what happens at any particular instant, for the applied E.M.F. is not constant, but is a function of the time. For the time being, assume that the pulsating E.M.F. follows a sine law.

Both the electrostatic and the electromagnetic fields vary with the time. Owing to the pulsation of the electrostatic field, energy is consumed in the insulating media. This causes a loss-current, which is in phase with the potential difference at the respective point. The presence of foreign bodies in the field causes an increase in these displacement currents; to this also belongs *electrostatic influence.* The displacement currents can be resolved into two components, one of which is in phase with the difference of potential, and the other displaced 90° from it.

The alternating electromagnetic field induces E.M.F.'s both in the line itself and in outside conductors. The E.M.F.'s induced in the line, i.e. the E.M.F.'s of self-induction, can, under certain conditions, cause an unequal distribution of the current over the cross-section, which will cause an increase in the ohmic resistance of the line (*skin-effect*). The closed conductors lying in the electromagnetic field act as transformer secondaries with the transmission line as primary. Hence in the closed secondaries current will flow which will react on the main conductor (mutual induction).

These E.M.F.'s of mutual induction can also be resolved into an energy component in phase with the current, and an idle component at 90° to the same. The latter component decreases the apparent self-induction of the line. Eddy currents can also be added to the currents in adjacent conductors.

The electromagnetic field produces losses in magnetic bodies due to hysteresis; these losses can be approximately allowed for by an increase in the ohmic resistance, since the field strength is nearly proportional to the current, so long as the field is weak.

Fig. 128 shews a two-wire transmission scheme, representing the effects that have just been discussed above.

We now make the assumption, without which calculation is difficult, that the conductor is uniform, so that the constants of the conductor per unit length can be given. The calculation of these constants is

complicated and inexact, since they depend on the frequency, the pressure and the atmospheric conditions.

*Franke** and *Breisig†* have shewn, however, how these constants can be determined by simple measurements. A conductor can be represented by four constants, which we may suppose to have been experimentally determined.

Since these measurements and calculations must serve as the foundation in working out new installations, we shall briefly summarise them here, and shew the influence which the constants exert.

r_a denotes the equivalent ohmic resistance per kilometre by which the current I must be multiplied in order to obtain the pressure in phase with the current. This pressure drop is due to the ohmic resistance of the line and the watt components of the pressures induced by the resultant electromagnetic field.

x_a denotes the equivalent reactance per kilometre by which the current I must be multiplied in order to obtain the E.M.F.'s which the current leads by 90°. These E.M.F.'s are the wattless components of the E.M.F.'s induced by the resultant electromagnetic field.

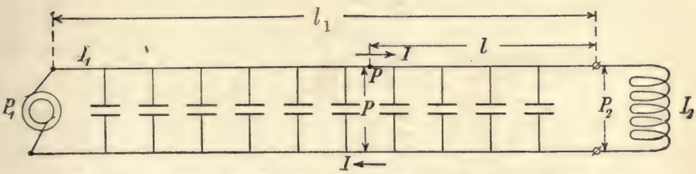


FIG. 128.—Single-phase Transmission Line with Distributed Capacity.

g_l denotes the equivalent conductance per kilometre by which the pressure P must be multiplied in order to get the currents in phase with the pressure. These currents are due to the losses in the insulation and the air, and to the watt components of the displacement currents induced by the electrostatic field.

b_l denotes the equivalent susceptance per kilometre by which the pressure P must be multiplied, in order to get the currents which lag 90° behind the pressure. These currents are the wattless components of the displacement currents induced by the resultant electrostatic field.

Writing symbolically, we get

$$z_a = (r_a - jx_a) l_1,$$

$$y_l = (g_l - jb_l) l_1,$$

where l_1 is the single length of the line in kilometres.

Let the pressure at the receiver terminals be

$$p_2 = P_2 \sqrt{2} \sin \omega t,$$

and the constants of the line and load, i.e. g_2 and b_2, be given; we can then calculate the pressure, the current and their phase displacement

* *E.T.Z.* 1891, Heft 35. † *E.T.Z.* 1899, Heft 10.

at any point in the line. When this is done, we shall be able to find the load on the supply station.

At a point P distant l from the receiver station, we have a pressure $p=P\sqrt{2}\sin(\omega t+\psi)$ and a current $i=I\sqrt{2}\sin(\omega t+\psi-\phi)$.

Since the sinusoidal terminal pressure, under steady conditions, always produces sinusoidal currents and pressures throughout the whole system, it is not necessary to deal with momentary values in this case, so that, for the sake of simplicity, we will introduce the symbolic expressions $\dot{P}$ and $\dot{I}$ and use these for the preliminary calculations. In the formulae deduced, we can then return to the instantaneous values, where these assist in explaining the same.

Let l be negative when taken in the direction of the flow of energy, and positive when taken in the opposite direction; then, in the element dl of the conductor, the *increase of current* is

$$d\dot{I}=\dot{P}\frac{\dot{y}_l}{l_1}dl \quad \text{or} \quad \frac{d\dot{I}}{dl}=\dot{P}\frac{\dot{y}_l}{l_1}.$$

Further, the *increase of pressure* in the conductor-element dl due to the current $\dot{I}$ is

$$d\dot{P}=\dot{I}\frac{\dot{z}_d}{l_1}dl \quad \text{or} \quad \frac{d\dot{P}}{dl}=\dot{I}\frac{\dot{z}_d}{l_1}.$$

By differentiating these two equations, we get

$$\left.\begin{aligned}\frac{d^2\dot{I}}{dl^2}&=\frac{d\dot{P}}{dl}\frac{\dot{y}_l}{l_1}=\dot{I}\frac{\dot{z}_d\dot{y}_l}{l_1^2}\\ \text{and}\quad \frac{d^2\dot{P}}{dl^2}&=\frac{d\dot{I}}{dl}\frac{\dot{z}_d}{l_1}=\dot{P}\frac{\dot{z}_d\dot{y}_l}{l_1^2}\end{aligned}\right\}\quad\text{(75)}$$

These two equations are homogeneous linear differential equations of the second order, and their indefinite integrals are:

$$\dot{P}=\dot{A}\epsilon^{\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}}+\dot{B}\epsilon^{-\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}}$$

and

$$\dot{I}=\sqrt{\frac{\dot{y}_l}{\dot{z}_d}}\left(\dot{A}\epsilon^{\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}}-\dot{B}\epsilon^{-\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}}\right),$$

where $\dot{A}$ and $\dot{B}$ are the constants of integration. These can be determined thus:

At $\quad l=0, \quad \dot{P}=\dot{P}_2 \quad\text{and}\quad \dot{I}=\dot{I}_2.$

Substituting these in the above:

$$\dot{P}_2=\dot{A}+\dot{B}$$

and

$$\dot{I}_2=\sqrt{\frac{\dot{y}_l}{\dot{z}_d}}(\dot{A}-\dot{B}),$$

or
$$A = \frac{P_2 + I_2\sqrt{\frac{z_d}{y_l}}}{2}$$

and
$$B = \frac{P_2 - I_2\sqrt{\frac{z_d}{y_l}}}{2}.$$

Hence
$$P = \tfrac{1}{2}P_2\left(\epsilon^{\sqrt{y_l z_d}\frac{l}{l_1}} + \epsilon^{-\sqrt{y_l z_d}\frac{l}{l_1}}\right) + \tfrac{1}{2}I_2\sqrt{\frac{z_d}{y_l}}\left(\epsilon^{\sqrt{y_l z_d}\frac{l}{l_1}} - \epsilon^{-\sqrt{y_l z_d}\frac{l}{l_1}}\right) \quad \text{......(76)}$$

and
$$I = \tfrac{1}{2}I_2\left(\epsilon^{\sqrt{y_l z_d}\frac{l}{l_1}} + \epsilon^{-\sqrt{y_l z_d}\frac{l}{l_1}}\right) + \tfrac{1}{2}P_2\sqrt{\frac{y_l}{z_d}}\left(\epsilon^{\sqrt{y_l z_d}\frac{l}{l_1}} - \epsilon^{-\sqrt{y_l z_d}\frac{l}{l_1}}\right). \quad \text{..... (77)}$$

Substituting, for the sake of brevity,
$$C = \tfrac{1}{2}\left(\epsilon^{\sqrt{y_l z_d}} + \epsilon^{-\sqrt{y_l z_d}}\right),$$
we can now measure the values of P_1 and I_1 at the supply terminals, where $l = l_1$ for the following two cases as suggested by *A. Franke*:

(1) At *no-load*, i.e. the receiver terminals are open and the current I_2 in the receiver circuit is zero,
$$\frac{I_{10}}{P_{10}} = y_0 = \sqrt{\frac{y_l}{z_d}}\,\frac{\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}}{\epsilon^{\sqrt{y_l z_d}} + \epsilon^{-\sqrt{y_l z_d}}}$$
y_0 can be called the apparent admittance of the conductor. Further, the pressure at the receiver terminals at no-load is
$$P_2 = \frac{P_{10}}{C}. \quad \text{.................................(78)}$$

(2) At *short-circuit*, i.e. the resistance and therefore the pressure between the receiver terminals is zero; i.e.
$$P_2 = 0$$
and
$$\frac{I_{1K}}{P_{1K}} = z_K = \sqrt{\frac{z_d}{y_l}}\,\frac{\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}}{\epsilon^{\sqrt{y_l z_d}} + \epsilon^{-\sqrt{y_l z_d}}}.$$
z_K may be called the apparent impedance of the conductor. The short-circuit current at the receiver terminals is
$$I_2 = \frac{I_{1K}}{C}. \quad \text{.................................(79)}$$

By the division and multiplication of z_K and y_0, we get
$$\frac{z_K}{y_0} = \frac{z_d}{y_l} \text{ and } z_K y_0 = 1 - \frac{1}{C^2}. \quad \text{....................(80)}$$

Then, by introducing $\underset{.}{C}$, y_0 and z_K, we get the equations for the supply pressure and current:

$$P_1 = C(P_2 + z_K I_2) = C P_2 (1 - z_K y_0) + z_K I_1$$

$$= \frac{P_2}{C} + z_K I_1, \quad \text{.......................................(81)}$$

and

$$I_1 = C(I_2 + y_0 P_2) = C I_2 (1 - z_K y_0) + y_0 P_1$$

$$= \frac{I_2}{C} + y_0 P_1, \quad \text{..(82)}$$

or

$$\left.\begin{aligned} P_2 &= C(P_1 - z_K I_1) \\ I_2 &= C(I_1 - y_0 P_1). \end{aligned}\right\} \quad \text{..........................(83)}$$

and

Since these equations hold in general for the pressures and currents in any particular part of the conductor, and the constants C, y_0 and z_K of this part are independent of anything that lies beyond its limits, it will be seen that the equations are sufficient for calculating the electric conditions at any part of the line.

It is sufficient therefore to know the constants r_a, x_a, g_l, b_l or C, y_0, z_K *and the electric conditions at any point of the conductor, in order to be able to calculate the electric conditions for any other point of the conductor.* The three characteristic magnitudes of the conductor C, y_0 and z_K are determined by the short-circuit and no-load experiments.

The calculation of these three quantities can then be carried out either graphically or analytically. In both cases we start from $\epsilon^{\sqrt{y_l z_a}}$. We have

$$\epsilon^{\sqrt{y_l z_a}} = \epsilon^{\sqrt{(g_l - jb_l)(r_a - jx_a)}\, l_1} = \epsilon^{(\lambda - j\mu) l_1}.$$

Working out the root, we get

$$\lambda^2 - \mu^2 = g_l r_a - b_l x_a,$$

$$2\lambda\mu = g_l x_a + b_l r_a$$

and

$$\lambda^2 + \mu^2 = \sqrt{(g_l^2 + b_l^2)(r_a^2 + x_a^2)},$$

from which we get the following expressions for λ and μ:

$$\left.\begin{aligned} \lambda &= \sqrt{\tfrac{1}{2}\{\sqrt{(g_l^2 + b_l^2)(r_a^2 + x_a^2)} + (g_l r_a - b_l x_a)\}} \\ \mu &= \sqrt{\tfrac{1}{2}\{\sqrt{(g_l^2 + b_l^2)(r_a^2 + x_a^2)} - (g_l r_a - b_l x_a)\}}. \end{aligned}\right\} \quad \text{............(84)}$$

and

The quantities λ and μ depend only on the electric properties of the line per unit length and the frequency, and for a system with uniform conductors can be calculated once for all.

Since b_l is a capacity susceptance and $y_l = (g_l - jb_l) l_1$, then b_l is always positive. μ, whose sign is determined by the product $2\lambda\mu$, will then also be positive as a rule.

In calculating the phenomena in long conductors, it is also useful to know the following ratio:

$$\sqrt{\frac{y_l}{z_d}} = \sqrt{\frac{y_l\epsilon^{-j\psi_l}}{z_d\epsilon^{-j\psi_d}}} = \sqrt{\frac{y_l}{z_d}}\epsilon^{j\frac{1}{2}(\psi_d - \psi_l)},$$

where ψ_l is a positive angle.

43. Current and Pressure Distribution in Lines with Uniformly Distributed Capacity. By means of the constants λ and μ, the value of the current and pressure along the line can also be calculated. For this, it is best to start from the equations

$$\begin{aligned} \dot{P} &= \dot{A}\epsilon^{\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}} + \dot{B}\epsilon^{-\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}} \\ &= \dot{A}\epsilon^{(\lambda - j\mu)l} + \dot{B}\epsilon^{-(\lambda - j\mu)l} \end{aligned}$$

and

$$\begin{aligned} \dot{I} &= \sqrt{\frac{\dot{y}_l}{\dot{z}_d}}\left(\dot{A}\epsilon^{\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}} - \dot{B}\epsilon^{-\sqrt{\dot{y}_l\dot{z}_d}\frac{l}{l_1}}\right) \\ &= \sqrt{\frac{\dot{y}_l}{\dot{z}_d}}\left(\dot{A}\epsilon^{(\lambda - j\mu)l} - \dot{B}\epsilon^{-(\lambda - j\mu)l}\right), \end{aligned}$$

and use the transformation

$$\epsilon^{\pm(\lambda - j\mu)l} = \epsilon^{\pm\lambda l}\epsilon^{\mp j\mu l} = \epsilon^{\pm\lambda l}\cos(\mu l \mp j\sin\mu l).$$

We then get the following expressions for the pressure and current at any point in the line:

$$\dot{P} = (\dot{A}\epsilon^{\lambda l} + \dot{B}\epsilon^{-\lambda l})\cos\mu l - j(\dot{A}\epsilon^{\lambda l} - \dot{B}\epsilon^{-\lambda l})\sin\mu l$$

and

$$\dot{I} = \sqrt{\frac{\dot{y}_l}{\dot{z}_d}}\{(\dot{A}\epsilon^{\lambda l} - \dot{B}\epsilon^{-\lambda l})\cos\mu l - j(\dot{A}\epsilon^{\lambda l} + \dot{B}\epsilon^{-\lambda l})\sin\mu l\}.$$

The two constants $\dot{A}$ and $\dot{B}$ represent pressure vectors, and can be written

$$\dot{A} = \frac{\dot{P}_2 + \sqrt{\frac{\dot{z}_d}{\dot{y}_l}}\dot{I}_2}{2} = P_A\epsilon^{j\psi_A}$$

and

$$\dot{B} = \frac{\dot{P}_2 - \sqrt{\frac{\dot{z}_d}{\dot{y}_l}}\dot{I}_2}{2} = P_B\epsilon^{j\psi_B}.$$

Substituting these expressions in the equations for $\dot{P}$ and $\dot{I}$, we get

$$\begin{aligned} \dot{P} &= P_A\epsilon^{(\lambda - j\mu)l + j\psi_A} + P_B\epsilon^{-(\lambda - j\mu)l + j\psi_B} \\ &= P_A\epsilon^{\lambda l}\epsilon^{-j(\mu l - \psi_A)} + P_B\epsilon^{-\lambda l}\epsilon^{j(\mu l + \psi_B)} \end{aligned}$$

and

$$\begin{aligned} \dot{I} &= \sqrt{\frac{\dot{y}_l}{\dot{z}_d}}\{P_A\epsilon^{(\lambda - j\mu)l + j\psi_A} - P_B\epsilon^{-(\lambda - j\mu)l + j\psi_B}\} \\ &= \sqrt{\frac{y_l}{z_d}}\{P_A\epsilon^{\lambda l}\epsilon^{-j[\mu l - \psi_A - \frac{1}{2}(\psi_d - \psi_l)]} - P_B\epsilon^{-\lambda l}\epsilon^{j[\mu l + \psi_B + \frac{1}{2}(\psi_d - \psi_l)]}\}. \end{aligned}$$

Turning now from the symbolic expressions to the momentary values, the pressure will be

$$p = P_A \epsilon^{\lambda l} \sin(\omega t + \mu l - \psi_A) + P_B \epsilon^{-\lambda l} \sin(\omega t - \mu l - \psi_B)$$

and the current

$$i = P_A \sqrt{\frac{y_l}{z_a}} \epsilon^{\lambda l} \sin\left[\omega t + \mu l - \psi_A - \tfrac{1}{2}(\psi_a - \psi_l)\right]$$

$$- P_B \sqrt{\frac{y_l}{z_a}} \epsilon^{-\lambda l} \sin\left[\omega t - \mu l - \psi_B - \tfrac{1}{2}(\psi_a - \psi_l)\right].$$

These equations shew that at any instant both p and i vary along the conductor after a sine wave. If we consider the momentary values at the end of the line and at a distance $\frac{\pi}{\mu}$ from the end, it will be seen that these have opposite values. This shews that, in very long conductors, at different points the pressures oppose one another and the currents flow in opposite directions.

Since the currents and pressures at points along the line $l = \frac{2\pi}{\mu}$ apart have the same phase, the length of the current and pressure waves is $\frac{2\pi}{\mu}$. From this it is further seen that the waves require a complete period $\left(T = \frac{1}{c}\right)$ to traverse the distance $\frac{2\pi}{\mu}$, and since the frequency is c cycles per second, the speed at which the wave travels is $v = \frac{2\pi c}{\mu} = \frac{\omega}{\mu}$.

Hence the currents and pressures in long lines travel at finite velocities which depend only on the constants of the line.

If we neglect the losses in the line, i.e. put $g_l = 0$ and $r_a = 0$, we have

$$\mu = \sqrt{b_l x_a} = 2\pi c \sqrt{L_a C_l},$$

and the speed at which the waves travel will be

$$v = \frac{2\pi c}{\mu} = \frac{1}{\sqrt{L_a C_l}} \text{ km/sec.},$$

where L_a and C_l represent the self-induction and capacity of the line per kilometre.

As will be seen later on, the speed at which the electric waves travel along a conductor approaches the velocity of light, viz. 300,000 km/sec.

Thus the current and pressure waves pass along a long transmission line of 100 km in 1/3000 sec., i.e. with a frequency of 50, during $\frac{c}{3000} = \frac{1}{60}$ cycle, which corresponds to a phase displacement of 6° between the momentary values at the two ends.

The expressions for p and i are made up of two parts, one of which

increases with the distance l from the receiver terminals, whilst the other decreases in the same direction.

The phase displacement between these two waves at a point along the line is $\psi_B - \psi_A + 2\mu l$, i.e. it increases with the distance from the receiver terminals.

The second wave, therefore, can be regarded as the reflection of the first wave, the point of reflection lying beyond the receiver terminals. The second wave lags $\psi_B - \psi_A$ more behind the first than the lag corresponding to the time during which the wave travels from the point in question to the receiver terminals and back.

It is also interesting to note that the resultant pressure wave is formed from the sum of the outgoing and the reflected pressure waves, whilst the resultant current wave equals the difference between the outgoing and reflected current waves.

This is also clear, for at any point in the line the pressures must add, whilst the current must be the difference between that flowing towards the receiver terminals and the reflected current flowing back to the generator.

In addition, each current wave lags $\frac{1}{2}(\psi_a - \psi_l)$ in phase behind the pressure wave producing it.

Since the two separate waves move along the conductor like waves on the surface of water, they can be regarded as **progressive** waves, while the resultant waves are similar in character to a **stationary** wave.

(a) In the special case where the *receiver terminals are open and the line losses negligible,*

$$\lambda = 0, \quad \mu = 2\pi c\sqrt{L_a C_l},$$

$$\dot{I}_2 = 0, \quad \dot{P}_A = \dot{P}_B = \frac{\dot{P}_2}{2} = \tfrac{1}{2}P_2$$

and

$$\sqrt{\frac{\dot{y}_l}{\dot{z}_a}} = \sqrt{\frac{jb_l}{jx_a}} = \sqrt{\frac{C_l}{L_a}};$$

thus, at any point in the line,

$$p = \tfrac{1}{2}P_2 \sin(\omega t + \mu l) + \tfrac{1}{2}P_2 \sin(\omega t - \mu l) = P_2 \sin \omega t \cos \mu l$$

and $$i = \tfrac{1}{2}P_2\sqrt{\frac{C_l}{L_a}}[\sin(\omega t + \mu l) - \sin(\omega t - \mu l)] = P_2\sqrt{\frac{C_l}{L_a}} \cos \omega t \sin \mu l,$$

whence follows $$I = P\sqrt{\frac{C_l}{L_a}} \tan \mu l.$$

In this special case, therefore, the resultant current and pressure waves possess the same properties as **stationary waves with nodes and loops** well known in acoustics. At the points

$$l = 0, \ \frac{\pi}{\mu}, \ \frac{2\pi}{\mu}, \ \frac{3\pi}{\mu}, \ \frac{4\pi}{\mu}, \ \cdots,$$

the current is always zero, whilst between these points it pulsates between a maximum and minimum. At the first points we have nodes, at the others loops of the current wave.

The pressure wave which leads the current wave by 90° both in space and time has its nodes at the positions $l=\frac{\pi}{2\mu}, \frac{3\pi}{2\mu}, \frac{5\pi}{2\mu}, \ldots$, and its loops at $l=0, \frac{\pi}{\mu}, \frac{2\pi}{\mu}, \frac{3\pi}{\mu}, \ldots$.

If the length of the line is $l_1=\frac{5\pi}{2\mu}$, as in Fig. 129, then in this special case, where $\lambda=0$ and I_2 is zero, no applied pressure is necessary to

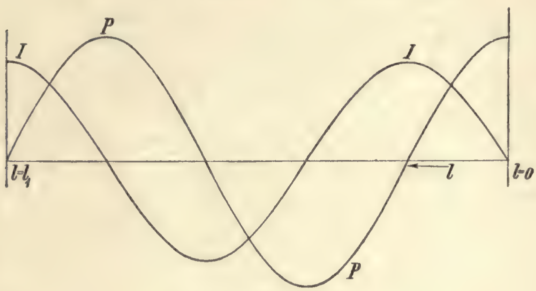


FIG. 129.

produce large current and pressure waves in the line; a condition we have already denoted as *pressure resonance.*

It may also be mentioned that the ratio of the current to the pressure waves is the same everywhere, viz. $\sqrt{\frac{C_l}{L_d}}$.

(*b*) We will also consider the opposite case to no-load in the receiver circuit, namely that in which the *receiver terminals are short-circuited and the line losses negligible.*

$$\lambda=0, \quad \mu=2\pi c\sqrt{L_d C_l}, \quad P_2=0,$$

$$P_A=-P_B=\frac{1}{2}\sqrt{\frac{z_d}{y_l}}I_2=\frac{1}{2}\sqrt{\frac{L_d}{C_l}}I_2;$$

hence
$$p=I_2\sqrt{\frac{L_d}{C_l}}\cos\omega t\sin\mu l,$$

$$i=I_2\sin\omega t\cos\mu l$$

and
$$P=I\sqrt{\frac{L_d}{C_l}}\tan\mu l.$$

In this case also we get a stationary wave, as shewn in Fig. 130, in which the current loops occur at $l=0, \frac{\pi}{\mu}, \frac{2\pi}{\mu}, \frac{3\pi}{\mu}$, etc., and the pressure loops at $l=\frac{\pi}{2\mu}, \frac{3\pi}{2\mu}, \frac{5\pi}{2\mu}$, etc. For a conductor whose length is $\frac{5}{4}$ of the wave-length, no current will flow in the short-circuited

conductor even with a large applied pressure at the terminals. This condition corresponds to *current-resonance.*

From the above it follows that *stationary waves can only be produced with the receiver circuit either open or short-circuited and negligible line losses.* When one of these conditions is not fulfilled, the current and pressure waves travel along the conductor at a speed approaching that of light in a vacuum.

With normal loads, therefore, it is best to deal with the outgoing and reflected waves, and from the ratio between the amplitudes of these two waves at the receiver terminals and their phase displacement $(\psi_B - \psi_A)$, calculate the current and pressure waves over the whole line. At the receiver terminals, where $l = 0$, the relation between the amplitudes of the reflected and outgoing waves is

$$\frac{\dot{P}_A}{\dot{P}_B} = \frac{\dot{P}_2 - \sqrt{\frac{\dot{z}_d}{\dot{y}_l}}\,\dot{I}_2}{\dot{P}_2 + \sqrt{\frac{\dot{z}_d}{\dot{y}_l}}\,\dot{I}_2} = \frac{P_B}{P_A}\,\epsilon^{j(\psi_B - \psi_A)}$$

or

$$= \frac{P_B}{P_A}\left[\cos(\psi_B - \psi_A) + j\sin(\psi_B - \psi_A)\right].$$

The formula shews also that the reflection is only complete when $P_A = \pm P_B$ and $\psi_B = \psi_A$, which is only the case at no-load or short-circuit.

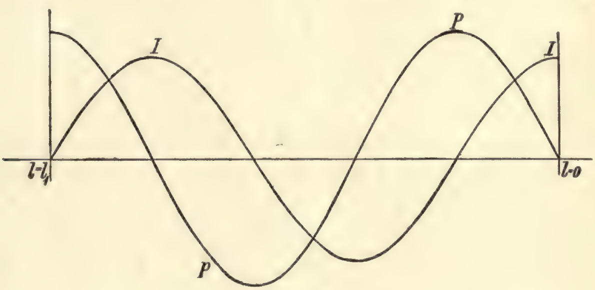


FIG. 130.

The kind of reflection under normal conditions depends both on the load at the receiver terminals and on the line constants.

For the case when the ratio of the resistance of the line to the self-induction is the same as that of the conductance to the capacity, i.e. when $\frac{r_d}{x_d} = \frac{g_l}{b_l}$, then

$$\sqrt{\frac{\dot{z}_d}{\dot{y}_l}} = \sqrt{\frac{z_d}{y_l}} = \sqrt{\frac{x_d}{b_l}} = \sqrt{\frac{L_d}{C_l}} \quad \text{and} \quad \psi_d - \psi_l = 0.$$

Such a line is by *O. Heaviside* termed *distortionless.*

Suppose, further, the load in the receiver circuit is non-inductive; then

$$\frac{P_B}{P_A} = \frac{P_2 - I_2\sqrt{\frac{L_a}{C_l}}}{P_2 + I_2\sqrt{\frac{L_a}{C_l}}} \quad \text{and} \quad \psi_B = \psi_A = \psi_2 = 0.$$

Under these conditions, the outgoing waves are reflected at the same angle as they arrive at the receiver terminals. The reflected waves are weaker, however, the greater the load, and vanish entirely when

$$P_2^2 C_l = I_2^2 L_a,$$

that is, when the electrostatic energy due to the receiver pressure equals the electromagnetic energy due to the receiver current and stored around the line. For this special case where the reflected wave vanishes,

$$p = \left(P_2 + I_2\sqrt{\frac{L_a}{C_l}}\right)\epsilon^{\lambda l}\sin(\omega t + \mu l),$$

$$i = \left(I_2 + P_2\sqrt{\frac{C_l}{L_a}}\right)\epsilon^{\lambda l}\sin(\omega t + \mu l)$$

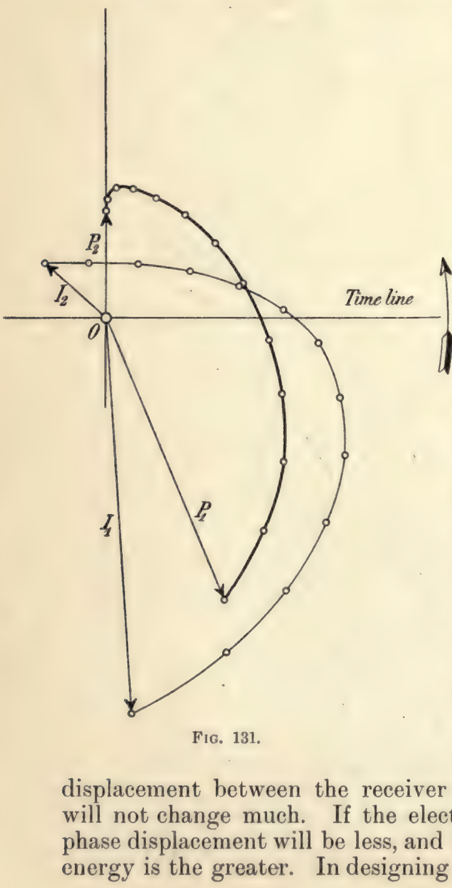


Fig. 131.

and

$$\frac{P}{I} = \frac{P_2 + I_2\sqrt{\frac{L_a}{C_l}}}{I_2 + P_2\sqrt{\frac{C_l}{L_a}}} = \sqrt{\frac{L_a}{C_l}}.$$

It follows further that the angle of phase displacement between current and pressure is zero, i.e. $\cos\phi = 1$, at every point in the line, which distinguishes the progressive wave in the circuit free from disturbance from the stationary wave. We also see that the phase displacement chiefly depends on the phase difference in the receiver circuit and to a much less extent on the relation between the electrostatic and the electromagnetic energy stored in the fields around the conductors at a given load. If these two quantities of energy are kept equal, the phase displacement between the receiver station and the generator station will not change much. If the electrostatic energy preponderates, the phase displacement will be less, and vice versa when the electromagnetic energy is the greater. In designing long lines, therefore, it is necessary

to see that these two quantities of energy are such as to give the best conditions of working. In Chap. IX. we shall see that the efficiency of such a transmission line is highest when $P_2^2 g_l = I_2^2 r_a$, that is, when the no-load losses with normal receiver pressure equal the short-

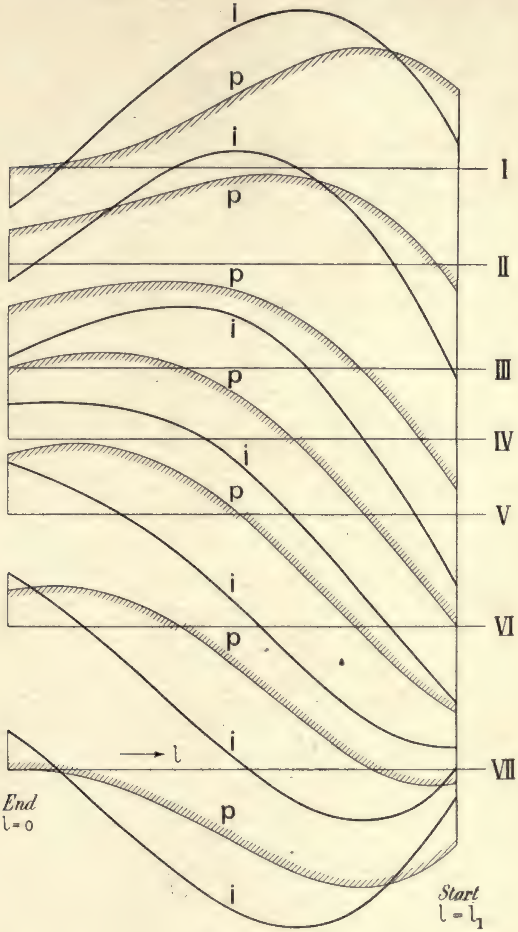


FIG. 132.

circuit losses with normal receiver current, and this is the case when, as above, the power factor throughout the line is unity.

In Fig. 131 the values of I and P are set off both in magnitude and direction along the polar co-ordinates for a power transmission line with abnormal conditions. The plotted points correspond to $\mu l = 15^\circ$. The pressure P_2 at the end of the line coincides with the ordinate axis.

The vector I_2 lags ϕ_2 behind P_2. By projecting the radii-vectores of these two curves on to the rotating time-line, we get the momentary values of the pressures and currents at every point along the line. These instantaneous values are represented in Fig. 132 as functions of the length of the line for six different instants of time taken $\frac{1}{12}$ of a complete period from one another.

From these curves it is clearly seen that the pressure and current vary after a sine law along the line, and at the same time we see how the pressure and current waves progress along the line.

44. Transmission of Energy over Quarter- and Half-wave Lines. We have just seen that very long lines with negligibly small line losses have certain peculiarities. The current and pressure waves are stationary when the receiver terminals are either short-circuited or open. We will now see how these lines behave when line losses are present.

Quarter-wave Transmission Line. We will first consider a line whose length is a quarter of the wave-length of the current and pressure waves. Such a line we can call a quarter-wave transmission line.

Then
$$\mu l_1 = \frac{\pi}{2},$$
whilst λ is not zero.

It then follows
$$\epsilon^{\pm(\lambda l_1 - j\mu l_1)} = \epsilon^{\pm\lambda l_1}(\cos \mu l_1 \mp j \sin \mu l_1) = j\epsilon^{\pm\lambda l_1},$$
and the constant $\underset{.}{C}$ of the line will be
$$\underset{.}{C} = \frac{\epsilon^{(\lambda - j\mu)l_1} + \epsilon^{-(\lambda - j\mu)l_1}}{2} = j\frac{-\epsilon^{\lambda l_1} + \epsilon^{-\lambda l_1}}{2} = -\sin(j\lambda l_1),$$

whilst
$$\underset{.}{C}z_K = \sqrt{\frac{z_d}{y_l}}\,\frac{\epsilon^{(\lambda - j\mu)l_1} - \epsilon^{-(\lambda - j\mu)l_1}}{2}$$
$$= -\sqrt{\frac{z_d}{y_l}}\,j\,\frac{\epsilon^{\lambda l_1} + \epsilon^{-\lambda l_1}}{2}$$
$$= -\sqrt{\frac{z_d}{y_l}}\,j\cos(j\lambda l_1)$$

and
$$\underset{.}{C}y_0 = \sqrt{\frac{y_l}{z_d}}\,\frac{\epsilon^{(\lambda - j\mu)l_1} - \epsilon^{-(\lambda - j\mu)l_1}}{2} = \sqrt{\frac{y_l}{z_d}}\,j\cos(j\lambda l_1).$$

The pressure and current at the supply terminals will then be, from equations 81 and 82,
$$\underset{.}{P}_1 = \underset{.}{C}\underset{.}{P}_2 + \underset{.}{C}z_K\underset{.}{I}_2 = -\underset{.}{P}_2\sin(j\lambda l_1) - j\underset{.}{I}_2\sqrt{\frac{z_d}{y_l}}\cos(j\lambda l_1)$$
and
$$\underset{.}{I}_1 = \underset{.}{C}\underset{.}{I}_2 + \underset{.}{C}y_0\underset{.}{P}_2 = -\underset{.}{I}_2\sin(j\lambda l_1) - j\underset{.}{P}_2\sqrt{\frac{y_l}{z_d}}\cos(j\lambda l_1).$$

From this it is easy to see the influence of the load in the receiver circuit and that of the line losses on the load in the supply circuit. If we put, for example, $\lambda = 0$; by making the line losses negligible, then

$$\dot{P}_1 = -j\dot{I}_2\sqrt{\frac{z_d}{y_l}} = -j\dot{I}_2\sqrt{\frac{L_d}{C_l}},$$

$$\dot{I}_1 = -j\dot{P}_2\sqrt{\frac{y_l}{z_d}} = -j\dot{P}_2\sqrt{\frac{C_l}{L_d}}$$

and
$$\frac{\dot{P}_1}{\dot{I}_1} = \frac{\dot{I}_2}{\dot{P}_2}\frac{L_d}{C_l}.$$

Such a line therefore behaves like Boucherot's condenser transformer, converting a constant pressure into a constant current, and conversely. Thus if we wish to increase the current in the receiver circuit, the supply pressure must be raised; whilst if the receiver pressure is to be raised, the current in the supply circuit must be increased accordingly. Since no losses occur in the line, the supplied energy equals the received energy, and since, further, $\dot{P}_1\dot{I}_1 = -\dot{P}_2\dot{I}_2$, the phase displacement in the supply station equals that in the receiver station.

Examining the effect of the line losses on the load in the supply station, these occur in the first two terms of the expressions for $\dot{P}_1$ and $\dot{I}_1$, viz. in

$$\dot{P}_2 \sin(j\lambda l_1) \quad \text{and} \quad \dot{I}_2 \sin(j\lambda l_1).$$

Since λl_1 is comparatively small, the sine can be replaced by the angle, and we get for the two loss components, $\dot{P}_2 j\lambda l_1$ and $\dot{I}_2 j\lambda l_1$. The line losses are thus directly proportional to λl_1, a quantity which can be calculated as follows:

$$\lambda l_1 = \frac{\lambda l_1}{\mu l_1}\mu l_1 = \frac{\lambda}{\mu}\frac{\pi}{2}.$$

Since
$$2\lambda\mu = g_l x_d + b_l r_d$$

and
$$\mu^2 \simeq b_l x_d,$$

then
$$\frac{2\lambda}{\mu} \simeq \frac{r_d}{x_d} + \frac{g_l}{b_l}.$$

Thus
$$\lambda l_1 = \frac{\pi}{4}\left(\frac{r_d}{x_d} + \frac{g_l}{b_l}\right),$$

where b_l is to be taken as positive. Since we also put $\cos(j\lambda l_1) = 1$,

then
$$\dot{P}_1 = j\left[\dot{P}_2\frac{\pi}{4}\left(\frac{r_d}{x_d} + \frac{g_l}{b_l}\right) + \dot{I}_2\sqrt{\frac{z_d}{y_l}}\right]$$

and
$$\dot{I}_1 = j\left[\dot{I}_2\frac{\pi}{4}\left(\frac{r_d}{x_d} + \frac{g_l}{b_l}\right) + \dot{P}_2\sqrt{\frac{z_d}{y_l}}\right].$$

If the load in the receiver circuit is non-inductive, which is best in such long transmission lines, and we take $\frac{r_d}{x_d}=\frac{g_l}{b_l}$, then $\sqrt{\frac{z_d}{y_l}}=\sqrt{\frac{L_d}{C_l}}$, and the phase displacement at the supply station will thus be zero also, i.e. $\cos\phi=1$ in both the supply and receiver stations.

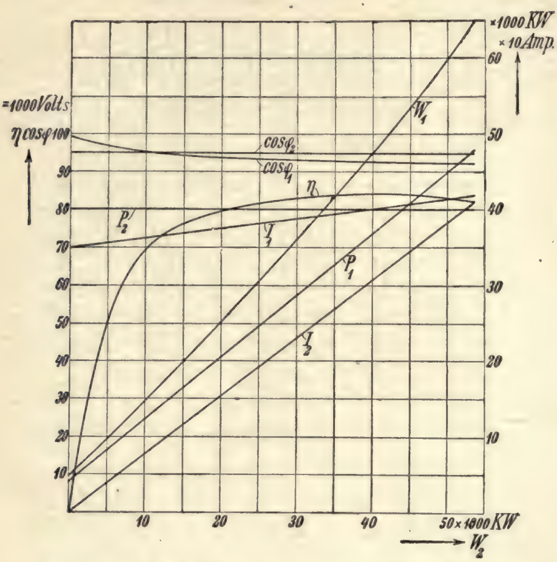


FIG. 133.—Load Curves of a Quarter-wave Transmission Line.

In Fig. 133, the load curves of the supply and receiver stations of a quarter-wave transmission line are shewn for constant receiver pressure, and $\cos\phi_2=0{\cdot}95$.

The supply pressure increases rapidly along a straight line with increasing load, whilst the supply current only increases slightly, but also along a straight line. The increase in current serves to cover the line losses as they increase with the load. This method of transmission has recently been fully treated by *Steinmetz*, who illustrated its practical value for very long lines. At 50 cycles, the length of the transmission by means of the quarter-wave line is about

$$\frac{300000}{4\times 50}=1500 \text{ km}.$$

Half-wave Transmission Line. Here the length of the line equals half a wave-length, i.e.

$$\mu l_1=\pi,$$

whilst λ is not zero.

It then follows that

$$\epsilon^{\pm(\lambda l_1-j\mu l_1)}=\epsilon^{\pm\lambda l_1}(\cos\mu l_1\mp j\sin\mu l_1)=-\epsilon^{\pm\lambda l_1}.$$

The line constant is then

$$C = \frac{\epsilon^{(\lambda - j\mu)l_1} + \epsilon^{-(\lambda - j\mu)l_1}}{2} = -\frac{\epsilon^{\lambda l_1} + \epsilon^{-\lambda l_1}}{2} = -\cos(j\lambda l_1),$$

whilst

$$Cz_K = \sqrt{\frac{z_d}{y_l}}\,\frac{\epsilon^{(\lambda - j\mu)l_1} - \epsilon^{-(\lambda - j\mu)l_1}}{2}$$

$$= -\sqrt{\frac{z_d}{y_l}}\,\frac{\epsilon^{\lambda l_1} - \epsilon^{-\lambda l_1}}{2}$$

$$= j\sqrt{\frac{z_d}{y_l}}\sin(j\lambda l_1)$$

and

$$Cy_0 = j\sqrt{\frac{y_l}{z_d}}\sin(j\lambda l_1).$$

The pressure and current at the supply terminals are, accordingly,

$$P_1 = CP_2 + Cz_K I_2$$

$$= -P_2\cos(j\lambda l_1) + jI_2\sqrt{\frac{z_d}{y_l}}\sin(j\lambda l_1),$$

and

$$I_1 = CI_2 + Cy_0 P_2$$

$$= -I_2\cos(j\lambda l_1) + jP_2\sqrt{\frac{y_l}{z_d}}\sin(j\lambda l_1).$$

If the line losses were negligible, i.e. $\lambda = 0$, then we should have

$$P_1 = -P_2 \quad \text{and} \quad I_1 = -I_2.$$

Consequently, the line behaves under steady working conditions like a line possessing no resistance, inductance or capacity. Taking the line losses into account and making the same assumptions as above,

$$\cos(j\lambda l_1) = 1$$

and

$$\sin(j\lambda l_1) \simeq j\lambda l_1 = j\frac{\pi}{2}\left(\frac{r_d}{x_d} + \frac{g_l}{b_l}\right),$$

then

$$P_1 = -P_2 - I_2\sqrt{\frac{z_d}{y_l}}\frac{\pi}{2}\left(\frac{r_d}{x_d} + \frac{g_l}{b_l}\right)$$

and

$$I_1 = -I_2 - P_2\sqrt{\frac{y_l}{z_d}}\frac{\pi}{2}\left(\frac{r_d}{x_d} + \frac{g_l}{b_l}\right),$$

which are quite obvious. At 50 cycles, the length of a half-wave line is about $\frac{300000}{2 \times 50} = 3000$ km. The current and pressure vectors in Fig. 131 correspond to a half-wave transmission line where the electrostatic energy predominates. The current in the receiver circuit lags, whilst that in the supply circuit leads. The losses in this line are chosen unduly large, as clearly seen from the relation between $I_2 P_2$ and $I_1 P_1$.

45. Equivalent Circuit of a Power Transmission Scheme containing Uniformly Distributed Capacity in the Line.

First Form of Equivalent Circuit. In Section 40, the total capacity in the line was replaced by a capacity concentrated at the centre of gravity. We shall now shew that this is allowable in the case of a uniform conductor, provided the capacity and the impedances of the equivalent circuit are properly chosen.

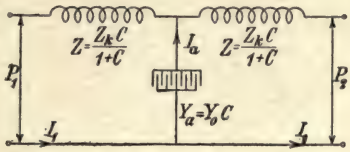


FIG. 134.

Consider the circuit in Fig. 134—we have the following equations:

$$I_a = y_0 C\left(P_2 + I_2 \frac{z_K C}{1+C}\right)$$

and

$$I_1 = I_a + I_2 = C y_0 P_2 + I_2 \frac{y_0 z_K C^2}{1+C} + I_2,$$

or

$$I_1 = C(I_2 + P_2 y_0).$$

Similarly for the supply pressure:

$$P_1 = P_2 + I_2 \frac{z_K C}{1+C} + I_1 \frac{z_K C}{1+C}$$

$$= P_2 + I_2 \frac{z_K C}{1+C} + P_2 \frac{z_K y_0 C^2}{1+C} + I_2 \frac{z_K C^2}{1+C},$$

or, putting $z_K y_0 = 1 - \frac{1}{C^2}$,

$$P_1 = C(P_2 + I_2 z_K).$$

Thus we get the same equations for the circuit in Fig. 134 as for the uniform power transmission line with uniformly distributed capacity (cp. equations (81) and (82)), and the effects of the latter can nearly all be simply deduced from the equivalent circuit.

The admittance y_a of the equivalent circuit is

$$y_a = C y_0 = \tfrac{1}{2}\sqrt{\frac{y_l}{z_d}}\left(\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}\right) \quad \text{.................(85)}$$

and the impedance z is

$$z = \frac{C z_K}{1+C} = \frac{\sqrt{\frac{z_d}{y_l}}\left(\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}\right)}{2 + \epsilon^{\sqrt{y_l z_d}} + \epsilon^{-\sqrt{y_l z_d}}}. \quad \text{.................(86)}$$

Second Form of Equivalent Circuit. The equivalent circuit just deduced has the form of a three-phase star, as we shall see in Chap. XVII. It is shewn there that every star-system can be reduced to an equivalent mesh-system. Such a mesh-system is shewn in Fig. 135. Every

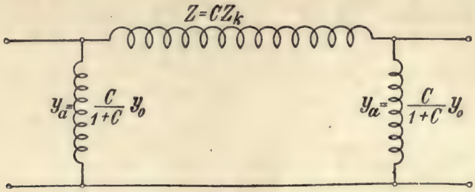


FIG. 135.

uniform transmission line, therefore, can be replaced by a circuit like that in Fig. 135. The three branches of this circuit have the constants

$$z = Cz_K,$$
$$y_a = y_a = \frac{C}{1+C} y_0.$$

To prove this we derive the following equations for the equivalent circuit:

$$I_1 = I_2 + P_2 y_a + [(I_2 + P_2 y_a) z + P_2] y_a$$
$$= I_2(1 + z y_a) + P_2 y_a (2 + z y_a).$$

Here $$1 + z y_a = 1 + z_K y_0 \frac{C^2}{1+C}$$ (see eq. 80, p. 131.)

Hence we have

$$I_1 = C I_2 + (1 + C) P_2 y_a = C(I_2 + P_2 y_0).$$

Similarly, for the supply pressure:

$$P_1 = P_2 + (I_2 + P_2 y_a) z$$
$$= C P_2 + I_2 z = C(P_2 + I_2 z_K).$$

We thus get the same equations (81 and 82) for this circuit as those deduced for the transmission line. The following formulae serve to determine the constants of this equivalent circuit:

$$\left.\begin{aligned} z &= C z_K = \tfrac{1}{2}\sqrt{\frac{z_d}{y_l}}\left(\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}\right), \\ y_a &= \frac{C}{1+C} y_0 = \frac{\sqrt{\dfrac{y_l}{z_d}}\left(\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}\right)}{2 + \epsilon^{\sqrt{y_l z_d}} + \epsilon^{-\sqrt{y_l z_d}}}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(87)$$

where, as before from eq. 80,

$$C^2 = \frac{1}{1 - z_K y_0}.$$

From this equivalent circuit, we see that every uniform transmission line with inductance, resistance and capacity behaves like a line possessing only inductance and resistance, provided that two equal capacities are assumed to be in parallel with the load and the generators respectively.

If the current diagram of the load for such a line is given for a definite P_2, we must first add the constant current $P_2 y_a$ to the load current and invert the current curve thus obtained by the rules already given, and then add the impedance z. By a further inversion and adding the current $P_1 y_a$, the diagram for the current at the supply terminals is obtained for a constant supply pressure P_1. One advantage of this equivalent circuit over that deduced above is due to the fact that only two inversions are required for the current diagram, whilst four are necessary in the other. On the other hand, if we have to find the pressure diagram of the generators from the load pressure diagram, the first form is preferable.

46. Uniformly Distributed Capacity in Transformers and Alternating-current Machines. Not only in high-tension transmission lines, but also in high-pressure windings of electromagnetic apparatus, distributed capacity is met with. Under ordinary working conditions, however, this is chiefly confined to transformers for very high pressures. In machines, distributed capacity only becomes dangerous on switching in and when sudden load variations occur. Since, for the moment, we are only dealing with steady conditions, we shall only deal here with high-pressure transformers.

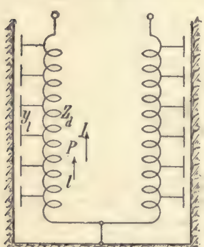


FIG. 136.

(*a*) Under steady working conditions, individual coils in the transformer windings assume potentials considerably higher than that of the surrounding iron, which is usually connected to earth; as a consequence of this, condenser action takes place between the high tension coils and the earthed masses of iron. The insulation of the winding and the oil or air act as dielectrics. Assuming that the middle point of the high-tension winding of the transformer is earthed, we get the following equivalent scheme for the distributed capacity in the transformer (Fig. 136), when the capacity between the several parts of the winding is neglected. As in the high-tension lines, we denote the impedance per unit length of the winding by $r_a - jx_a = \frac{z_a}{l_1}$ and the admittance per unit length of the winding by

$$g_l - jb_l = \frac{y_l}{l_1}.$$

In addition, we have here an E.M.F. E_a induced per unit length of the winding. Denote the pressure at distance l from the earthed

point by P, and the current at the same point by I; then in the element dl, the current increase will be

$$dI = \mp P\frac{y_l}{l_1}dl,$$

and the increase of pressure

$$dP = \left(E_d \mp I\frac{z_d}{l_1}\right)dl,$$

where the first sign refers to the secondary winding of a transformer and the armature windings of a generator, whilst the other sign refers to the primary of a transformer and the stator winding of a motor. By differentiating the last equation and eliminating I, we get

$$\frac{d^2P}{dl^2} = P\frac{z_d y_l}{l_1^2},$$

that is, the same differential equation as for transmission lines.

The E.M.F. E_d, induced in the winding from outside, has no effect therefore on the form of the differential equation, but makes its appearance in the limiting conditions. Similarly, by differentiating the first equation, we have

$$\frac{d^2I}{dl^2} = I\frac{z_d y_l}{l_1} \mp E_d\frac{y_l}{l_1},$$

which differs from the current equation for transmission lines.

It is best therefore to start from the pressure equation. The solution of this is (see p. 130)

$$P = A\epsilon^{\sqrt{y_l z_d}\frac{l}{l_1}} + B\epsilon^{-\sqrt{y_l z_d}\frac{l}{l_1}}.$$

The limits are

$$l = 0, \quad P = 0,$$

$$l = l_1, \quad P = P_1,$$

and give

$$0 = A + B,$$

$$P_1 = A\left(\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}\right);$$

hence

$$A = \frac{P_1}{\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}}$$

and

$$P = P_1\frac{\epsilon^{\sqrt{y_l z_d}\frac{l}{l_1}} - \epsilon^{-\sqrt{y_l z_d}\frac{l}{l_1}}}{\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}}.$$

Inserting this value of P in the equation

$$I\frac{z_d}{l_1} = \mp\left(\frac{dP}{dl} - E_d\right),$$

we get the following expression for I:

$$I = \mp P_1 \sqrt{\frac{y_l}{z_d}} \frac{\epsilon^{\sqrt{y_l z_d}\frac{l}{l_1}} + \epsilon^{-\sqrt{y_l z_d}\frac{l}{l_1}}}{\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}} \pm \frac{E_a l_1}{z_d}.$$

If no point of the transformer winding were connected to earth, then any point in it might assume earth-potential, and from this both l and P would then be calculated.

In this connection it must also be remembered that the several parts of the low-tension winding in a high-tension transformer are statically charged, since they act as the second plate of a condenser, the first plate of which is formed by the high-tension winding. These charges, however, neutralise one another when the potential of the high-pressure winding is symmetrically distributed with respect to the neutral point. If the high-tension winding is not earthed and its potential is not symmetrical with respect to the neutral point, the electrostatic charges in the secondary winding do not neutralise one another, and the whole secondary winding can assume a fairly high static pressure with respect to earth, when the secondary winding is well insulated from earth. When the low-tension windings of high-tension transformers are not earthed, it is still advisable to earth their neutral point through a pressure safety device, such as a water-spray, etc.

Assume further that the winding is the secondary of a transformer on no-load; then the current at the terminals is

$$I_1 = 0,$$

i.e.

$$0 = P_{10} \sqrt{\frac{y_l}{z_d}} \frac{\epsilon^{\sqrt{y_l z_d}} + \epsilon^{-\sqrt{y_l z_d}}}{\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}} + \frac{E_a l_1}{z_d}.$$

Thus the pressure at the secondary terminals of a transformer on no-load, which possesses distributed capacity, is

$$P_{10} = \frac{E_a l_1}{\sqrt{y_l z_d}} \frac{\epsilon^{\sqrt{y_l z_d}} + \epsilon^{-\sqrt{y_l z_d}}}{\epsilon^{\sqrt{y_l z_d}} - \epsilon^{-\sqrt{y_l z_d}}}.$$

Let us consider the simple case when the resistance r_d and the conductance g_l of the winding are negligible; then

$$\sqrt{y_l z_d} = \sqrt{(-j\omega C)(-j\omega L)} = j\omega\sqrt{LC}$$

and

$$P_{10} = \frac{E_a l_1}{\omega\sqrt{LC}} \tan \omega\sqrt{LC}.$$

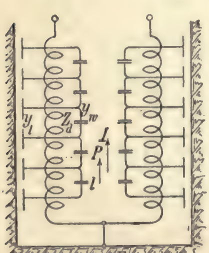


FIG. 137.

Since $\tan \omega\sqrt{LC}$ is greater than $\omega\sqrt{LC}$ for values of $\omega\sqrt{LC}$ less than $\frac{\pi}{2}$, the pressure at the terminals will always be greater than the E.M.F. induced in the winding.

(*b*) We now proceed a step further and consider the capacities which exist between the several turns and coils; they

act like shunted condensers to the winding elements, as depicted in Fig. 137. Let the condensers be denoted by the admittances $g_w - jb_w = \frac{y_w}{l_1}$ per unit length; then the increase of current in a winding element will be

$$dI = \mp\left(P\frac{y_l}{l_1} - \frac{y_w}{l_1}\frac{d^2P}{dl^2}\right)dl$$

and the increase of pressure

$$dP = \left(E_d \mp I\frac{z_d}{l_1}\right)dl.$$

Thus we get the two differential equations for pressure and current,

$$\left(1 + \frac{y_w z_d}{l_1^2}\right)\frac{d^2P}{dl^2} = P\frac{z_d y_l}{l_1^2}$$

or

$$\frac{d^2P}{dl^2} = P\frac{z_d}{l_1}\frac{\frac{y_l}{l_1}}{1 + \frac{y_w z_d}{l_1^2}},$$

and similarly,

$$\frac{d^2I}{dl^2} = I\frac{z_d}{l_1}\frac{\frac{y_l}{l_1}}{1 + \frac{y_w z_d}{l_1^2}} \mp E_d\frac{\frac{y_l}{l_1}}{1 + \frac{y_w z_d}{l_1^2}}.$$

Since these two differential equations only differ from the former by the factor

$$\frac{y_l'}{l_1} = \frac{\frac{y_l}{l_1}}{1 + \frac{y_w z_d}{l_1^2}},$$

instead of $\frac{y_l}{l_1}$, all the formulae deduced above can also be used for this case, if we substitute y_l' for y_l.

Thus the capacity C_w between turns and coils acts like an increase in the capacity C with respect to earth. In all the formulae, instead of y_l we have

$$y_l' = \frac{y_l}{1 + \frac{y_w}{l_1}\frac{z_d}{l_1}} = \frac{y_l}{1 + (g_w - jb_w)(r_d - jx_d)}.$$

For the case when g_w and r_d are very small,

$$y_l' = \frac{y_l}{1 - x_d b_w} = -j\frac{\omega C}{1 - \omega^2 L C_w}$$

and the secondary pressure of the transformer on no-load

$$P_{10}=\frac{E_d l_1\sqrt{1-\omega^2 LC_w}}{\omega\sqrt{LC}}\tan\frac{\omega\sqrt{LC}}{\sqrt{1-\omega^2 LC}}.$$

So long as $\omega^2 LC_w<1$, this expression is of the same nature as that found without considering the capacity between the conductors. In transformers the capacity C_w between the turns is usually much larger than the capacity with respect to earth, although in high-pressure machines C can assume high values compared with C_w.

47. Distributed Capacity in Lightning-protecting Apparatus. The multi-gap lightning-arrester of the General Electric Co. of Schenectady—as shewn in Fig. 138—consists of one or more series of metallic cylinders or rollers insulated from one another, the first of which is connected to the line to be protected and the last to earth, either directly or through a resistance When the line is charged to a high potential by atmospheric electricity, the rollers, which can be regarded as the elements of several condensers in series, all become charged.

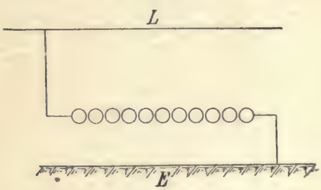


FIG. 138.

If now the pressure between two cylinders becomes greater than the break-down pressure for the air-gap between, a spark will jump across and then across the other cylinders, whereby the line is discharged to a lower potential. If the potential of the line and also the charge of the first cylinder be steady and uniformly directed, then all the cylinders take up the same steady charge, and the pressure between the line and earth distributes itself uniformly over all the gaps, so that the potential across all the rollers can be represented by the dotted straight line I in Fig. 139. Since, however, the metallic cylinders possess not only mutual capacity, but also, with respect to earth, all the cylinders do not take up the same charge, but the charge on the cylinders decreases towards earth, instead of the dotted straight line we get the full-line potential curve II. If, in addition to the capacities, the conductance from cylinder to cylinder and from cylinders to earth is considered, we get Fig. 140 as shewing completely the circuit of the lightning arrester.

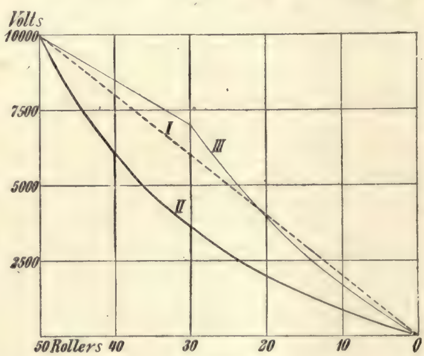


FIG. 139.

(*a*) As this circuit is similar in character to the transmission line

in Fig. 128, the equations deduced for the latter may also be used for the mathematical investigation of the roller lightning arrester. Naturally the differential equations for the transmission line are deduced for an alternating potential P_1 in the line and not for

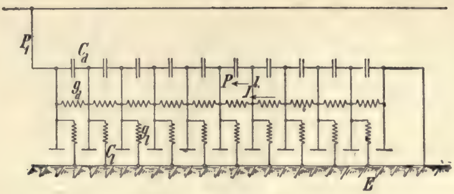


FIG. 140.

a steady one. Since, however, an alternating potential occurs as frequently as a steady, and further, since the differential equations for the former can be suitably simplified for the case of a steady potential, we shall start with the general differential equations for an alternating potential.

These are
$$\frac{d^2P}{dl^2} = P\frac{z_d y_l}{l_1^2}$$
and
$$\frac{d^2I}{dl^2} = I\frac{z_d y_l}{l_1^2}.$$
Here
$$y_l = (g_l - jb_l)\,l_1 = (g_l - j\omega C_l)\,l_1$$
and
$$\frac{z_d}{l_1} = \frac{l_1}{y_d} = \frac{1}{(g_d - jb_d)} = \frac{1}{(g_d - j\omega C_d)},$$
where all constants refer to one roller and l therefore is expressed as the number of rollers by which the respective point is away from the roller connected to earth. Usually C_d is of the order 10^{-11} farad, whilst C_l is about $\frac{1}{100}$ of C_d. As seen from these expressions, we have neglected the small inductance L of the rollers, which is only of the order $2 \,.\, 10^{-8}$ henry, and consequently only begins to have an influence on the pressure conditions when the frequency $c = \frac{\omega}{2\pi}$ approaches the order $\frac{1}{2\pi\sqrt{LC_d}}$, i.e. about 35 millions.

We can therefore neglect self-induction entirely, and thus obtain the following differential equations
$$\frac{d^2P}{dl^2} = P\frac{g_l - j\omega C_l}{g_d - j\omega C_d}$$
and
$$\frac{d^2I}{dl^2} = I\frac{g_l - j\omega C_l}{g_d - j\omega C_d}.$$

The solutions of these equations are

$$\dot{P} = \dot{A}\epsilon^{\sqrt{\frac{y_l}{y_d}}l} + \dot{B}\epsilon^{-\sqrt{\frac{y_l}{y_d}}l},$$

$$\dot{I} = \frac{\sqrt{y_l y_d}}{l_1}\left(\dot{A}\epsilon^{\sqrt{\frac{y}{y_d}}l} - \dot{B}\epsilon^{-\sqrt{\frac{y_l}{y_d}}l}\right).$$

Inserting the limits

$$l = 0, \quad \dot{P} = 0,$$

$$l = l_1, \quad \dot{P} = \dot{P}_1,$$

we get
$$0 = \dot{A} + \dot{B}$$

and
$$\dot{P}_1 = \dot{A}\epsilon^{\sqrt{\frac{y_l}{y_d}}l_1} + \dot{B}\epsilon^{-\sqrt{\frac{y_l}{y_d}}l_1};$$

hence
$$\dot{P} = \dot{P}_1 \frac{\epsilon^{\sqrt{\frac{y_l}{y_d}}l} - \epsilon^{-\sqrt{\frac{y_l}{y_d}}l}}{\epsilon^{\sqrt{\frac{y_l}{y_d}}l_1} - \epsilon^{-\sqrt{\frac{y_l}{y_d}}l_1}}$$

and
$$\dot{I} = \dot{P}_1 \frac{\sqrt{y_l y_d}}{l_1} \frac{\epsilon^{\sqrt{\frac{y_l}{y_d}}l} + \epsilon^{-\sqrt{\frac{y_l}{y_d}}l}}{\epsilon^{\sqrt{\frac{y_l}{y_d}}l_1} - \epsilon^{-\sqrt{\frac{y_l}{y_d}}l_1}}.$$

(*b*) Consider first the simple case where the conductance g_l bears the same relation to the capacity C_l as the conductance g_d to the capacity C_d; then the ratio

$$\frac{y_l}{y_d} = \frac{C_l}{C_d} = \frac{g_l}{g_d}$$

is a positive real number, and

$$\dot{P} = \dot{P}_1 \frac{\epsilon^{\sqrt{\frac{C_l}{C_d}}l} - \epsilon^{-\sqrt{\frac{C_l}{C_d}}l}}{\epsilon^{\sqrt{\frac{C_l}{C_d}}l_1} - \epsilon^{-\sqrt{\frac{C_l}{C_d}}l_1}}.$$

The pressure, therefore, follows a curve independent of the frequency, which also holds for a continuous (steady) pressure.

In Fig. 139 the potential curve II. is calculated for the case of a roller lightning arrester, where $C_d = 400C$ and $l_1 = 50$ cylinders. Then

$$\sqrt{\frac{C}{C_d}}l_1 = \frac{50}{\sqrt{400}} = 2{\cdot}5.$$

With the assumption $\frac{g_l}{C_l}=\frac{g_d}{C_d}$ the current will be

$$I=P_1(g_d-j\omega C_d)\sqrt{\frac{C_l}{C_d}}\,\frac{\epsilon^{\sqrt{\frac{C_l}{C_d}}l}+\epsilon^{-\sqrt{\frac{C_l}{C_d}}l}}{\epsilon^{\sqrt{\frac{C_l}{C_d}}l_1}-\epsilon^{-\sqrt{\frac{C_l}{C_d}}l_1}}.$$

It increases with the frequency, i.e. with ω.

For a steady (continuous) pressure, $\omega=0$, and the current is a minimum,

$$I=P_1\sqrt{g_lg_d}\,\frac{\epsilon^{\sqrt{\frac{g_l}{g_d}}l}+\epsilon^{-\sqrt{\frac{g_l}{g_d}}l}}{\epsilon^{\sqrt{\frac{g_l}{g_d}}l_1}-\epsilon^{-\sqrt{\frac{g_l}{g_d}}l_1}}.$$

The pressure between two cylinders is, in general,

$$\Delta P=-\frac{dP}{dl}=-P_1\sqrt{\frac{C_l}{C_d}}\,\frac{\epsilon^{\sqrt{\frac{C_l}{C_d}}l}+\epsilon^{-\sqrt{\frac{C_l}{C_d}}l}}{\epsilon^{\sqrt{\frac{C_l}{C_d}}l_1}-\epsilon^{-\sqrt{\frac{C_l}{C_d}}l_1}}.$$

This is greatest between the first two cylinders, that is between the two nearest the line, and for the example in question approximately equal to $P_1\sqrt{\frac{C_l}{C_d}}=\frac{P_1}{20}$. If this pressure exceeds the break-down pressure, a spark passes between the first two rollers, and so on along the whole series, for when the pressure breaks down across the first two, the pressure between the second and third is increased, and so on. In lightning arresters which consist of many cylinders, it is often observed that the sparks vanish before all the rollers have been passed. This is due to the fact that the charge which the spark carries with it becomes less from roller to roller, the decrease being caused partly by the conductance to earth and partly by the capacity of the rollers with respect to earth.

From the foregoing, it is seen that—contrary to a popular view—the distribution of the potential over the gaps of roller lightning arresters, not only with rapidly alternating potentials, but also with steady potentials, is quite unsymmetrical, and consequently the potential curve in both cases deviates considerably from a straight line.

(*c*) We will now return to the general case where there is no definite relation between the capacity and conductance. Here the potential does not always follow the difference of two exponential curves, but under certain conditions the difference of two sine curves, whose amplitudes decrease according to an exponential curve. In this case,

$$\epsilon^{\sqrt{\frac{y_l}{y_d}}l}=\epsilon^{(\lambda-j\mu)l}=\epsilon^{\lambda l}(\cos\mu l-j\sin\mu l),$$

where
$$\lambda = \sqrt{\frac{1}{2}\left(\sqrt{\frac{g_l^2 + b_l^2}{g_a^2 + b_a^2}} + \frac{g_l g_a + b_l b_a}{g_a^2 + b_a^2}\right)}$$

and
$$\mu = \sqrt{\frac{1}{2}\left(\sqrt{\frac{g_l^2 + b_l^2}{g_a^2 + b_a^2}} - \frac{g_l g_a + b_l b_a}{g_a^2 + b_a^2}\right)}.$$

There is, however, a case when the potential curve follows nearly a straight line, namely when y_l is very small compared with y_a; then we put $\frac{d^2P}{dl^2} = 0$, and therefore $P = P_1 \frac{l}{l_1}$.

This occurs when either the conductance and capacity to earth are very small, or when the conductance from roller to roller is very large. This latter is the case when sparks pass between the cylinders, for then the resistance of the air-gaps becomes a minimum, due to ionisation of the air. Consequently, across the rollers where small sparks pass, the potential curve follows a straight line. It ceases to be a straight line, however, where the sparks disappear—from this point the curve follows the general equation. This phenomenon was first noticed by Rushmore and Dubois,* and is represented in Fig. 139 by curve III.

Usually the conductance g_l to earth in relation to the capacity C_l is much smaller than g_a to C_a. Consequently, λ, and with it the drop of potential ΔP between the first cylinders, increases with the frequency. This explains why a potential at a high frequency discharges itself across a roller lightning arrester more easily than the same potential at a lower frequency.

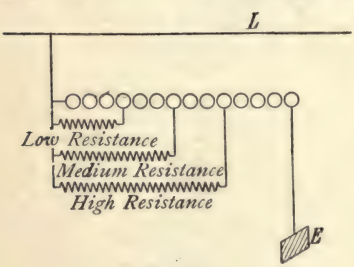


Fig. 141.

According to Rushmore and Dubois, an ideal lightning arrester should behave the same with all potentials independently of the frequency. This is the case with a roller lightning protector when the potential curve follows a straight line. This can be obtained, as is done by the G.E.C., by placing several resistances of different values between the line and the first rollers, as shewn in Fig. 141. By this means, y_a is made much greater than y_l, and the potential curve follows a broken curve, which does not deviate largely from the straight line I (Fig. 139). The same authors have also shewn that the discharge currents of low frequency pass along the largest resistance, whilst the high frequency discharge currents pass along the rollers.

* *Proceedings A.I.E.E.* 1907.

CHAPTER IX.

NO-LOAD AND SHORT-CIRCUIT DIAGRAMS.

48. The No-load and Short-circuit Constants of an Electric Circuit.

(*a*) *Main Equations of the General Circuit.* In the previous chapters we have discussed different kinds of electric circuits: firstly, ordinary conductors containing resistance and inductance; then, electromagnetic apparatus whose circuits are magnetically interlinked; and lastly, circuits containing uniformly distributed capacity. Moreover, we have seen how all these circuits can be replaced by a simple circuit containing an impedance in series with two parallel admittances. This naturally suggests that all circuits are governed by the same laws, which is actually the case, whilst to find these laws we have but to apply the generalised form of Kirchhoff's Laws and the Law of Superposition.

In what follows it will be necessary to show the importance of the no-load and short-circuit constants of the general electric circuit, and this we shall do under the assumption that the Law of Superposition is always applicable; that is to say, the effect produced in a circuit by any cause is independent of any other causes which may be at work at the same time in the circuit. Thus, a pressure produces the same currents in a circuit whether other pressures are present or not, or a current causes the same drop of pressure when other currents are present as when no other currents are present. Further, for the time being, we shall assume the applied pressure $\dot{P}_1$ is a sine wave.

Fig. 142 shews the diagram of a general circuit, which may contain transformers, converters or any other kind of alternating-current apparatus.

Let the supply pressure $\dot{P}_1$ act at the terminals PP of the circuit, whilst at the terminals SS at any part of the circuit suppose we have a

load W_2. We must now study the effect of this load—which depends on the pressure P_2 and the current I_2—on the electric properties of the circuit. The two vectors P_2 and I_2 are at an angle ϕ_2 to one another, so that the power $W_2 = P_2 I_2 \cos\phi_2$.

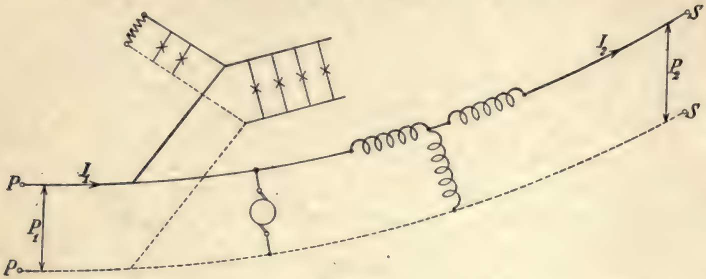


FIG. 142.

First let the whole circuit be unloaded, and the terminals SS open, and let the supply pressure P_{10} be so regulated that the pressure P_2, corresponding to the load W_2, acts at the terminals SS.

When this is the case, the installation is said to be on *no-load* and a current I_0 will be taken by the circuit.

We can then write $$P_{10} = C_1 P_2$$

and $$I_0 = P_{10} y_0,$$

where all the quantities are to be taken as symbolic. $C_1 = C_1 \epsilon^{j\psi_1}$ is a complex number expressing the relation between the two vectors P_{10} and P_2. y_0 is a measure of the electric conductance of the circuit, and can be called its *admittance*. Thus

$$y_0 = g_0 + jb_0 = y_0 \epsilon^{j\phi_0}.$$

I_0 is the no-load current of the circuit, and has the watt component $P_{10} g_0$ and the wattless component $P_{10} b_0$. The losses due to the no-load current I_0 are then $$W_0 = P_{10}^2 g_0.$$

We now *connect* the terminals SS by a conductor, whose resistance is zero, and so regulate the supply pressure P_K that the current I_2 corresponding to the load W_2 flows between the terminals SS. Under these conditions—known as *short-circuit*—a current I_{1K} is taken by the circuit. Symbolically, $$I_{1K} = C_2 I_2$$

and $$P_K = I_{1K} z_K.$$

$C_2 = C_2 \epsilon^{j\psi_2}$ is, like C_1, complex and expresses the relation between the current vectors I_{1K} and I_2. z_K is a measure of the apparent electric resistance of the circuit, and can be called its *impedance*. Thus:

$$z_K = r_K - jx_K = z_K \epsilon^{-j\phi_K}.$$

P_K is the short-circuit pressure of the circuit with respect to the

terminals SS, and has the watt component $I_{1K}r_K$ and the wattless component $I_{1K}x_K$. The losses due to the short-circuit current are then

$$W_K = I_{1K}^2 r_K.$$

Having considered these two extremes—no-load and short-circuit—we now pass on to the normal-load condition. For this purpose, we can start either from no-load, $I_2 = 0$, and gradually increase the current passing between the terminals SS, without altering the pressure P_2; or from the short circuit, and gradually increase the pressure P_2 at the terminals SS, without altering the current I_2. The pressure P_2 at the terminals SS requires at the terminals PP a pressure vector C_1P_2 and a current vector $I_0 = P_{10}y_0$. Similarly the current I_2 in the receiver circuit SS requires at the supply terminals PP a current vector C_2I_2 and a pressure vector $P_K = I_{1K}z_K$. Now, since any two conditions of the circuit are independent of one another, the load condition can be obtained by superposing the no-load and short-circuit conditions. Hence, when the circuit is on load, the pressure at the supply terminals is

$$P_1 = P_{10} + P_K = C_1P_2 + I_{1K}z_K$$

and the current

$$I_1 = I_0 + I_{1K} = P_{10}y_0 + C_2I_2,$$

or, since

$$I_{1K} = C_2I_2 \quad \text{and} \quad P_{10} = C_1P_2,$$

then

$$P_1 = C_1P_2 + C_2I_2z_K \qquad \text{(88)}$$

and

$$I_1 = C_2I_2 + C_1P_2y_0. \qquad \text{(89)}$$

These two equations are the *chief equations* of the circuit, and by means of them the conditions in the circuit for any load W_2—P_2, I_2—can be determined. As is seen from these two equations (88 and 89) every circuit is determined by four constants C_1, C_2, y_0 and z_K.

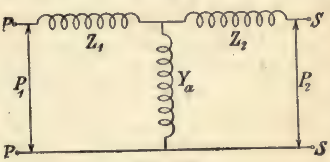


FIG. 143.

It can be shewn, however, that a definite relation exists between these four magnitudes, so that only three constants are sufficient to determine the characteristics of a circuit. Consider the circuit represented in Fig. 143 having the constants z_1, z_2 and y_a—we can then calculate the constants z_K, y_0, C_1 and C_2 for this circuit as follows:

At no-load this circuit takes a current I_0, where:

$$I_0 = \frac{P_{10}}{z_1 + \frac{1}{y_a}} = \frac{P_{10}y_a}{1 + z_1y_a} = P_{10}y_0.$$

The receiver pressure P_2 is

$$P_2 = P_{10} - I_0z_1 = P_{10} - \frac{P_{10}y_az_1}{1 + z_1y_a} = \frac{P_{10}}{1 + z_1y_a} = \frac{P_{10}}{C}.$$

Hence, for this circuit, $$C_1 = 1 + z_1 y_a \quad \text{.........(90)}$$

and $$y_0 = \frac{y_a}{C_1}. \quad \text{.........(91)}$$

At short-circuit the supply current is

$$I_{1K} = I_2 + I_2 z_2 y_a = I_2(1 + z_2 y_a) = I_2 C_2$$

and the short-circuit pressure

$$P_K = I_2 z_2 + I_{1K} z_1 = I_{1K}\left(z_1 + \frac{z_2}{C_2}\right) = I_{1K} z_K,$$

whence $$C_2 = 1 + z_2 y_a \quad \text{.........(92)}$$

and $$z_K = z_1 + \frac{z_2}{C_2}. \quad \text{.........(93)}$$

From equations (90) to (93), we get by multiplying z_K and y_0:

$$z_K y_0 = \left(z_1 + \frac{z_2}{C_2}\right)\frac{y_a}{C_1} = \frac{y_a z_1}{C_1} + \frac{y_a z_2}{C_1 C_2}$$

$$= \frac{C_1 - 1}{C_1} + \frac{C_2 - 1}{C_1 C_2} = 1 - \frac{1}{C_1 C_2}$$

or $$C_1 C_2(1 - y_0 z_K) = 1. \quad \text{.........(94)}$$

We have thus the relation between y_0, z_K, C_1 and C_2. Such a relation might have been predicted, from the fact that the four constants y_0, z_K, C_1 and C_2 can be expressed by the three magnitudes z_1, z_2 and y_a.

(*b*) *Determination of the Constants of a General Circuit by Measurement.* Every circuit can be defined by the four constants C_1, C_2, y_0 and z_K; and since these can be expressed by three independent constants z_1, z_2 and y_a, it is possible to replace every circuit by an equivalent circuit similar to that in Fig. 143. The above relation (eq. 94) holds for this circuit, and can therefore be applied generally.

Hence $$C_1 C_2(1 - y_0 z_K) = 1$$

is the third chief equation of an electric circuit.

From this we see that only three measurements are necessary for determining the constants C_1, C_2, y_0 and z_K.

From equation (94) we get

$$C_1 C_2 = C_1 C_2 \epsilon^{j(\psi_1 + \psi_2)} = \frac{\dfrac{P_1}{z_K}}{\dfrac{P_1}{z_K} - P_1 y_0} = \frac{I_K}{I_K - I_0},$$

where I_0 and I_K denote the no-load and short-circuit currents for one and the same supply pressure P_1.

In Fig. 144 let $\overline{OP_0}$ be the no-load and $\overline{OP_K}$ the short-circuit current; then

$$C_1C_2 = \frac{\overline{OP_K}}{\overline{P_0P_K}} \simeq \frac{I_K}{I_K - I_0\cos(\phi_0 - \phi_K)}. \quad \text{(95)}$$

Also
$$\psi_1 + \psi_2 = \angle\, OP_KP_0,$$

$$\left.\begin{aligned} \tan(\psi_1+\psi_2) &= \frac{I_0\sin(\phi_0-\phi_K)}{I_K - I_0\cos(\phi_0-\phi_K)}, \\ \psi_1+\psi_2 &\simeq \frac{57\cdot 3 I_0 \sin(\phi_0-\phi_K)}{I_K - I_0\cos(\phi_0-\phi_K)}. \end{aligned}\right\} \quad \text{(96)}$$

From this it is seen that the greater the ratio of the no-load current I_0 to the short-circuit current I_K, the greater C_1C_2 will be; the angle $(\psi_1+\psi_2)$, on the contrary, depends chiefly on the difference $(\phi_0 - \phi_K)$ of the phase-displacement angles at no-load and short-circuit. If, in addition to y_0 and z_K, either $C_1 = C_1\epsilon^{j\psi_1}$ or $C_2 = C_2\epsilon^{j\psi_2}$ is measured, the other constants can easily be calculated from formulae (94) and (95).

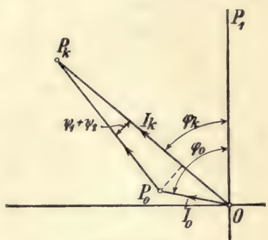


Fig. 144.

In many cases it is impossible, and under any conditions difficult, to measure C_1 and C_2 directly, since they are both complex quantities. The absolute magnitudes of the same can be found from the no-load and short-circuit measurements,

$$C_1 = \frac{P_{10}}{P_2} \quad \text{and} \quad C_2 = \frac{I_{1K}}{I_2}.$$

The angle ψ_1 is the phase-displacement angle between the supply and receiver pressures at no-load, and the angle ψ_2 is the phase-displacement angle between the supply and receiver currents at short-circuit.

These phase-displacement angles are small, and consequently not easy to measure. When the supply and the receiver terminals are a long distance apart, it is even impossible to determine these angles exactly by direct measurement. Hence we shall shew how these two angles can be simply determined by indirect measurement.

From the three chief equations, we get, by simple transpositions,

$$P_1 - I_1 z_K = C_1 P_2 (1 - y_0 z_K) = \frac{P_2}{C_2}$$

or
$$P_2 = C_2(P_1 - I_1 z_K), \quad \text{(88a)}$$

and
$$I_1 - P_1 y_0 = C_2 I_2 (1 - y_0 z_K) = \frac{I_2}{C_1}$$

or
$$I_2 = C_1(I_1 - P_1 y_0). \quad \text{(89a)}$$

The two equations (88a) and (89a) are in every way equivalent and analogous to the chief equations (88) and (89). Whilst, however, by means of equations (88) and (89), it is possible to calculate the supply current I_1 and pressure P_1 for a given receiver load (P_2, I_2), the equations (88a) and (89a) enable us to calculate P_2 and I_2, when the load (P_1, I_1) at the supply station is known.

Let the pressure P_2 act at the receiver terminals with the supply circuit open; then the current I_1 in the supply circuit is zero, and the current at the receiver terminals is

$$I_2 = -C_1 P_1 y_0,$$

whilst the receiver pressure $P_2 = C_2 P_1$. From the receiver terminals, a current $I_{2\,0} = -I_2 = C_1 P_1 y_0 = P_{2\,0}\frac{C_1}{C_2} y_0 = P_{2\,0} y'_0$ will flow into the circuit, and the pressure at the receiver terminals is

$$P_{2\,0} = C_2 P_1.$$

From this, we get

$$\frac{C_1}{C_2} = \frac{y'_0}{y_0},$$

where y'_0 is the admittance of the circuit when the supply terminals are open.

If we now short-circuit the supply terminals ($P_1 = 0$) and apply the pressure at the receiver terminals, the current will be

$$I_{2K} = C_1 I_1$$

and the short-circuit pressure at the receiver terminals is

$$P_{2K} = -P_2 = C_2 I_1 z_K = I_{2K}\frac{C_2 z_K}{C_1} = I_{2K} z'_K.$$

z'_K is the impedance of the circuit when the supply terminals are short-circuited, and we have

$$\frac{C_1}{C_2} = \frac{z_K}{z'_K}.$$

Hence, from the three chief equations we have the following relation:

$$\frac{C_1}{C_2} = \frac{z_K}{z'_K} = \frac{y'_0}{y_0}; \quad \text{.............................(97)}$$

whence we get

$$\Delta\psi = \psi_1 - \psi_2 = \phi_K - \phi'_K = \phi'_0 - \phi_0 \quad \text{....................(98)}$$

or

$$\Delta\psi = \tfrac{1}{2}(\phi_K - \phi'_K + \phi'_0 - \phi_0). \quad \text{......................(98}a\text{)}$$

From formulae (96) and (98) ψ_1 and ψ_2 can now be easily calculated—for we have

$$\psi_1 = \tfrac{1}{2}(\psi_1 + \psi_2 + \Delta\psi)$$

and

$$\psi_2 = \tfrac{1}{2}(\psi_1 + \psi_2 - \Delta\psi).$$

In order to determine $\underset{\cdot}{C}_1$, $\underset{\cdot}{C}_2$, $\underset{\cdot}{y}_0$ and $\underset{\cdot}{z}_K$, it is best to carry out three of the following four measurements. As a check, it is also desirable to carry out all four.

1. With open receiver terminals, measure the supply pressure P_{10}, the no-load current I_0, the no-load losses W_0 and the receiver pressure P_2.

Then, since $y_0 = \frac{I_0}{P_{10}}$ and $\phi_0 = \cos^{-1}\left(\frac{W_0}{I_0 P_{10}}\right)$, we can now find

$$\underset{\cdot}{y}_0 = g_0 + jb_0 = y_0 \epsilon^{j\phi_0}.$$

Further, $$C_1 = \frac{P_{10}}{P_2}.$$

2. With the receiver terminals short-circuited, measure the supply pressure P_K, the short-circuit current I_{1K}, the short-circuit losses W_K and the receiver current I_2. Then, since

$$z_K = \frac{P_K}{I_{1K}} \quad \text{and} \quad \phi_K = \cos^{-1}\left(\frac{W_K}{P_K I_{1K}}\right),$$

we can find $$\underset{\cdot}{z}_K = r_K - jx_K = z_K \epsilon^{-j\phi_K}.$$

Further, $$C_2 = \frac{I_{1K}}{I_2}.$$

3. With the supply terminals open, measure the pressure P_{20}, the current I_{20} and the power W_0' at the receiver terminals, and the pressure P_1 at the supply terminals. From the first three measurements we get

$$\phi_0' = \cos^{-1}\left(\frac{W_0'}{P_{20} I_{20}}\right),$$

and further, $$C_2' = \frac{P_{20}}{P_1}.$$

4. Short-circuit the supply terminals and measure the pressure P_{2K}, the current I_{2K} and the power W_K' at the receiver terminals, and the supply current I_1. We then get

$$\phi_K' = \cos^{-1}\left(\frac{W_K'}{P_{2K} I_{2K}}\right)$$

and $$C_1 = \frac{I_{2K}}{I_1}.$$

From the four phase-displacement angles ϕ_0, ϕ_K, ϕ_0' and ϕ_K' we get the angle $\Delta\psi$ in accordance with formula (98).

It often happens that the pressure acting on the supply circuit is transformed before reaching the receiver circuit. In this case, P_2 and I_2 in the above formulae denote the receiver pressure and current reduced to the supply system. By this means, the ratio of conversion of the transformer is completely removed from all further calculations.

(*c*) *Chief Equations of a Symmetrical Circuit.* Considering again the circuit represented in Fig. 143 with the constants $\underset{.}{z}_1$, $\underset{.}{z}_2$ and $\underset{.}{y}_a$, we see from formulae (90) and (92) that

$$\underset{.}{C}_1 = 1 + \underset{.}{z}_1\underset{.}{y}_a = 1 + \underset{.}{z}\underset{.}{y}_a = \underset{.}{C}$$

and
$$\underset{.}{C}_2 = 1 + \underset{.}{z}_2\underset{.}{y}_a = 1 + \underset{.}{z}\underset{.}{y}_a = \underset{.}{C}$$

are equal when $\underset{.}{z}_1 = \underset{.}{z}_2 = \underset{.}{z}$, i.e. when the circuit is symmetrical about its centre. This holds generally, even for complicated circuits, and we then get

$$\underset{.}{C}^2(1 - \underset{.}{y}_0 z_K) = 1 \quad \text{..........................(94b)}$$

or
$$\underset{.}{C}^2 = \frac{1}{1 - \underset{.}{y}_0 z_K}.$$

If z_K and y_0 are known, $\underset{.}{C} = C\epsilon^{j\psi}$ can be found from the relation between y_0, z_K and $\underset{.}{C}$. The two magnitudes z_K and y_0 can easily be found by measuring the pressure, current and power at no-load and short-circuit.

We have

$$C \simeq \frac{1}{\sqrt{1 - y_0 z_K \cos(\phi_0 - \phi_K)}} = \sqrt{\frac{I_K}{I_K - I_0 \cos(\phi_0 - \phi_K)}} \quad \text{.........(99)}$$

and
$$\tan 2\psi = \frac{y_0 z_K \sin(\phi_0 - \phi_K)}{1 - y_0 z_K \cos(\phi_0 - \phi_K)} = \frac{I_0 \sin(\phi_0 - \phi_K)}{I_K - I_0 \cos(\phi_0 - \phi_K)}, \quad \text{.........(100)}$$

or, measured in degrees,

$$\psi^0 \simeq 28{\cdot}65 \frac{I_0 \sin(\phi_0 - \phi_K)}{I_K - I_0 \cos(\phi_0 - \phi_K)}.$$

For this symmetrical circuit, the chief equations are:

$$\underset{.}{P}_1 = \underset{.}{C}(\underset{.}{P}_2 + \underset{.}{I}_2 z_K), \quad \text{..........................(88b)}$$

$$\underset{.}{I}_1 = \underset{.}{C}(\underset{.}{I}_2 + \underset{.}{P}_2 \underset{.}{y}_0) \quad \text{..........................(89b)}$$

and
$$\underset{.}{C}^2(1 - \underset{.}{y}_0 z_K) = 1. \quad \text{..........................(94b)}$$

These hold for the usual cases met with in practice, for example transformers, induction motors and many power transmission schemes.

We shall now shew how the magnitudes P_K, I_{1K} and ϕ_K obtained from the short-circuit diagram can be used for finding the percentage rise of pressure, and the magnitudes P_{10}, I_0 and ϕ_0 obtained from the no-load diagram, the percentage change of current in a circuit, whilst both can be used for the determination of the change in the phase angle ϕ.

49. Determination of the Pressure Rise in a Circuit by means of the Short-circuit Diagram. If the pressure P_2 at the receiver terminals *SS* is to remain constant from no-load to full-load, W_2, the supply pressure must be varied accordingly. This pressure variation

is best expressed as a percentage of the no-load pressure P_{10}. The change is generally an increase, whence we define

$$\frac{P_1 - P_{10}}{P_{10}} 100 = \epsilon \%$$

as the percentage rise of pressure. To calculate this for a symmetrical circuit with $C_1 = C_2$, we proceed graphically, as in Fig. 145. Set off $\dot{I}_2$

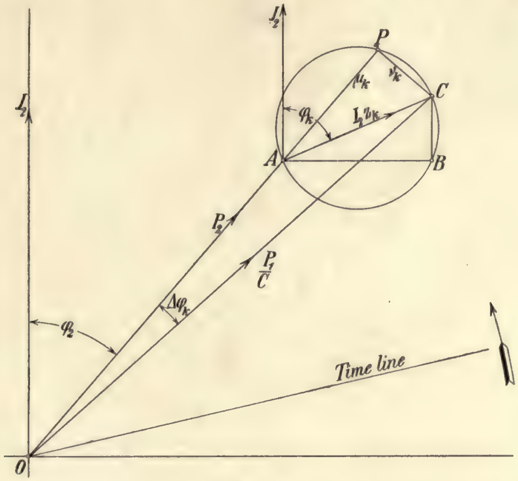


FIG. 145.

along the ordinate axis and $\dot{P}_2 = \overline{OA}$ at angle ϕ_2 to $\dot{I}_2$. Set off vector $\overline{AC} = \dot{I}_2 z_K$ at angle ϕ_K to the ordinate axis, where $\phi_K = \tan^{-1} \frac{x_K}{r_K}$; then

$$\overline{OC} = \dot{P}_2 + \dot{I}_2 z_K = \frac{\dot{P}_1}{\dot{C}} \quad \text{(see eq. 88}b\text{).}$$

Then, since $$\overline{OA} = \dot{P}_2 = \frac{\dot{P}_{10}}{\dot{C}},$$

the percentage pressure rise $\epsilon \%$ can be expressed thus:

$$\epsilon \% = \frac{P_1 - P_{10}}{P_{10}} 100 = \frac{\overline{OC} - \overline{OA}}{\overline{OA}} 100.$$

On $\overline{AC}$ as diameter describe a circle and produce $\overline{OA}$ to cut this circle in P; then $\overline{AB} = \dot{I}_2 x_K$ and $\overline{BC} = I_2 r_K$.

Let $$\overline{AP} = \mu_K \overline{OA} \quad \text{and} \quad \overline{CP} = \nu_K \overline{OA};$$

then, from Fig. 145, we have

$$\epsilon = \frac{\overline{OC} - \overline{OA}}{\overline{OA}} = \sqrt{(1 \pm \mu_K)^2 + \nu_K^2} - 1 = \sqrt{1 \pm 2\mu_K + \mu_K^2 + \nu_K^2} - 1;$$

and working out this root,

$$\epsilon = \frac{\pm 2\mu_K + \mu_K^2 + \nu_K^2}{2} - \frac{4\mu_K^2 \pm 4\mu_K(\mu_K^2 + \nu_K^2) + (\mu_K^2 + \nu_K^2)^2}{8} + \dots$$

$$= \pm\mu_K + \frac{\nu_K^2}{2} \pm \frac{\mu_K(\mu_K^2 + \nu_K^2)}{2} - \dots .$$

When $\mu_K = \nu_K = 0{\cdot}2$, the last term $\mu_K^3 = \frac{8}{1000}$, and is therefore generally negligible.

If we write $\overline{AP} = \frac{\mu_K}{100}\overline{OA}$ and $\overline{CP} = \frac{\nu_K}{100}\overline{OA}$,

where μ_K and ν_K are not to be taken as ratios but as percentages, then the percentage rise of pressure will be

$$\epsilon\,\% = \frac{P_1 - P_{10}}{P_{10}}100 = \pm\mu_K + \frac{\nu_K^2}{200}. \qquad \text{(101)}$$

The negative sign before μ_K is for the case when the phase angle ϕ_2 leads and is greater than $\frac{\pi}{2} - \phi_K$. Hence, to determine the percentage rise of pressure, we set off (Fig. 146) $\overline{AC} = I_2 z_K$ as a percentage of P_2

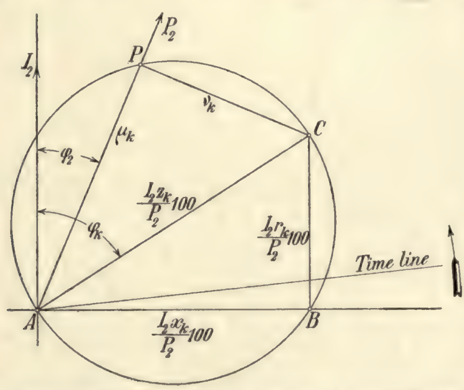


FIG. 146.—Short-circuit Diagram of a Symmetrical Circuit for determining the Percentage Pressure Rise.

at an angle ϕ_K to the ordinate axis, describe a circle on the same as diameter, and draw $\overline{AP}$ at an angle ϕ_2 to the ordinate axis: we then get

$$\overline{AB} = \frac{I_2 x_K}{P_2}100; \quad \overline{BC} = \frac{I_2 r_K}{P_2}100.$$

and the percentage pressure rise

$$\epsilon\,\% = \pm\overline{AP} + \frac{\overline{CP}^2}{200}.$$

This is a maximum when $\phi_2 = \phi_K$.

When $\phi_2 = 0$, $\mu_K = \frac{I_2 r_K}{P_2} 100$ and $\nu_K = \frac{I_2 x_K}{P_2} 100$.

Thus, in this case, $\epsilon \% = 100 \left\{ \frac{I_2 r_K}{P_2} + \frac{1}{2} \left(\frac{I_2 x_K}{P_2} \right)^2 \right\}$.

Fig. 146 can be appropriately called the *short-circuit diagram.*

If we are not dealing with a symmetrical circuit, but with the general case, for which the constants $\dot{C}_1$ and $\dot{C}_2$ may considerably differ, we substitute the actual receiver pressure $\dot{P}_2$ with the phase displacement ϕ_2 by a fictitious pressure $\dot{P}'_2 = \frac{\dot{C}_1}{\dot{C}_2} \dot{P}_2$ displaced from the receiver current $\dot{I}_2$ by the angle $\phi_2 - \Delta\psi$. Then, since, from equation (88),

$$\frac{\dot{P}_1}{\dot{C}_2} = \frac{\dot{C}_1}{\dot{C}_2} \dot{P}_2 + \dot{I}_2 \dot{z}_K = \dot{P}'_2 + \dot{I}_2 \dot{z}_K,$$

the above formula (101) holds also for the rise of pressure in the general case, provided that we use $P'_2 = P_2 \frac{C_1}{C_2}$ instead of P_2 and $\phi'_2 = \phi_2 - \Delta\psi$ instead of ϕ_2 in the short-circuit diagram. Fig. 147 represents the

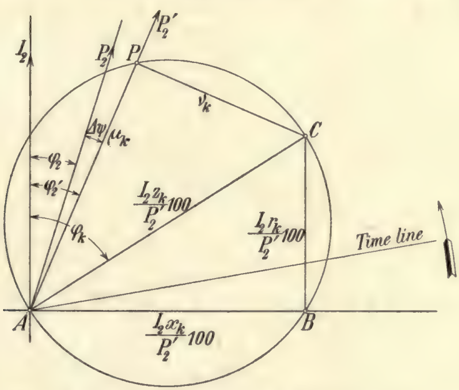


FIG. 147.—Short-circuit Diagram of the General Circuit.

short-circuit diagram of a general circuit—in this $I_2 \dot{z}_K$ is set off as a percentage of $\dot{P}'_2$.

50. Determination of the Change of Current in a Circuit by means of the No-load Diagram. The pressure P_2 at the terminals SS requires, as we have seen, a no-load current. On account of this no-load current, the load current I_1 is greater than the short-circuit current I_{1K}. Starting from short circuit, let the pressure be gradually increased—then I_1 will also increase, and we have now to calculate

the percentage increase of current in passing from short-circuit to full-load. This is,

$$j\% = \frac{I_1 - I_{1K}}{I_{1K}} 100.$$

For a symmetrical circuit,

$$\frac{\dot{I}_1}{\dot{C}} = \dot{I}_2 + P_2 \dot{y}_0 \quad \text{(see eq. 89}b\text{)}.$$

This equation also can be expressed graphically. In Fig. 148 set

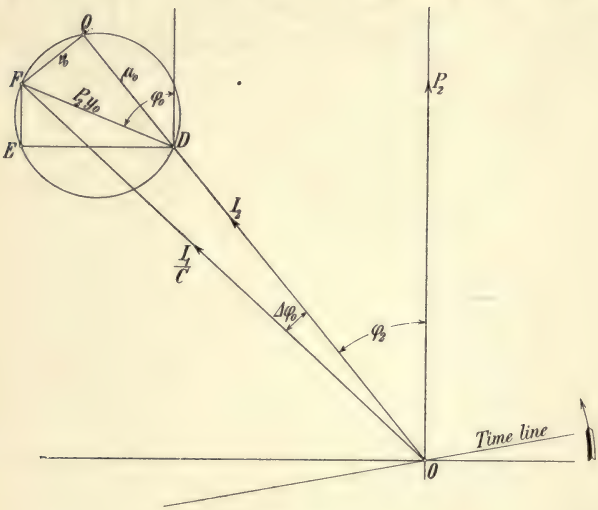


FIG. 148.

off $\dot{P}_2$ along the ordinate axis and $\dot{I}_2 = \overline{OD}$ at angle ϕ_2 to $\dot{P}_2$. Set off further $\overline{DF} = P_2 \dot{y}_0$ at angle ϕ_0 to the ordinate axis, where $\phi_0 = \tan^{-1}\frac{b_0}{g_0}$, so that

$$\overline{OF} = \frac{\dot{I}_1}{\dot{C}}.$$

Further, since $$\overline{OD} = \dot{I}_2 = \frac{\dot{I}_{1K}}{\dot{C}},$$

the percentage increase of current $j\%$ can be written

$$j\% = \frac{I_1 - I_{1K}}{I_{1K}} 100 = \frac{\overline{OF} - \overline{OD}}{\overline{OD}} 100.$$

On $\overline{DF}$ describe a circle and produce OD to cut the circle in Q; then

$$DE = P_2 b_0 \quad \text{and} \quad EF = P_2 g_0.$$

Let $\overline{DQ}=\frac{\mu_0}{100}\overline{OD}$ and $\overline{FQ}=\frac{\nu_0}{100}\overline{OD}$;

then the percentage increase of current is

$$j\,\%=\frac{I_1-I_{1K}}{I_{1K}}100=\pm\mu_0+\frac{\nu_0^2}{200}. \qquad (102)$$

The negative sign before μ_0 is for the case when the phase angle ϕ_2 leads and is greater than $\frac{\pi}{2}-\phi_0$.

Hence, to find the percentage current increase, set off (Fig. 149) $\overline{DF}=P_2y_0$ as a percentage of I_2 at an angle ϕ_0 to the ordinate axis,

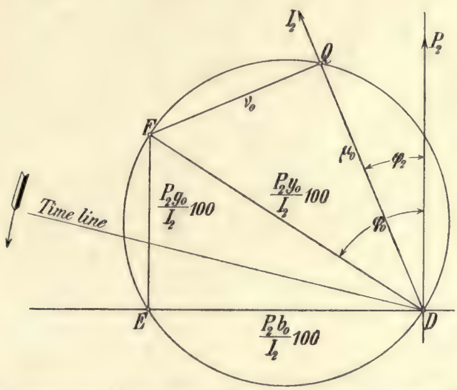


FIG. 149.—No-load Diagram of a Symmetrical Circuit for determining the Percentage Change of Current.

describe a circle on the same as diameter and draw $\overline{DQ}$ at an angle ϕ_2 to the ordinate axis.

Then we have $\overline{DE}=\frac{P_2b_0}{I_2}100$; $\overline{EF}=\frac{P_2g_0}{I_2}100$,

and the percentage increase of current

$$j\,\%=\frac{I_1-I_{1K}}{I_{1K}}100=\pm\overline{DQ}+\frac{\overline{FQ}^2}{200}.$$

This is a maximum when $\phi_2=\phi_0$. When $\phi_2=0$,

$$\mu_0=\frac{P_2g_0}{I_2}100 \quad \text{and} \quad \nu_0=\frac{P_2b_0}{I_2}100.$$

Hence, in this case

$$j\,\%=100\left\{\frac{P_2g_0}{I_2}+\frac{1}{2}\left(\frac{P_2b_0}{I_2}\right)^2\right\}.$$

We can appropriately call the diagram in Fig. 149 the *no-load* diagram.

In an unsymmetrical circuit,

$$\frac{\dot{I}_1}{\dot{C}_2} = \dot{I}_2 + \frac{\dot{C}_1}{\dot{C}_2} P_2 y_0 = \dot{I}_2 + P'_2 y_0.$$

Consequently the no-load diagram and formula (102) hold for any circuit, provided we use $P'_2 = \frac{C_1}{C_2} P_2$ instead of P_2 and $\phi'_2 = \phi_2 - \Delta\psi$ instead of ϕ_2. This is done in the no-load diagram in Fig. 150, which accordingly holds quite generally.

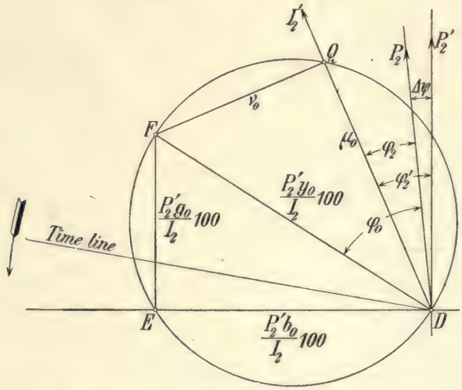


FIG. 150.—No-load Diagram of the General Circuit.

When the conditions are such that the results yielded by the short-circuit and no-load diagrams are inaccurate, we can use an alternative method, and find the pressure P_1 and current I_1 in the supply circuit by means of the load diagrams shewn in Figs. 145 and 148.

51. Change in Phase Displacement. The phase displacement between the pressure and current in a circuit changes as we pass from the receiver terminals to the supply terminals. This displacement is determined by the vector $\dot{P}_K = \dot{C}_2 \dot{I}_2 \dot{z}_K$ of the short-circuit pressure and the vector $\dot{I}_0 = \dot{C}_1 \dot{P}_2 \dot{y}_0$ of the no-load current. The angle of phase displacement of the load at *SS* has been denoted by ϕ_2 in the above—similarly we can denote that at the supply terminals *PP* by ϕ_1.

Then $$\phi_1 = \sphericalangle(\dot{P}_1, \dot{I}_1) = \sphericalangle\left(\frac{\dot{P}_1}{\dot{C}}, \frac{\dot{I}_1}{\dot{C}}\right),$$

for the two vectors $\frac{\dot{P}_1}{\dot{C}}$ and $\frac{\dot{I}_1}{\dot{C}}$ are rotated through the same angle in respect to the vectors $\dot{P}_1$ and $\dot{I}_1$.

From Figs. 145 and 148 we see that

$$\sphericalangle\left(\frac{\dot{P}_1}{\dot{C}}, \frac{\dot{I}_1}{\dot{C}}\right) = \sphericalangle\left(\frac{\dot{P}_1}{\dot{C}}, \dot{P}_2\right) + \sphericalangle(\dot{P}_2, \dot{I}_2) + \sphericalangle\left(\dot{I}_2, \frac{\dot{I}_1}{\dot{C}}\right)$$

or
$$\phi_1 = \Delta\phi_K + \phi_2 + \Delta\phi_0.$$

In order to find the phase-displacement angle at the supply terminals for a symmetrical circuit, we must therefore calculate the two angles $\Delta\phi_K$ and $\Delta\phi_0$.

From Fig. 145 we have $\sin(\Delta\phi_K) = \frac{\overline{PC}}{\overline{OC}}.$

Denoting the ratio $\frac{\overline{OA}}{\overline{OC}} = \frac{P_{10}}{P_1}$ by α, we get

$$\alpha = \frac{1}{1 + \frac{\epsilon\,\%}{100}} = \frac{1}{1+\epsilon}$$

and
$$\sin(\Delta\phi_K) = \frac{\overline{PC}}{\overline{OA}}\alpha = \frac{\nu_K \alpha}{100}.$$

We can express $\sin(\Delta\phi_K)$ in the form of a series, thus:

$$\sin(\Delta\phi_K) = \Delta\phi_K - \frac{(\Delta\phi_K)^3}{3!} + \ldots = \frac{\nu_K \alpha}{100}.$$

$\frac{(\Delta\phi_K)^3}{6}$ is negligible compared with $\Delta\phi_K$, so long as $\Delta\phi_K \leqq 0{\cdot}25$, which is usually the case where $\Delta\phi_K$ is expressed in circular measure; or when measured in degrees we have

$$\Delta\phi_K = \frac{\nu_K \alpha}{100} \cdot \frac{180}{\pi},$$

$$\text{i.e. } \Delta\phi_K = 0{\cdot}573\nu_K\alpha = \frac{0{\cdot}573\nu_K}{1+\epsilon}.$$

In a similar manner, from Fig. 148,

$$\sin(\Delta\phi_0) = \frac{\overline{QF}}{\overline{OF}},$$

or, denoting $\frac{\overline{OD}}{\overline{OF}} = \frac{I_{1K}}{I_1}$ by β, we get

$$\beta = \frac{1}{1 + \frac{j\,\%}{100}} = \frac{1}{1+j}$$

and
$$\Delta\phi_0 = 0{\cdot}573\nu_0\beta = \frac{0{\cdot}573\nu_0}{1+j},$$

whence the angle of phase displacement at the supply terminals is

$$\phi_1 = \phi_2 + 0{\cdot}573\left(\frac{\nu_K}{1+\epsilon} + \frac{\nu_0}{1+j}\right). \quad \text{..................(103)}$$

In this formula, ν_K and ν_0 are to be taken negative when the points P and Q respectively lie on the arcs BC and EF; this is the case when the angle of lag ϕ_2 is greater than ϕ_K or ϕ_0 respectively.

In the case of the general unsymmetrical circuit, we must substitute $\phi_2' = \phi_2 - \Delta\psi$ for ϕ_2 in formula (103), where ϕ_2' is the angle between the imaginary receiver pressure P_2' and the receiver current I_2.

It has already been shewn that $\Delta\psi = \psi_1 - \psi_2$; hence, for any circuit, the phase-displacement angle at the supply terminals is

$$\phi_1 = \phi_2 + (\psi_2 - \psi_1) + 0{\cdot}573\left(\frac{\nu_K}{1+\epsilon} + \frac{\nu_0}{1+j}\right). \quad \text{..........(103a)}$$

52. Maximum Power and Efficiency. With constant supply pressure P_1 and load power-factor (i.e. $\cos\phi_2 = \text{const.}$), it is only possible to transmit a certain maximum power to the receiver circuit. If we try to go beyond this by increasing the load admittance y_2, the receiver pressure P_2 will fall more rapidly than the receiver current I_2 will rise. This maximum is naturally reached when the drop of pressure $\dot{I}_2\dot{z}_K$ in the circuit itself equals the receiver pressure $\dot{P}_2' = \frac{\dot{C}_1}{\dot{C}_2}\dot{P}_2$.

From the equation

$$\frac{\dot{P}_1}{\dot{C}_2} = \frac{\dot{C}_1}{\dot{C}_2}\dot{P}_2 + \dot{I}_2\dot{z}_K = \frac{\dot{C}_1}{\dot{C}_2} I_2 z_2 + \dot{I}_2\dot{z}_K,$$

it follows that, when ϕ_2 is constant, the power given out at the receiver terminals

$$W_2 = I_2 P_2 \cos\phi_2$$

is a maximum when the product of the two absolute values $\frac{\dot{C}_1}{\dot{C}_2} I_2 z_2$ and $I_2 z_K$ is a maximum. Since the sum $\frac{\dot{P}_1}{\dot{C}_2}$ of these two vectors is constant, the product of their absolute values is a maximum when they are equal.

Hence the condition for maximum power is

$$\frac{\dot{C}_1}{\dot{C}_2}\dot{P}_2 = I_2\dot{z}_K$$

or

$$\frac{\dot{C}_1}{\dot{C}_2} z_2 = z_K.$$

In this case the receiver current is

$$\dot{I}_2 = \frac{\frac{1}{\dot{C}_2}\dot{P}_1}{\frac{\dot{C}_1}{\dot{C}_2} z_2 + z_K}.$$

The vectors z_K and $\frac{C_1}{C_2} z_2$ are displaced from one another by the angle $\phi_K - \phi_2 + \Delta\psi$.

Hence the receiver current at maximum load is

$$I_2 = \frac{P_1}{2C_2 z_K \cos \frac{1}{2}(\phi_K - \phi_2 + \Delta\psi)},$$

and therefore $$W_{2\max} = I_2^2 z_2 \cos \phi_2 = I_2^2 \frac{C_2}{C_1} z_K \cos \phi_2$$

$$= \frac{P_1^2 \cos \phi_2}{2C_1 C_2 z_K \{1 + \cos(\phi_K - \phi_2 + \Delta\psi)\}}.$$

Now, from Eq. 95, p. 159,

$$\frac{1}{C_1 C_2} \simeq \frac{I_K - I_0 \cos(\phi_0 - \phi_K)}{I_K}.$$

Hence, $$W_{2\max} = \frac{P_1\{I_K - I_0 \cos(\phi_0 - \phi_K)\} \cos \phi_2}{2\{1 + \cos(\phi_K - \phi_2 + \Delta\psi)\}}. \quad \text{.............(104)}$$

Since $P_2' = \frac{P_{10}}{C_2}$ and $I_2 z_K = \frac{P_K}{C_2}$, the conditions for maximum power can be written: *With constant supply pressure and load power-factor, we get maximum power for the load, whose no-load and short-circuit pressures are equal.*

Proceeding further, we can now find the load power-factor $\cos \phi_2$ necessary for obtaining the maximum power at the receiver terminals.

By differentiating Eq. 104, we find that this happens when

$$-\{1 + \cos(\phi_K - \phi_2 + \Delta\psi)\} \sin \phi_2 - \cos \phi_2 \sin(\phi_K - \phi_2 + \Delta\psi) = 0$$

or when $$-\phi_2 = \phi_K + \Delta\psi.$$

Introducing this value of ϕ_2 into the expression for $W_{2\max}$, we get

$$W_{2\max} = \frac{P_1^2 \cos \phi_2}{2C_1 C_2 z_K (1 + \cos 2\phi_2)}$$

$$= \frac{P_1^2}{4C_1^2 z_2 \cos \phi_2} = \frac{P_1^2}{4C_1^2 r_2}. \quad \text{......................(104a)}$$

To find the *efficiency* of the general circuit, we calculate the power W_1 supplied to the circuit at the terminals PP and divide this into the power W_2 taken out at the receiver terminals. The supply power is most easily obtained from the real part of the product of P_1 and the conjugate vector of I_1. The supplied power W_1 is

$$W_1 = W_0 + W_K + sP_2 I_2,$$

where $$W_0 = P_{10}^2 g_0 = C_1^2 P_2^2 g_0$$

is the no-load loss when P_2 acts at the receiver terminals, and

$$W_K = I_{1K}^2 r_K = C_2^2 I_2^2 r_K$$

is the short-circuit loss when the current in the receiver circuit is I_2.

$$s = p\cos\phi_2 + q\sin\phi_2,$$

where
$$p = \frac{I_K + I_0\cos(\phi_0+\phi_K)}{I_K - I_0\cos(\phi_0-\phi_K)}$$

and
$$q = \frac{I_0\sin(\phi_0+\phi_K)}{I_K - I_0\cos(\phi_0-\phi_K)},$$

or
$$s = \frac{I_K\cos(\phi_2-\Delta\psi) + I_0\cos(\phi_0+\phi_K-\phi_2+\Delta\psi)}{I_K - I_0\cos(\phi_0-\phi_K)},$$

and depends only on the kind of load, i.e. on $\cos\phi_2$.

Since the power at the receiver terminals is

$$W_2 = I_2^2 r_2,$$

we get the percentage efficiency

$$\eta\,\% = \frac{W_2}{W_1}100 = \frac{W_2}{W_0 + W_K + \dfrac{s}{\cos\phi_2}W_2}100. \quad\ldots\ldots\ldots\ldots(105)$$

Both at no-load and short-circuit the efficiency is zero, for in the first case the useful current is zero and in the second case the useful pressure. In the former case, the sum of all the losses is W_0 and in the latter W_K.

Starting from no-load and keeping the load power-factor constant, the efficiency and the heating losses W_K gradually rise as the load is gradually increased, whilst the no-load losses W_0 decrease. When $W_K = W_0$, the efficiency will be a maximum, for, with a given loss $W_0 + W_K = \text{const.}$, the product $W_0 W_K = C_1^2 C_2^2 g_0 r_K P_2^2 I_2^2$ is a maximum when the two losses are equal. Thus we see:

For a given kind of load in a circuit, the efficiency is a maximum when the short-circuit losses corresponding to the load current equal the no-load losses corresponding to the load pressure.

The maximum efficiency for a given load power-factor is

$$\eta'_{\max}\,\% = \frac{W_2}{2W_0 + \dfrac{s}{\cos\phi_2}W_2}100. \quad\ldots\ldots\ldots\ldots\ldots\ldots(106)$$

Further, we find by differentiation that the power factor $\cos\phi_2$, for which the efficiency is a maximum, occurs when

$$W_0 = W_K$$

and $(W_0 + W_K)\sin\phi_2 = 2W_0\sin\phi_2 = -P_2 I_2\{p\sin(\Delta\psi) + q\cos(\Delta\psi)\}$,

and the maximum efficiency is

$$\eta_{\max}\,\% = \frac{W_2}{W_2\{p\cos(\Delta\psi) - q\sin(\Delta\psi)\} + 2W_0\cos^2\phi_2}. \quad\ldots\ldots(107)$$

Considering equation (107) more closely, we see that the load current I_2 at the absolute maximum efficiency is displaced in phase with respect to the receiver pressure. In general it will be found that I_2 lags behind or leads P_2 almost as much as I_1 leads or lags behind P_1.

53. A Transmission Scheme. As an example of the application of the no-load and short-circuit diagrams to a symmetrical circuit, we will consider a transmission line. This consists of a supply station where the pressure is transformed up, the transmission line and the receiver station where the pressure is transformed down. We will assume both supply and receiver transformers to have the same ratio of transformation.

(1) No-load measurements:

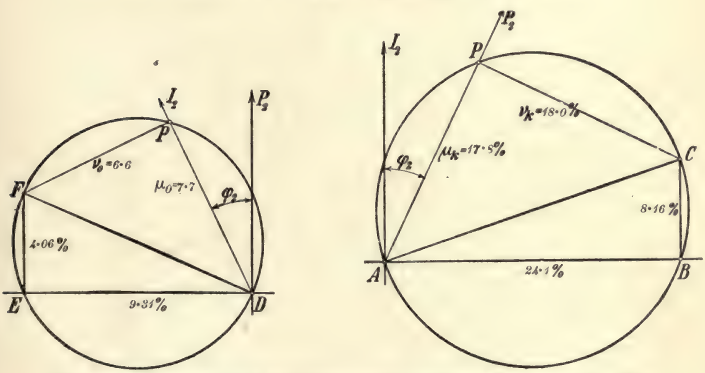
$P_1 = 1000$ volts, $I_0 = 100$ amps., $W_0 = 40$ K.W., $P_2 = 985$ volts.

(2) Short-circuit measurements:

$I_K = 1000$ amps., $P_K = 250$ volts, $W_K = 80$ K.W., $I_2 = 985$ amps.

We get
$$C_1 = C_2 = C = \frac{1000}{985} = 1{\cdot}015,$$

$$\psi_1 + \psi_2 = 2\psi = 57{\cdot}3\,\frac{I_0 \sin(\phi_0 - \phi_K)}{I_K - I_0 \cos(\phi_0 - \phi_K)} = 0^\circ{\cdot}12.$$



FIGS. 151*a* and *b*.—No-load and Short-circuit Diagrams of Transmission Line.

For a load current $I_2 = 985$ amps., the watt and wattless components of the no-load current are, in percentages,

$$I_{0W}\,\% = \frac{1}{9{\cdot}85}\,\frac{W_0}{P_0} = 4{\cdot}06\,\%,$$

$$I_{0WL}\,\% = \frac{1}{9{\cdot}85}\sqrt{100^2 - 40^2} = 9{\cdot}31\,\%.$$

The no-load diagram is drawn in Fig. 151*a*.

For a power factor $\cos\phi_2=0{\cdot}9$ in the receiver circuit, the percentage current increase is

$$j\,\%=\mu_0+\frac{\nu_0^2}{200}=7{\cdot}95\,\%.$$

At short-circuit, the watt component of the supply pressure is

$$P_{KW}=\frac{W_K}{I_K}=80 \text{ volts}$$

or $\frac{80}{9{\cdot}85}=8{\cdot}12\,\%$ of the constant receiver pressure $P_2=985$ volts. The wattless component is

$$P_{KWL}=\sqrt{250^2-80^2}=237 \text{ volts},$$

corresponding to $24{\cdot}1\,\%$ of P_2. The short-circuit diagram is shewn in Fig. 151*b*. The percentage pressure increase with $\cos\phi_2=0{\cdot}9$ is

$$\epsilon\,\%=\mu_K+\frac{\nu_K^2}{200}=19{\cdot}4\,\%.$$

In the transmission scheme, the phase displacement between current and pressure is increased by the angle

$$\Delta\phi_0+\Delta\phi_K=0{\cdot}573\left(\frac{\nu_0}{1+j}+\frac{\nu_K}{1+\epsilon}\right)=12^\circ{\cdot}25.$$

Hence the supply phase-displacement is

$$\phi_1=\phi_2+\Delta\phi_0+\Delta\phi_K=25^\circ{\cdot}85+12^\circ{\cdot}25=38^\circ{\cdot}1$$

and the power factor at the supply terminals $\cos\phi_1=0{\cdot}785$.

The efficiency of the transmission scheme is

$$\eta=\frac{P_2I_2\cos\phi_2}{W_0+W_K+sP_2I_2},$$

where

$$s=\frac{I_K\cos\phi_2+I_0\cos(\phi_0+\phi_K-\phi_2)}{I_K-I_0\cos(\phi_0-\phi_K)}$$

$$=\frac{4000\times 0{\cdot}9+100\cos 111^\circ{\cdot}85}{4000-100\cos 4^\circ{\cdot}9}=0{\cdot}945,$$

whence

$$\eta\,\%=\frac{985\times 985\times 0{\cdot}9}{40000+80000+0{\cdot}945\times 985\times 985}$$

$$=\frac{873}{1035}=84{\cdot}2\,\%.$$

54. A Single-phase Transformer. As a further example, we will take the single-phase transformer, which represents the simplest form of all electromagnetic apparatus and machines. The no-load measurements taken on a 50 K.V.A. single-phase transformer were:

$$P_{10}=5000 \text{ volts},\quad I_0=0{\cdot}4 \text{ amp.},\quad W_0=750 \text{ watts},$$

and at short-circuit

$$I_{1K}=10 \text{ amps.},\quad P_K=250 \text{ volts},\quad W_K=1000 \text{ watts.}$$

Hence the no-load watt current is

$$I_{0W}=\frac{W_0}{P_{10}}=\frac{750}{5000}=0{\cdot}15 \text{ amp.},$$

and the no-load wattless current

$$I_{0WL}=\sqrt{I_0^2-I_{0W}^2}=\sqrt{0{\cdot}4^2-0{\cdot}15^2}=0{\cdot}37 \text{ amp.}$$

I_{0W} is 1·5 % and I_{0WL} is 3·7 % of the load current (10 amps.); from these two magnitudes the no-load diagram (Fig. 152*a*) is obtained. At $\cos\phi_2=0{\cdot}9$ the percentage current increase is

$$j\,\%=\mu_0+\frac{v_0^2}{200}=2{\cdot}97+\frac{2{\cdot}67^2}{200}=3{\cdot}0\,\%.$$

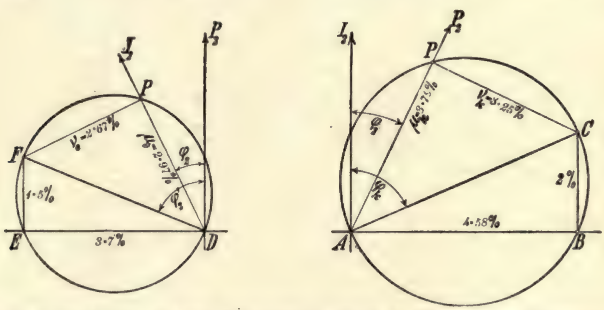
With normal short-circuit current, the watt component of the primary pressure is

$$P_{KW}=\frac{W_K}{I_{1K}}=\frac{1000}{10}=100 \text{ volts},$$

i.e. 2 % of the normal pressure. The wattless component is

$$P_{KWL}=\sqrt{P_K^2-P_{KW}^2}=\sqrt{250^2-100^2}=229 \text{ volts}$$

or 4·58 % of the normal pressure. From these two values we obtain



FIGS. 152*a* and *b*.—No-load and Short-circuit Diagrams of Single-phase Transformer.

the short-circuit diagram (Fig. 152*b*). At $\cos\phi_2=0{\cdot}9$ the percentage pressure rise is

$$\epsilon\,\%=\mu_K+\frac{v_K^2}{200}=3{\cdot}79+\frac{3{\cdot}25^2}{200}=3{\cdot}84\,\%.$$

The increase in the phase displacement between pressure and current due to transformation is

$$\Delta\phi_0+\Delta\phi_K=0{\cdot}573\left(\frac{v_0\,\%}{1+j}+\frac{v_K\,\%}{1+\epsilon}\right)=3^\circ{\cdot}28.$$

The angle of phase difference on the primary side is then

$$\phi_1 = \phi_2 + \Delta\phi_0 + \Delta\phi_K = 25^\circ\cdot 85 + 3^\circ\cdot 28 = 29^\circ\cdot 13$$

and the power factor at the primary terminals,

$$\cos\phi_1 = 0\cdot 871,$$

whilst $$\cos\phi_2 = 0\cdot 900.$$

In Fig. 153 the percentage increase of pressure and current and the increase in phase displacement with constant pressure and current on

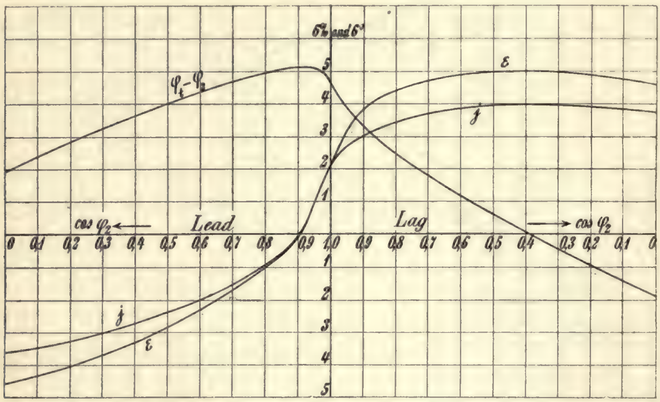


FIG. 153.

the secondary side are also shewn as functions of $\cos\phi_2$. It is seen that all three magnitudes vary most in the neighbourhood of unity power factor, i.e. $\cos\phi_2 = 1\cdot 0$.

CHAPTER X.

THE LOAD DIAGRAM.

55. Load Diagram of an Electric Circuit. As we have seen, the no-load and short-circuit diagrams are well suited for investigating the working of a short transmission line or modern transformer. For representing the phenomena, however, which occur in a long transmission line or in motors where electric energy is transformed into mechanical, these diagrams are less suitable. If we have, for example, a motor fed from mains whose pressure is kept constant, we require a diagram which will enable us to see directly how large a watt current and how large a wattless current will be taken by the same at any given load. Further, the diagram must shew, at the same time, the efficiency and speed of the motor when working at this load and also its overload capacity. We shall now shew how to construct a diagram from which all these quantities can be accurately obtained. For this purpose we start from the equations (88) and (89) of the general electric circuit, viz.

$$P_1 = C_1 P_2 + C_2 I_2 z_K$$

and

$$I_1 = C_2 I_2 + C_1 P_2 y_0.$$

From these, it follows:

$$P_1 - I_1 z_K = C_1 P_2 (1 - y_0 z_K) = \frac{P_2}{C_2},$$

and since $I_2 = \frac{P_2}{z_2}$, the current in the supply circuit will be:

$$I_1 = C_1 P_2 \left(\frac{C_2}{C_1 z_2} + y_0 \right) = (P_1 - I_1 z_K) \frac{y_0 + \dfrac{C_2}{C_1 z_2}}{1 - y_0 z_K}.$$

Put

$$\frac{y_0}{1 - y_0 z_K} = y_0 C_1 C_2 = y_a$$

and

$$\frac{C_2}{C_1 z_2 (1 - y_0 z_K)} = \frac{C_2^2}{z_2} = y_b;$$

then the current I_1 can be written:

$$I_1 = (P_1 - I_1 z_K)(y_a + y_b). \quad \text{.......................(108)}$$

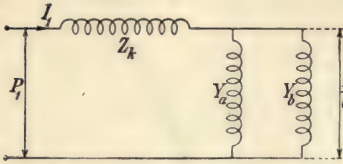


FIG. 154.—Equivalent Circuit of the General Electric Circuit.

This equation shews that every circuit can be replaced by that shewn in Fig. 154, since equation (108) also holds for this circuit. We must now find, however, to what the two parallel branches in the original circuit correspond. To the branch with admittance y_b a power W_b is supplied, where

$$W_b = P_2 I_2 \cos(\phi_2 + 2\psi_2) = P_2 I_2 \cos\phi_2 \frac{\cos(\phi_2 + 2\psi_2)}{\cos\phi_2} = W_2 \frac{\cos(\phi_2 + 2\psi_2)}{\cos\phi_2},$$

i.e. *the branch with admittance y_b corresponds with respect to power to the load circuit with impedance z_2.*

To the second branch with admittance y_a a power W_a is supplied, which, expressed symbolically, equals

$$W_a = (P_1 - I_1 z_K)^2 y_a = \frac{P_2^2}{C_2^2} C_1 C_2 y_0 = \frac{C_1}{C_2} P_2^2 y_0.$$

This corresponds to a loss which is proportional to the square of the pressure. This loss includes such losses as iron losses and those which occur in the dielectrics of electrical apparatus and machines. Considering finally the path with impedance z_K, we find in it the loss

$$W_1 = I_1^2 z_K.$$

This is the copper loss in the circuit, and represents that part of the electrical energy which is dissipated in the form of heat.

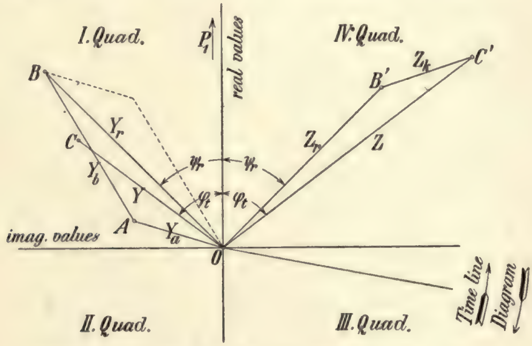


FIG. 155.—Diagram of the Equivalent Circuit in Fig. 154.

To find the current I_1 for a given impedance z_2, we first of all calculate y_a and y_b and set off the same in a rectangular co-ordinate system (Fig. 155). The negative part of the abscissa axis is taken for

the axis of the imaginary values. By adding y_a and y_b geometrically, we get the resultant admittance y_r. Since the admittances y_a and y_b are in series with the impedance z_K, we first find the impedance z_r corresponding to the admittance y_r. Thus:

$$y_r = g_r + jb_r = y_r \epsilon^{j\psi_r}$$

and

$$z_r = \frac{1}{y_r} = \frac{g_r}{y_r^2} - j\frac{b_r}{y_r^2} = r_r - jx_r.$$

Adding now the short-circuit impedance z_K to z_r, we get the resultant impedance z. The inversion of z gives the admittance y, which falls in the first quadrant. Finally, multiplying the admittance y by the terminal pressure P_1, we get the current I_1 in the supply circuit. As usual, let the pressure vector P_1 fall on the ordinate axis, so that the current vector I_1 coincides with the admittance y. Then the vector $\overline{OC}$ not only gives the direction of the current in the supply circuit, but also its magnitude to a certain scale.

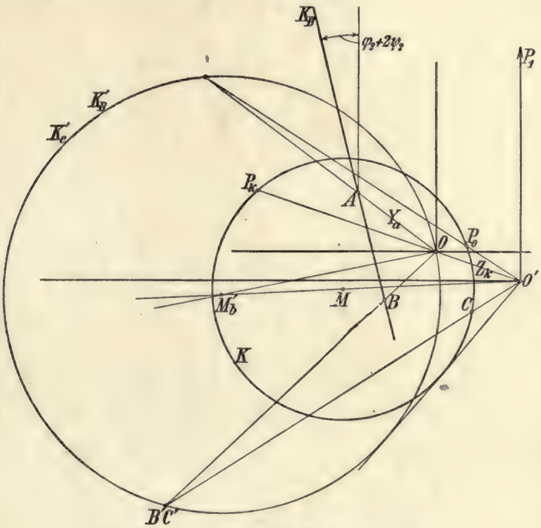


FIG. 156.—Construction of Current Diagram.

To determine the locus of the current vector I_1, we first find the curve traced out by the vector y_b when the load z_2 is varied. This is

$$y_b = \frac{C_2^2}{z_2} = C_2^2 y_2 = C_2^2 y_2 \epsilon^{j(\phi_2 + 2\psi_2)}.$$

Assume, by way of example, that the phase displacement ϕ_2 in the load circuit is constant. Then the locus of the admittance

$$y_b = C_2^2 y_2 \epsilon^{j(\phi_2 + 2\psi_2)}$$

is a straight line K_B (Fig. 156) making the angle $(\phi_2 + 2\psi_2)$ with the

ordinate axis. In order to draw the load diagram for this case, we first set off the constant admittance

$$\overline{OA} = y_a = C_1 C_2 y_0 = g_a + jb_a,$$

and draw through A a straight line K_B at angle $(\phi_2 + 2\psi_2)$ to the ordinate axis. The admittance y_r is then represented by the vector $\overline{OB}$ drawn to this line. Then, to find the impedance z_r corresponding to y_r, we find the inverse of the straight line K_B with the origin O as the centre of inversion. This inverse curve is not drawn in the fourth quadrant, but in the first, since we must return to this latter by a further inversion. Now, the inverse curve of a straight line is a circle passing through the centre of inversion; thus, in this case, the inversion circle is K'_B and the centre of inversion is the origin O. The centre of K'_B lies on a straight line passing through the inversion centre O and perpendicular to the line K_B. The radii-vectores of the circle K'_B from O then give the impedance z_r. We now add the short-circuit impedance z_K to z_r by moving the co-ordinate system to the right through a distance equal and parallel to z_K. The origin O' of the new co-ordinate system consequently falls in the third quadrant. Then the vectors drawn to the circle K'_B, or, as it is now, K'_C, from this new origin give the total impedance z of the whole circuit. Finally, still remaining in the same quadrant, let K, with centre M on the line $O'M'_b$, be the inverse circle of the circle K'_C, with O' as centre of inversion. Then the vectors drawn from O' to this circle K represent both the admittance y and to another scale the current I_1 supplied to the line in magnitude and phase, when the vector of supply pressure P_1 coincides with the ordinate axis.

The circle K is the desired current diagram, and on it lie the short-circuit point P_K and the no-load point P_0. The former is the inverse of the point O, and the latter is obtained by a double inversion of the point A.

In Fig. 157 the final current diagram K is drawn to another scale. All points on the upper part of the circle, lying between P_K and P_0, correspond to points on the straight line K_B above A, i.e. to load in the branch y_b; while points on the lower part of the circle correspond to points on the straight line K_B below A, i.e. y_b is then negative and the branch works as a generator. The ordinates of the circle K shew directly the watt currents I_W, which the circuit takes in or gives out. By multiplying these currents by P_1, we obtain the power consumed in the circuit.

The loss and power lines are now found in the same way as above. The line of supplied power $W_1 = P_1 I_W$ is simply the abscissa axis. The copper loss may be written

$$V_K = I_1^2 r_K = B_K \mathsf{V}_K,$$

where $\mathsf{V}_K = 0$ is the equation of the loss-line and B_K is a constant. This loss-line is the semi-polar of the circle with respect to the origin, as shewn previously. The distance $\overline{PR}$ from a point P on the circle to this loss-line is proportional to the copper loss.

Consider the triangle OP_KP. The two sides $\overline{OP_K}$ and $\overline{OP}$ represent the short-circuit current I_K and the supplied current I_1 respectively. Let each side of the triangle be multiplied by z_K, then $\overline{OP_K}=I_Kz_K$ represents the terminal pressure P_1 and $\overline{OP}=I_1z_K$ the pressure consumed in z_K. Since the three pressure vectors P_1, I_1z_K and $\frac{P_2}{C_2}$ form a closed triangle, the line $\overline{P_KP}$ will represent the pressure vector $\frac{P_2}{C_2}$ to the same scale of pressure. This pressure causes a loss V_a in the branch whose admittance is $y_a=C_1C_2y_0$.

As before, we can write

$$V_a=\left(\frac{P_2}{C_2}\right)^2 g_a=B_a\mathsf{V}_a,$$

where $\mathsf{V}_a=0$ is the equation of the loss-line. This loss-line is tangent to the circle at the point P_K and the loss V_a for a point P on the circle is proportional to $\overline{PS}$, the distance of P from this loss-line.

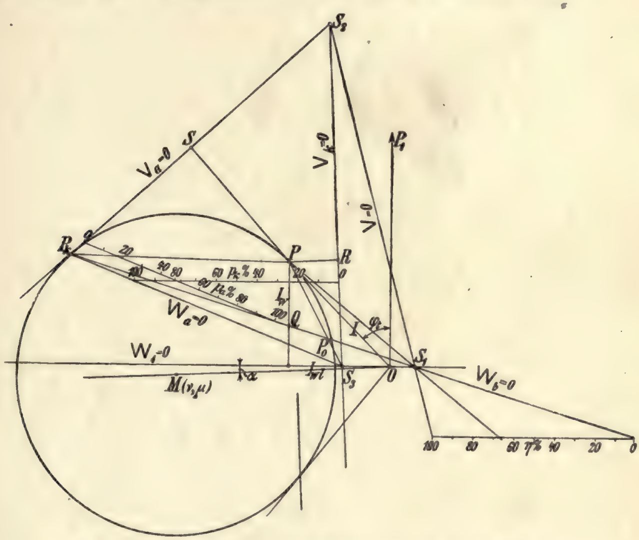


FIG. 157.—Current Diagram of Equivalent Circuit in Fig. 154.

The power-line can now easily be determined. Denoting W_1-V_K by W_a, we have

$$W_a=A_1\mathsf{W}_1-B_K\mathsf{V}_K=A_a\mathsf{W}_a,$$

where we write

$$W_1=A_1\mathsf{W}_1$$

to obtain symmetrical notation, $\mathsf{W}_1=0$ being the equation of the abscissa axis.

The line $\mathsf{W}_a=0$ for the remaining power after subtracting the copper losses, clearly passes through S_3, the point of intersection of

the abscissa axis with the line of copper loss $\mathsf{V}_K=0$. Further, since W_a is zero at short-circuit, the power-line $\mathsf{W}_a=0$ passes through the short-circuit point P_K.

The power consumed in the branch of admittance y_b is

$$W_b=W_1-V_K-V_a=A_1\mathsf{W}_1-B\mathsf{V}=A_b\mathsf{W}_b,$$

where $$B\mathsf{V}=B_K\mathsf{V}_K+B_a\mathsf{V}_a=V$$

denotes the sum of the losses, which are represented by the line $\mathsf{V}=0$. As the equations shew, this resultant loss-line must pass through S_2, the intersection of the two loss-lines $\mathsf{V}_K=0$ and $\mathsf{V}_a=0$, and we know this point, since we have found both these loss-lines. The resultant loss-line $\mathsf{V}=0$ must further pass through the intersection of the abscissa axis $\mathsf{W}_1=0$ with the resultant power-line $\mathsf{W}_b=0$. This resultant power-line contains the points for which the power in the branch y_b is zero, which only occurs at no-load and at short-circuit. Hence, the resultant power-line passes through the points P_0 and P_K. From this we can find S_1, the intersection of the power-line $\mathsf{W}_b=0$ with the abscissa axis, and the resultant loss-line $\mathsf{V}=0$ can be drawn through the points S_1 and S_2.

In a branch of the equivalent circuit, the supplied power, the losses in this branch and the useful power, which is the difference of these two, can always be represented by three lines, which intersect in a point. It was shewn in Sect. 25, that a straight line drawn between two of these lines parallel to the third is divided in the ratio of the first two powers by a line from the point of intersection of the above three lines to a point on the circle. Such a line can, therefore, at once be used to represent the efficiency or the percentage loss in a branch of the equivalent circuit.

In Fig. 157 a line has been drawn parallel to the abscissa axis between the lines $\mathsf{V}_K=0$ and $\mathsf{W}_a=0$. A line $\overline{S_3P}$ then divides this line in the ratio $\frac{V_K}{W_a}$, the ratio of the part nearest the loss-line to the whole line being $\frac{V_K}{W_a+V_K}=\frac{V_K}{W_1}$ and the ratio of the part nearest the power-line to the whole line being $\frac{W_a}{W_a+V_K}=\frac{W_a}{W_1}$. Starting from the loss-line $\mathsf{V}_K=0$ and dividing the line drawn parallel to the abscissa axis into 100 parts, the division where $\overline{S_3P}$ meets this line gives the percentage loss in the branch z_K,

$$p_K\,\%=\frac{V_K}{W_a+V_K}\,100=\frac{V_K}{W_1}\,100.$$

In the same way (Fig. 157) a line is drawn parallel to $\mathsf{W}_a=0$ between $\mathsf{V}_a=0$ and $\mathsf{W}_b=0$, and the intersection of this with P_KP gives the percentage loss in the branch y_a,

$$p_a\,\%=\frac{V_a}{W_b+V_a}\,100=\frac{V_a}{W_a}\,100.$$

To obtain the efficiency of the whole equivalent circuit, the procedure is similar. Draw a line between $W_b = 0$ and $V = 0$ parallel to the abscissa axis and divide it into 100 parts, beginning at the power-line $W_b = 0$. Then the intersection of this line with $\overline{S_1 P}$ gives the efficiency,

$$\eta' \% = \frac{W_b}{W_b + V} 100 = \frac{W_b}{W_1} 100.$$

We will now consider the relation between the power W_b in the equivalent circuit and the power W_2 consumed in the original general circuit. At the beginning of this section we denoted the load-impedance of this original circuit by $z_2 = \frac{1}{y_2}$.

Also $$y_b = C_2^2 y_2 = C_2^2 y_2 \epsilon^{j(\phi_2 + 2\psi_2)}$$

and $$I_b = \frac{P_2}{C_2} y_b = P_2 C_2 y_2 = C_2 I_2.$$

Hence $$W_b = \frac{P_2}{C_2} I_b \cos \phi_b = P_2 I_2 \cos(\phi_2 + 2\psi_2)$$

and $$W_2 = P_2 I_2 \cos \phi_2 = W_b \frac{\cos \phi_2}{\cos(\phi_2 + 2\psi_2)}.$$

Therefore, the efficiency of the general circuit is

$$\eta \% = \frac{W_b}{W_1} 100 \frac{\cos \phi_2}{\cos(\phi_2 + 2\psi_2)} = \eta' \frac{\cos \phi_2}{\cos(\phi_2 + 2\psi_2)}.$$

Since $2\psi_2$ is usually a very small angle, η is only slightly greater than η'. If $2\psi_2$ is known, we can divide the horizontal between the power-line $W_b = 0$ and the loss-line $V = 0$ into $100 \frac{\cos \phi_2}{\cos(\phi_2 + 2\psi_2)}$ equal parts and read off η directly instead of η'.

As shewn above, the line $\overline{P_K P}$ serves for reading off the pressure in the receiver circuit for any load. The current I_2 in the receiver circuit can be obtained just as easily from the diagram. At any point P, we have

$$I_2 = C_1 \overline{P_0 P},$$

which can be proved as follows:

For any load, we have

$$I_b = C_2 I_2 = I_1 - (P_1 - I_1 z_K) y_a$$

and at no-load $$0 = I_0 - (P_1 - I_0 z_K) y_a.$$

Subtracting the second equation from the first, we obtain

$$C_2 I_2 = (1 + z_K y_a)(I_1 - I_0).$$

Since $$z_a + z_K = z_0,$$

we have $$1 + z_K y_a = \frac{1}{1 - z_K y} = C_1 C_2;$$

hence $$I_2 = C_1(I_1 - I_0),$$

or, in absolute values, $$I_2 = C_1\overline{P_0P}.$$

This diagram, which enables us to completely investigate the properties of any electric circuit and to study the working of the same, we shall refer to as the *load diagram* of the circuit. From it we can find directly for any load the following values: the current I_1 and phase displacement ϕ_1 in the supply circuit, the pressure P_2 in the receiver circuit, the total power W_1 supplied to the circuit, the power W_2 taken out of the circuit, i.e. the useful power, the efficiency η and the percentage losses in the copper and in the iron and dielectrics.

56. Simple Construction of the Load Diagram by Means of the No-load and Short-circuit Points. It now remains to be seen how the circle diagram K admits of a simple construction or calculation. Two points on the circle—namely, the no-load point P_0 and short-circuit

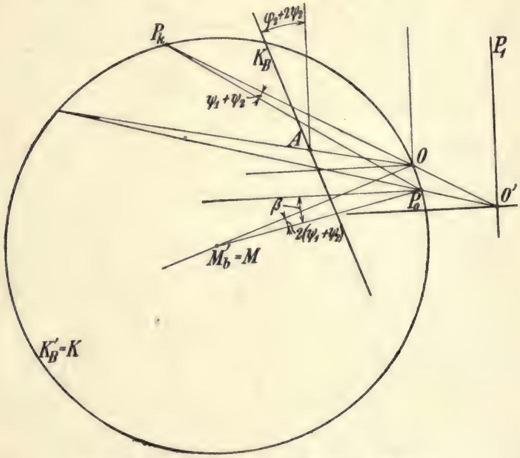


FIG. 158.—Construction of the Circle Diagram.

point P_K—are already known by experiment or otherwise. The perpendicular to the line joining these two points passes through the centre M of the circle. In addition to this, the direction of the line $\overline{P_0M}$ from the no-load point P_0 to the centre M of the circle can easily be determined as follows:

In Fig. 158 the straight line K_B represents the admittance $y_a + y_b$. This line, as shewn above, is inclined to the left of the ordinate axis at the angle $\phi_2 + 2\psi_2$. The corresponding impedance is represented by the circle K'_B. The line OM'_b falls below the abscissa axis, making an angle $\phi_2 + 2\psi_2$ with it. Then, after drawing OO' equal to z_K and taking the inversion of the circle K'_B to such a constant of inversion that K'_B represents its own inverse curve K, the points P_K and P_0 represent

respectively short-circuit and no-load in the circuit. Now, as shewn in Sect. 48, p. 159,
$$\angle OP_K P_0 = \psi_1 + \psi_2 = \tfrac{1}{2} \angle OMP_0.$$
Consequently the angle β which $\overline{P_0 M}$ makes below the abscissa axis is:
$$\begin{aligned}\beta &= \phi_2 + 2\psi_2 - 2(\psi_1 + \psi_2) = \phi_2 - 2\psi_1 \\ &= \phi_2 - (\psi_1 + \psi_2) - \Delta\psi \\ &= \phi_2 - \angle OP_K P_0 - \Delta\psi.\end{aligned}$$
Usually $\Delta\psi = \psi_1 - \psi_2$ is very small and may be neglected. When $\phi_2 = 0$ (i.e. non-inductive load) the radius $\overline{P_0 M}$ makes the angle $OP_K P_0 + \Delta\psi$ with the abscissa axis, and lies above it.

If the point P_0 lies above O on the circle, the opposite sign must be given to the angle $OP_K P_0$. This is the case when the phase displacement at no-load is smaller than that at short-circuit.

In Fig. 159, for the sake of clearness, only those lines are drawn which are necessary for the determination of the centre M of the circle,

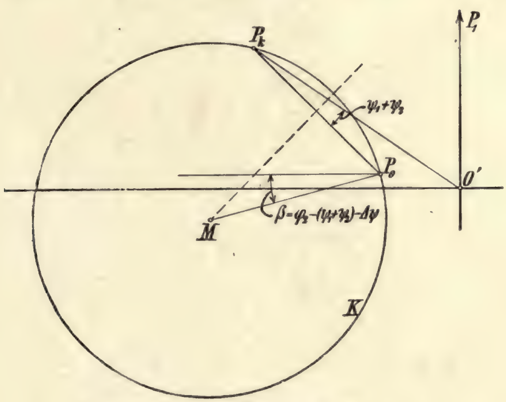


Fig. 159.—Determination of Centre of Circle.

and are obtained at once from the short-circuit and no-load points, when $\Delta\psi$ is known or negligible, as the case may be.

When ϕ_2 and $\Delta\psi$ are zero, the determination of the centre M of the circle is greatly simplified, as is shewn in Fig. 160. In this diagram, a vertical is drawn through the no-load point P_0 to cut the line $\overline{OP_K}$. The centre of this vertical is the same distance above the abscissa axis as the centre M.

From this construction the effect of disymmetry in the circuit on the position of M is clearly shewn. The greatest disymmetry occurs when $z_1 = 0$, i.e. when $\psi_1 = 0$ or $-(\psi_1 + \psi_2) - \Delta\psi = 0$ and the centre lies at $M_{(z_1=0)}$; or when $z_2 = 0$, i.e. $\psi_2 = 0$ or $-(\psi_1 + \psi_2) - \Delta\psi = -2\psi_1$ and the centre lies at $M_{(z_2=0)}$.

The centre M can also be obtained by another analytical graphical

method by using the line $\overline{MO'}$ through the origin as well as the line bisecting P_0P_K at right angles. This line makes an angle α with the abscissa axis, the tangent of which can be calculated from the following formula, which is deduced from Fig. 156:

$$\tan\alpha = \frac{-I_K\sin(\phi_2-\Delta\psi)+I_{0\,W}\sin(\phi_K-\phi_2+\Delta\psi)+I_{0\,WL}\cos(\phi_K-\phi_2+\Delta\psi)}{I_K\cos(\phi_2-\Delta\psi)+I_{0\,WL}\sin(\phi_K-\phi_2+\Delta\psi)-I_{0\,W}\cos(\phi_K-\phi_2+\Delta\psi)}.$$

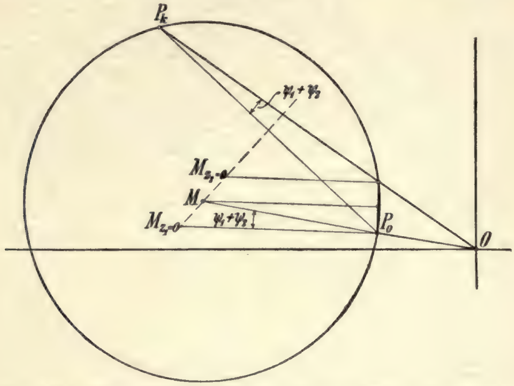


FIG. 160.—Simple Determination of Centre of Circle when $\phi_2=0$.

In most cases C_1 and C_2 are very little different from unity and still less from one another, and therefore $\Delta\psi=\psi_1-\psi_2$ is a very small angle, at most 5°. Hence, neglecting this angle, we obtain the following simple formula:

$$\tan\alpha = \frac{-I_K\sin\phi_2+I_{0\,W}\sin(\phi_K-\phi_2)+I_{0\,WL}\cos(\phi_K-\phi_2)}{I_K\cos\phi_2+I_{0\,WL}\sin(\phi_K-\phi_2)-I_{0\,W}\cos(\phi_K-\phi_2)}.$$

For a non-inductive load, $\phi_2=0$, and we have then,

$$\tan\alpha = \frac{I_{0\,W}\sin\phi_K+I_{0\,WL}\cos\phi_K}{I_K+I_{0\,WL}\sin\phi_K-I_{0\,W}\cos\phi_K}$$

$$= \frac{I_0\sin(\phi_0+\phi_K)}{I_K-I_0\cos(\phi_0+\phi_K)}.$$

57. Load Diagram of a Transmission Scheme. As examples of the application of the load diagram, we can consider first a transmission scheme, consisting of a supply station where the pressure is transformed up, a transmission line and a receiver station where the pressure is transformed down again.

The following readings are taken on no-load and short-circuit:

(1) No-load.

$$P_1=1000 \text{ volts},\quad I_0=325 \text{ amps.},\quad W_0=40 \text{ K.W.}$$

(2) Short-circuit.

$$P_K=1000 \text{ volts},\quad I_K=3000 \text{ amps.},\quad W_K=900 \text{ K.W.}$$

The transmission scheme works with a constant power-factor of $\cos \phi_2 = 0{\cdot}95$ in the receiver circuit.

The load diagram is shewn in Fig. 161 drawn to a scale of

$$1 \text{ mm} = 60 \text{ amperes.}$$

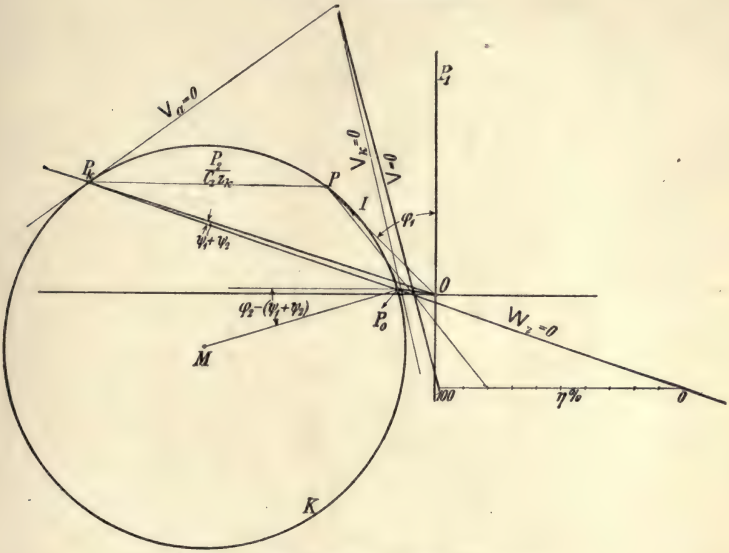


FIG. 161.—Load Diagram of Transmission Line.

Since such a scheme can in general be taken as symmetrical, we get, by direct measurement from the diagram,

$$C_1C_2 = C^2 = \frac{\overline{OP_K}}{\overline{P_0P_K}} = 1{\cdot}12,$$

$$C = 1{\cdot}06.$$

The line $\overline{OP_K}$, 50 mm in length, represents the primary pressure $P_1 = 1000$ volts to a scale of 1 mm = 20 volts. The receiver pressure on no-load is

$$P_{2\,0} = 20\ C\,.\,\overline{P_0P_K} = 20 \times 1{\cdot}06 \times 44{\cdot}5 = 945 \text{ volts.}$$

At P the receiver pressure is

$$P_2 = 20 \times 1{\cdot}06 \times 32{\cdot}7 = 695 \text{ volts.}$$

The short-circuit current in the receiver circuit is

$$I_{2\,K} = C\,.\,\overline{P_0P_K} = 1{\cdot}06 \times 60 \times 44{\cdot}5 = 2830 \text{ amps.}$$

58. Load Diagram of the General Transformer. The general transformer, whose method of working is described in Section 39, can be replaced by the equivalent circuit (Fig. 121, p. 120). r_1 and r_2

are the primary and secondary effective resistances, S_1 and S_2 the primary and secondary coefficients of stray induction of the transformer. The constants r_2 and $x_2 = 2\pi c S_2$ are both reduced to the primary circuit.

The usual case of the general transformer is the three-phase induction motor. The secondary power W_2 is here mechanical and equals

$$W_2 = I_2^2 r_2\left(\frac{1}{s} - 1\right),$$

where $r_2\left(\frac{1}{s} - 1\right)$ is the ohmic resistance equivalent to the load and is placed across the secondary terminals. The slip s gives the relative velocity of the rotary field relative to the secondary winding. Since the loss in the secondary circuit due to the rotor resistance r_2 is $I_2^2 r_2$, the total power supplied to the secondary circuit is $W = I_2^2 \frac{r_2}{s}$.

As all phases are alike in a polyphase motor, we need only consider one phase.

The following measurements were taken on such a motor:

(1) *No-load*, i.e. synchronism ($s = 0$),

the load resistance $r_2\left(\frac{1}{s} - 1\right)$ being infinite.

$$I_0 = 10{\cdot}1 \text{ amps.,} \quad P_1 = 110 \text{ volts,} \quad W_0 = 146{\cdot}5 \text{ watts.}$$

(2) *Short-circuit*, i.e. at rest or $s = 1$, since $r_2\left(\frac{1}{s} - 1\right)$ is zero.

$$I_K = 105 \text{ amps.,} \quad P_1 = 110 \text{ volts,} \quad W_K = 4040 \text{ watts.}$$

Hence, we get
$$\cos\phi_0 = \frac{146{\cdot}5}{110 \times 10{\cdot}1} = 0{\cdot}132,$$

$$\cos\phi_k = \frac{4040}{110 \times 105} = 0{\cdot}35.$$

In Fig. 162 the no-load point P_0 and the short-circuit point P_K are drawn to the scale 1 mm = 2 amps. For standard three-phase motors we can put $\Delta\psi = 0$, and further, since $\phi_2 = 0$, we get the centre of the circle by means of the construction in Fig. 160.

The lines of output $\mathsf{W}_2 = 0$ and of total loss $\mathsf{V} = 0$ can now be determined by means of Fig. 157, and from these the efficiency η is obtained.

Only the slip s, from which the speed of the motor can at once be determined, remains to be found from the diagram. This is

$$s = \frac{I_2^2 r_2}{I_2^2 \frac{r_2}{s}} = \frac{V_2}{W},$$

where W denotes the power supplied to the secondary circuit. Hence, the slip in per cent. is equal to the percentage copper loss in the secondary winding, and can be represented in a similar manner to that shewn in Fig. 157. The construction is as follows:

Draw the loss-line $\mathsf{V}_2=0$ for the loss $V_2=I_2^2 r_2$, and the power-line $\mathsf{W}=0$ for the power W supplied to the secondary circuit. Since, as we have shewn, the secondary current I_2 is proportional to $\overline{P_0P}$, the loss-line $\mathsf{V}_2=0$ is tangent to the circle at the no-load point P_0.

The power-line $\mathsf{W}=0$ passes through the no-load point P_0, since at this point $W=I_2^2\frac{r_2}{s}=0$ (because $I_2=0$), and through the point on the circle for which $s=\infty$. Since this latter point cannot be determined experimentally, we will employ the following approximation:

The line $\mathsf{V}_1=0$ for the primary copper loss I^2r_1 is identical with the loss-line $\mathsf{V}_K=0$. Neglecting the iron losses at short-circuit, we have this equation for the short-circuit point,

$$W_1=V_1+V_2=V_1+W.$$

Hence, if we draw a line $\overline{P_KC}$ through the short-circuit point parallel to the loss-line $\mathsf{V}_1=0$, i.e. a perpendicular to the line $\overline{OM}$, this line is

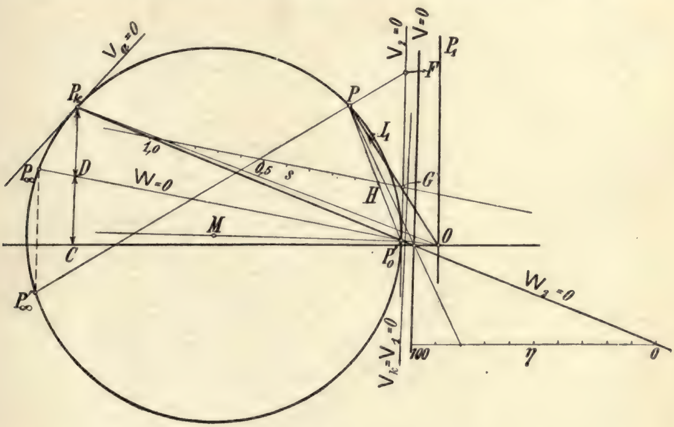


FIG. 162.—Load Diagram of the General Transformer.

divided by the abscissa axis ($\mathsf{W}_1=0$) and the power-line ($\mathsf{W}=0$) in the ratio $W_1:W$. That is, from Fig. 162 we have

$$\frac{\overline{P_KC}}{\overline{P_KD}}=\frac{W_1}{W}=\frac{I_K^2 r_K}{I_K^2 r_K-I_K^2 r_1}=\frac{r_K}{r_K-r_1}.$$

Substituting

$$r_K\simeq r_1+r_2,$$

we have

$$\frac{\overline{P_KD}}{\overline{DC}}=\frac{r_2}{r_1}.$$

Hence we can find the power-line $\mathsf{W}=0$ by drawing a line $\overline{P_K C}$ perpendicular to $\overline{OM}$ and dividing it at the point D in the ratio $\frac{r_2}{r_1}$. The line P_0D is then the power-line $\mathsf{W}=0$.

The slip s, or the percentage secondary heating loss, is now read off from the diagram by the point of intersection of the ray from P_0 and a line parallel to $\mathsf{W}=0$.

Drawing the image P'_∞ of the point P_∞ in the continuation of $\overline{P_0M}$, the slip can be measured by $\overline{P_0F}$, where F is the point where $\overline{P'_\infty P}$ produced cuts the loss-line $\mathsf{V}_2=0$. The scale of slip on the loss-line can best be found by determining the slip for any load-point by the first method and marking off the value on the loss-line $\mathsf{V}_2=0$. This construction for reading off the slip is clearly correct, since the two triangles $P'_\infty P_0F$ and P_0GH are similar.

The second method for determining the slip s is accurate and convenient for small slips.

CHAPTER XI.

ALTERNATING-CURRENTS OF DISTORTED WAVE-SHAPE.

59. Pressure Curves of Normal Alternators. 60. Fourier's Series. 61. Analytic Method for the Determination of the Harmonics of a Periodic Function. 62. Graphic Method for the Determination of the Harmonics of a Periodic Function. 63. Alternating-Currents of distorted Wave-Shape. 64. Power Yielded by an Alternating-Current of distorted Wave-Shape. 65. Effect of Wave-Shape on Measurements. 66. Resonance with Currents of distorted Wave-Shape. 67. Form Factor, Amplitude Factor and Curve Factor of an Alternating-Current.

59. Pressure Curves of Normal Alternators. In the preceding chapters we have dealt only with alternating-currents whose wave-shape is a sine curve. Strictly speaking, such currents are seldom met with in practice, for modern alternators would become much too expensive, if they were required to generate purely sinusoidal currents with all kinds of load. Consequently, we have to be contented when the wave-shape only deviates by a certain specified amount from a pure sine curve.

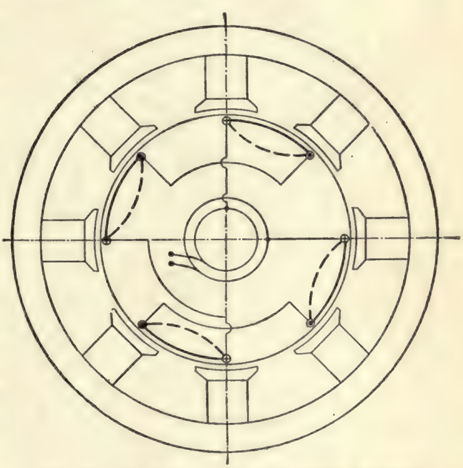


FIG. 163.—Diagram of Alternator with Revolving Armature.

Some 15 years ago, the question of the best shape of pressure curve was much discussed in technical circles. Some maintained that the peaked curve, as shewn in Fig. 172, p. 199, was the most favourable for transformers, since for a given effective pressure the hysteresis loss is then a minimum, and the efficiency consequently a maximum. This is, however, doubtful, because every deviation of

the current from a sine wave leads to an increase of the eddy losses in both iron and copper. Others, on the contrary, maintained that the peaked curve placed the greatest strain on the insulation, since for a given effective pressure this curve shape has the largest amplitude. Although many investigators at that time characterised this objection as groundless, it is nevertheless upheld nowadays. For lighting purposes, the flat-shaped curve (Fig. 172*c*, p. 199) was held to be the best,

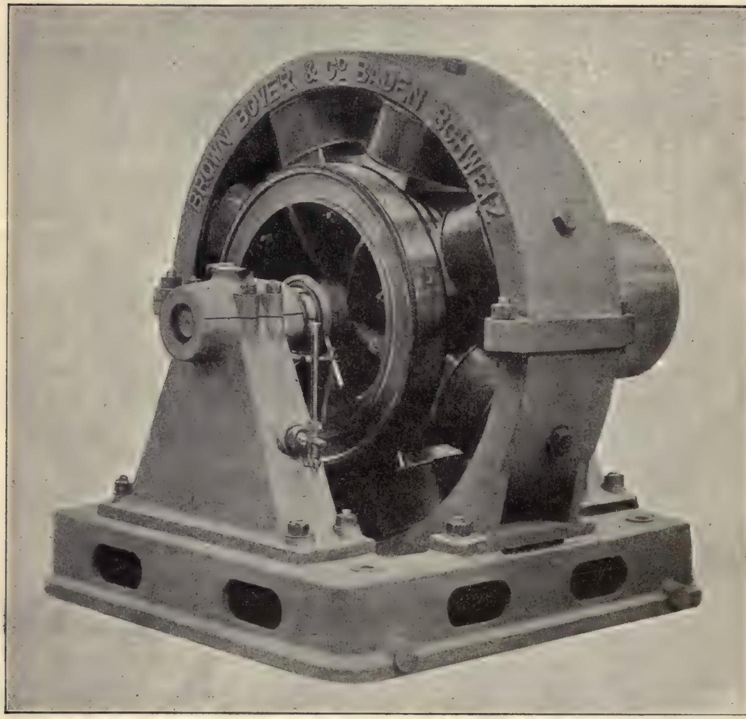


FIG. 164.

since in this case the current remains longest in the neighbourhood of its maximum, and therefore yields a steadier light.

At the present day, however, such opinions are rarely advanced, the prevailing opinion being strongly in favour of the sinusoidal pressure curve. In modern generators the greatest deviation from the fundamental is usually limited to 3 to 5 %. In Fig. 31 it was shewn how a purely sinusoidal pressure wave can be generated. The construction of such a generator, however, is very uneconomical. In order to employ a strong magnetic field, it is necessary to bed the winding—in which the current is to be induced—in iron, as shewn diagrammatically in Fig. 163.

This winding is fixed on a laminated armature and, in the case before us, rotates in a multi-polar field. The current is led off by means of

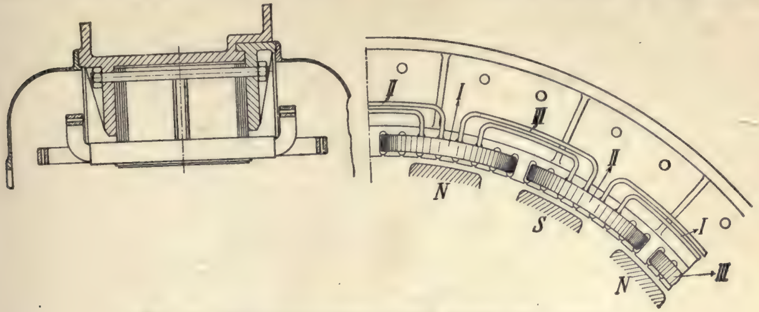


FIG. 165.—Diagram of Alternator with Stationary Armature.

slip-rings and brushes. Fig. 164 is a photograph of such an alternator with rotating armature. It is also possible, however, to have the armature fixed and let the magnets rotate—in which case we get

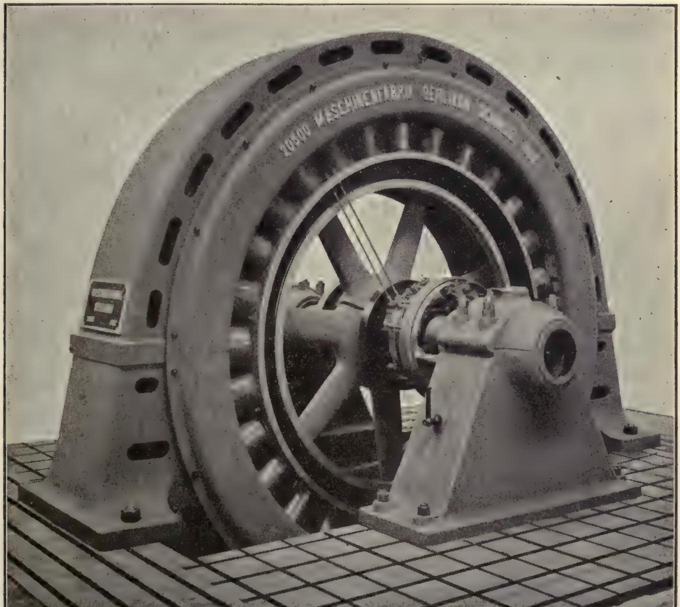


FIG. 166.

the arrangement in Fig. 165, a photograph of which is shewn in Fig. 166. The exciting current is led to the magnet coils through

slip-rings. In this arrangement, which is especially adapted for the production of high-pressure currents, the stationary armature is also referred to as the stator.

The pressure curve of these generators depends firstly on the shape of the pole-shoe, and secondly on the armature winding. If this latter is concentrated in one large closed slot per pole, the pressure curve will have the same shape as the field curve. This is represented in Figs. 167*a*, *b*, *c* for different kinds of loads. It is of especial interest to

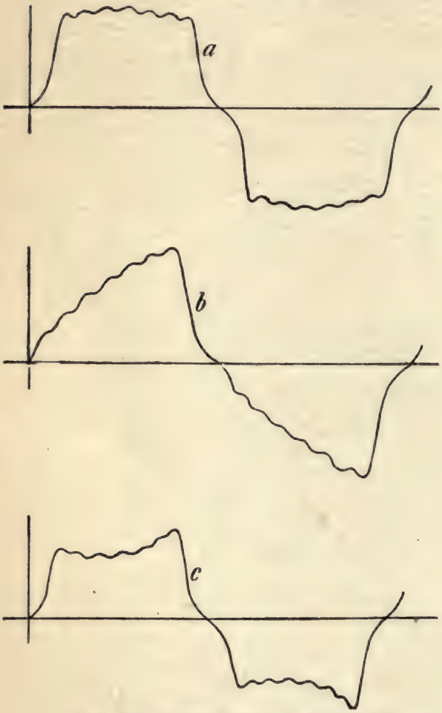


FIG. 167.—Field Curves of Alternator. (*a*) At No-load; (*b*) with Non-inductive Load; (*c*) with Inductive Load.

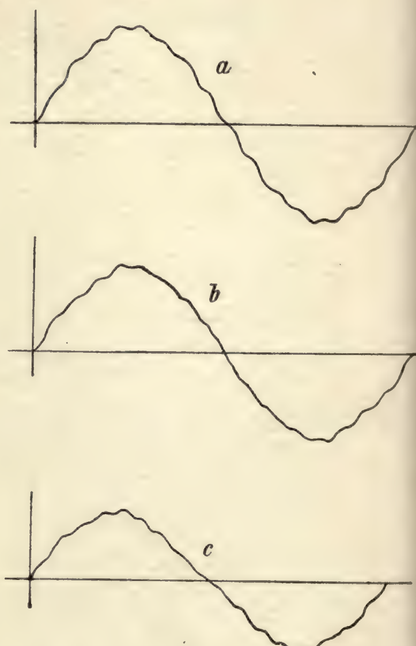


FIG. 168.—Pressure Curves of Alternator. (*a*) At No-load; (*b*) with Non-inductive Load; (*c*) with Inductive Load.

note the deviation of the curve at non-inductive load from that taken at no-load. The no-load curve is symmetrical, whilst the curve taken on load is distorted. This distortion is of course caused by the armature current, which reacts on the inducing field, and the difference represents the armature reaction. If the armature winding is distributed in several slots, the pressure curve will no longer follow the field curve, but will approach a sine wave, as is clearly seen from Fig. 168. These curves were taken on the same machine and under the same conditions as the above—except that the pressure of the whole winding was taken,

whilst the pressures shewn in Fig. 167 were taken from a single concentrated coil.

These last curves (for a distributed winding) are typical of the pressure curves of modern alternating-current generators, and it is clear that they deviate very little from sine waves.

60. Fourier's Series. As we have just mentioned, in practice we have to deal with alternating-currents whose momentary values, as functions of time, do not vary after a sine law, but according to some other periodic functions. In order to be able to carry out accurate calculations with such currents in a simple manner, it is best to analyse such a pressure curve into a sum of sine functions of different frequencies. The sine function possessing the lowest frequency is called the *first harmonic* or the *fundamental*, and all other sine functions, whose frequencies are multiples of that of the fundamental, are called *higher harmonics.* Since *Fourier* was the first to shew that every periodic function can be analysed into a series of sine functions, such series are generally termed *Fourier's Series.*

Before proceeding to deduce the same, however, we shall first quote a few integration formulae which will afterwards be needed.

These are as follows:

$$\int_{-\pi}^{+\pi} \sin mx \sin nx\, dx = \begin{cases} 0 \text{ when } m \lesseqgtr n, \\ 0 \text{ when } m = n = 0, \\ \pi \text{ when } m = n > 0, \end{cases} \quad \text{.........(109)}$$

where m and n are any positive integers.

Further,
$$\int_{-\pi}^{+\pi} \cos mx \sin nx\, dx = 0 \quad \text{.........(110)}$$

and
$$\int_{-\pi}^{+\pi} \cos mx \cos nx\, dx = \begin{cases} 2\pi \text{ when } m = n = 0, \\ \pi \text{ when } m = n > 0, \\ 0 \text{ when } m \gtreqless n. \end{cases} \quad \text{.........(111)}$$

In the interval, $-\pi$ to $+\pi$, let $f(x)$ be any continuous single-valued periodic function; we can then express the same by the following series—known as Fourier's Series:

$$f(x) = a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots + a_n \cos nx + b_n \sin nx + \dots.$$

The constant coefficients $a_1, a_2, a_3 \dots$ and $b_1, b_2, b_3 \dots$ are determined by multiplying both sides of the equation by $\cos(nx)dx$ and integrating from $-\pi$ to $+\pi$, whereby all terms on the right vanish except one. Thus, we get

$$\int_{-\pi}^{+\pi} f(x) \cos nx\, dx = a_n \int_{-\pi}^{+\pi} \cos^2(nx)\, dx = a_n \pi$$

or
$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx)\, dx.$$

Similarly, multiplying all through by $\sin(nx)dx$ and integrating between $-\pi$ and $+\pi$, we get

$$b_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)dx.$$

These two expressions for a_n and b_n can be somewhat transformed if we integrate first from $-\pi$ to 0 and then from 0 to $+\pi$, as follows:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos(nx)\,dx+\int_{0}^{+\pi} f(x)\cos(nx)\,dx\right\}.$$

In the first integral put $x=-y$; then

$$\int_{-\pi}^{0} f(x)\cos(nx)\,dx=\int_{-\pi}^{0} f(-y)\cos(-ny)\,d(-y)$$

$$=\int_{0}^{\pi} f(-y)\cos(ny)\,dy$$

or
$$\int_{-\pi}^{0} f(x)\cos(nx)\,dx=\int_{0}^{\pi} f(-x)\cos(nx)\,dx,$$

and we get
$$a_n=\frac{1}{\pi}\int_{0}^{\pi}[f(x)+f(-x)]\cos(nx)\,dx.$$

Similarly,
$$b_n=\frac{1}{\pi}\int_{0}^{\pi}[f(x)-f(-x)]\sin(nx)\,dx.$$

Example I. Find the value of i when the function $i=f(\omega t)$ traces out the rectangular curve represented in Fig. 169.

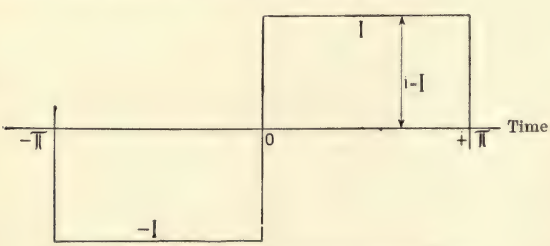


FIG. 169.—Rectangular Alternating-Current Curve.

From $\omega t=0$ to $\omega t=\pi$, $i=I$, and from $\omega t=0$ to $\omega t=-\pi$, $i=-I$.

Then, $a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} i\cos n\omega t\,d(\omega t)=\dfrac{1}{\pi}\displaystyle\int_{0}^{\pi}[I+(-I)]\cos n\omega t\,d(\omega t)=0$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} i \sin n\omega t \, d(\omega t) = \frac{1}{\pi}\int_0^{\pi} [I - (-I)] \sin n\omega t \, d(\omega t)$$

$$= \begin{cases} 0 \text{ when } n \text{ is even,} \\ \dfrac{4I}{n\pi} \text{ when } n \text{ is odd.} \end{cases}$$

Hence $$i = \frac{4I}{\pi}\left[\frac{\sin \omega t}{1} + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \ldots + \frac{\sin n\omega t}{n} + \ldots\right].$$

Example II. Find the value of i when the function $i = f(\omega t)$ traces out the triangular (saw-tooth) curve shewn in Fig. 170.

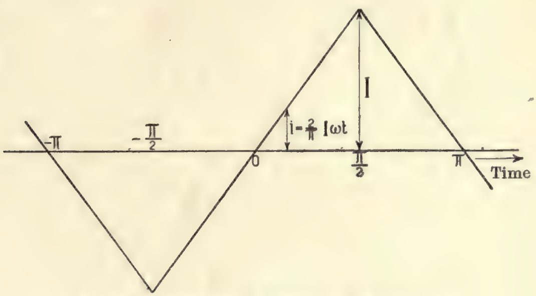


FIG. 170.—Triangular Alternating-Current Curve.

From $\omega t = 0$ to $\omega t = \frac{\pi}{2}$, $i = \frac{2}{\pi} I(\omega t)$.

,, $\omega t = 0$,, $\omega t = -\frac{\pi}{2}$, $i = -\frac{2}{\pi} I(\omega t)$.

,, $\omega t = \frac{\pi}{2}$,, $\omega t = \pi$, $i = \frac{2}{\pi} I(\pi - \omega t)$.

,, $\omega t = -\frac{\pi}{2}$,, $\omega t = -\pi$, $i = -\frac{2}{\pi} I(\pi - \omega t)$.

Accordingly,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} i \cos n\omega t \, dt$$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}} [I\,\omega t + (-I\,\omega t)] \cos n\omega t \, d(\omega t) + \int_{\frac{\pi}{2}}^{\pi} [I(\pi - \omega t) + \{-I(\pi - \omega t)\}] \cos n\omega t \, d(\omega t)\right\} = 0$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} i \sin n\omega t\, d(\omega t)$$

$$= \frac{2}{\pi^2}\int_0^{\frac{\pi}{2}} [I\,\omega t - (-I\,\omega t)] \sin n\omega t\, d(\omega t)$$

$$+ \frac{2}{\pi^2}\int_{\frac{\pi}{2}}^{\pi} [I(\pi - \omega t) - \{-I(\pi - \omega t)\}] \sin n\omega t\, d(\omega t).$$

If we put $$\omega t' = \pi - \omega t,$$

then the last integral becomes

$$+ \frac{2}{\pi^2}\int_0^{\frac{\pi}{2}} (I\,\omega t' + I\,\omega t') \sin (n\pi - n\omega t')\, d(\omega t').$$

For all even values of n,

$$\sin (n\pi - n\omega t') = -\sin n\omega t',$$

and for all odd values of n,

$$\sin (n\pi - n\omega t') = \sin n\omega t'.$$

Consequently, we get

$$b_n = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} 2I\omega t \sin n\omega t\, d(\omega t),$$

when n can only be an odd number; thus

$$b_n = \frac{8I}{\pi^2 n}\left\{-\omega t \cos n\omega t + \frac{\sin n\omega t}{n}\right\}_0^{\frac{\pi}{2}}$$

$$= \frac{8I}{\pi^2}\frac{\sin n\frac{\pi}{2}}{n^2},$$

i.e. $$b_1 = \frac{8I}{\pi^2}; \quad b_3 = -\frac{8I}{\pi^2}\frac{1}{9};$$

$$b_5 = \frac{8I}{\pi^2}\frac{1}{25}; \quad b_7 = -\frac{8I}{\pi^2}\frac{1}{49}.$$

Hence $$i = \frac{8I}{\pi^2}\left\{\frac{\sin \omega t}{1} - \frac{\sin 3\omega t}{9} + \frac{\sin 5\omega t}{25} - \frac{\sin 7\omega t}{49} + \dots\right\}.$$

In this example not only the $\cos n\omega t$ terms vanish, but also the terms $\sin n\omega t$, for which n is even.

This latter property is common to every curve whose two halves with respect to the abscissa axis are symmetrical, i.e. when the two halves coincide

when placed one above the other, as in Fig. 171. Or, considering the expressions,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{+\pi}[f(x) + f(x-\pi)\cos n\pi]\cos nx\,dx$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}[f(x) + f(x-\pi)\cos n\pi]\sin nx\,dx,$$

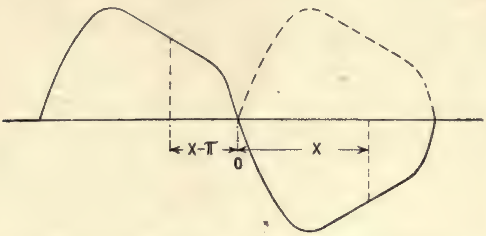


Fig. 171.—Symmetrical Curve with respect to Abscissa-axis.

we see for all even values of n, since $\cos\pi = +1$, that $a_n = 0$ and $b_n = 0$, provided that

$$f(x) = -f(x-\pi).$$

In practice, nearly all curves have this property; hence we can always omit those terms whose frequency is an even multiple of that

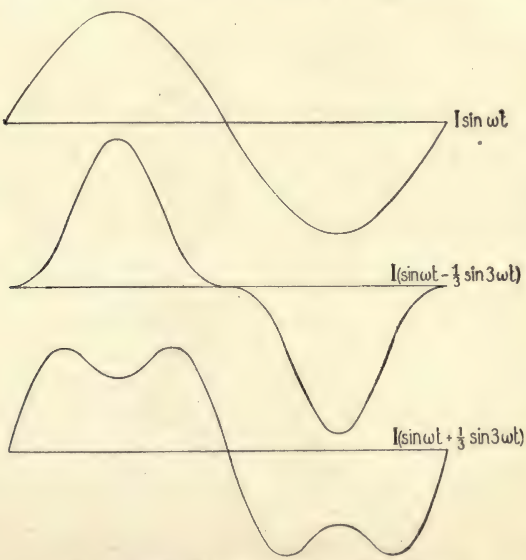


Fig. 172.—Effect of Third Harmonic on Wave-shape.

of the fundamental. An exception is the pressure curve of homopolar machines, which, however, are seldom used.

Considering again the expression

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{+\pi}[f(x)+f(-x)]\cos nx\,dx,$$

we see that a_n is always zero when

$$f(x) = -f(-x),$$

i.e. a_n vanishes, and consequently all the cosine terms of the series vanish, when the pressure curve is symmetrical about the origin.

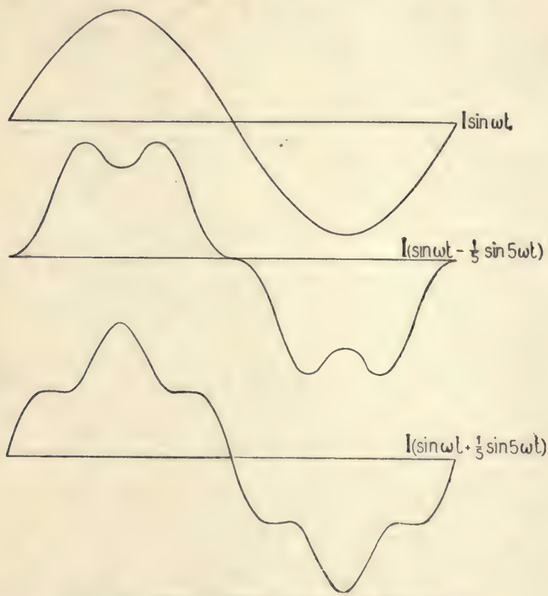


Fig. 173.—Effect of Fifth Harmonic on Wave-shape.

The curves in Figs. 172 and 173 shew the influence of the higher harmonics on the shape of a curve. Pressure curves similar to those shewn in Figs. 172 and 173 occur frequently in practice.

61. Analytic Method for the Determination of the Harmonics of a Periodic Function. If we are given a periodic curve taken by experiment (either by the point by point method or by an oscillograph) and wish to analyse the same, it is not possible as a rule to express it by means of a finite series, so that the above method cannot be used for determining the amplitudes a_n and b_n.

If the curve is taken by the point by point method and $2m$

momentary values have been measured at equidistant points in the whole period 2π, then we start from the equation

$$i = a_1 \cos \omega t + b_1 \sin \omega t + a_3 \cos 3\omega t + b_3 \sin 3\omega t + \ldots$$

and apply the Principle of the Least Squares, whereby the constants a_n and b_n must be so determined that $(i_{\text{calculated}} - i_{\text{measured}})^2$ is a minimum, i.e. we must have

$$\frac{\partial (i_{\text{calculated}} - i_{\text{measured}})^2}{\partial a_n} = 0$$

and $$\frac{\partial (i_{\text{calculated}} - i_{\text{measured}})^2}{\partial b_n} = 0,$$

and we get just as many linear equations as there are unknowns. Denoting the $2m$ measured values by $i_1, i_2, i_3 \ldots i_{2m}$, then

$$a_1 = \frac{2}{m}\left\{ i_1 \cos \frac{2\pi}{2m} + i_2 \cos \frac{4\pi}{2m} + i_3 \cos \frac{6\pi}{2m} + \ldots + i_{m-1} \cos \frac{2(m-1)\pi}{2m} - i_m \right\},$$

$$b_1 = \frac{2}{m}\left\{ i_1 \sin \frac{2\pi}{2m} + i_2 \sin \frac{4\pi}{2m} + i_3 \sin \frac{6\pi}{2m} + \ldots + i_{m-1} \sin \frac{2(m-1)\pi}{2m} \right\},$$

and in general

$$a_n = \frac{2}{m}\left\{ i_1 \cos n\frac{2\pi}{2m} + i_2 \cos n\frac{4\pi}{2m} + i_3 \cos n\frac{6\pi}{2m} + \ldots + i_{m-1} \cos n\frac{2(m-1)\pi}{2m} - i_m \right\},$$

$$b_n = \frac{2}{m}\left\{ i_1 \sin n\frac{2\pi}{2m} + i_2 \sin n\frac{4\pi}{2m} + i_3 \sin n\frac{6\pi}{2m} + \ldots + i_{m-1} \sin n\frac{2(m-1)\pi}{2m} \right\}.$$

In order to impress this method more clearly on the memory, its mathematical derivation can be considered from the following physical conception, more familiar to electrical engineers.

An E.M.F. $e_n = \cos n\omega t$ is induced in a circuit carrying a current represented by the curve

$$i = a_1 \cos \omega t + b_1 \sin \omega t + a_3 \cos 3\omega t + b_3 \sin 3\omega t + \ldots,$$

in which we require to find the n^{th} harmonic of the cosine terms. All the current harmonics must be wattless with the exception of that we are considering (the n^{th}), and the mean power is

$$W_{an} = \tfrac{1}{2} a_n.$$

On the other hand, the mean power is given by

$$W_{an} = \frac{1}{T}\int_0^T e_n i \, dt = \frac{1}{T}\int_0^T i \cos n\omega t \, dt$$

$$= \text{mean value of } (i \cos n\omega t).$$

Hence $$a_n = 2 \times \text{mean value of } (i \cos n\omega t),$$

and similarly, $$b_n = 2 \times \text{mean value of } (i \sin n\omega t).$$

This is the same result as that just obtained in another way.

If we take, for example, $2m = 24$, then the calculation can be carried out in the following tabular form:

Experimentally Determined Momentary Values.	Coefficients for Determining the Amplitudes.							
	a_1	b_1	a_3	b_3	a_5	b_5	a_7	b_7
i_1	0·966	0·259	0·707	0·707	0·259	0·966	−0·259	0·966
i_2	0·866	0·5	0	1·0	−0·866	0·5	−0·866	−0·5
i_3	0·707	0·707	−0·707	0·707	−0·707	−0·707	0·707	−0·707
i_4	0·5	0·866	−1·0	0	0·5	−0·866	0·5	0·866
i_5	0·259	0·966	−0·707	−0·707	0·966	0·259	−0·966	0·259
i_6	0	1·0	0	−1·0	0	1·0	0	−1·0
i_7	−0·259	0·966	0·707	−0·707	−0·966	0·259	0·966	0·259
i_8	−0·5	0·866	1·0	0	−0·5	−0·866	−0·5	0·866
i_9	−0·707	0·707	0·707	0·707	0·707	−0·707	−0·707	−0·707
i_{10}	−0·866	0·5	0	1·0	0·866	0·5	0·866	−0·5
i_{11}	−0·966	0·259	−0·707	0·707	−0·259	0·966	0·259	0·966
i_{12}	−1·0	0	−1·0	0	−1·0	0	−1·0	0

In the first column are the experimentally determined momentary values, taken 15° from one another. In the second column are the cosine values, by which i_1, $i_2 \ldots$ to i_m must be multiplied in order to find a_1; in the third column are the sine values, by which i_1, i_2, etc., must be multiplied in order to find b_1, etc.; in the next columns are the coefficients for determining a_3, b_3, a_5, b_5 and a_7, b_7.

It has here been assumed that the given curve is symmetrical about the abscissa axis, whence $i_1 = -i_{m+1}$, $i_2 = -i_{m+2}$, and so on. If this is not exactly the case, the mean value between i_1 and i_{m+1} must be taken in order to get i_1. Further, for symmetrical curves, the origin can always be chosen so that $i_m = 0$.

62. Graphic Method for the Determination of the Harmonics of a Periodic Function. Instead of the above analytic method, we can also proceed graphically, which is especially convenient when the whole curve, and not only a few points on it, is at hand. An example of such a method is that given by *Houston and Kennelly*, *El. World*, 1898, which depends on the following theorem:

"If an odd number w of half waves of a sine wave are divided into p sections by p vertical lines equidistant from one another, then, when $p > 1$ and p and w have no common factor greater than unity, the sum of the areas in the odd sections equals the sum of the areas in the even sections." In the summation, all surfaces above the zero line are taken as positive and below as negative.

To prove this theorem, divide the abscissa axis of the sine curve

from x to $x+w\pi$ into p equal parts, draw the ordinates through these points, and find the area of each section (see Fig. 174).

$$\int_\alpha^\beta \sin x\,dx = \cos\alpha - \cos\beta.$$

Now find the sums of the areas of the even and uneven sections, and

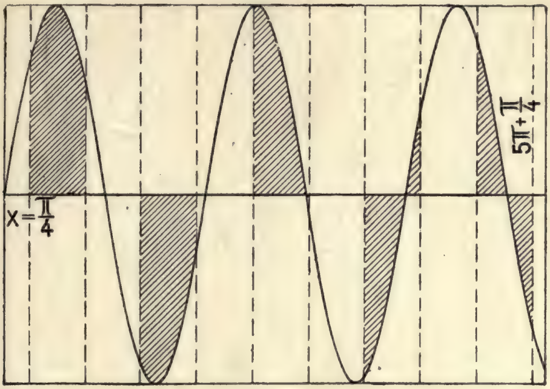


FIG. 174.

equate the difference of these two sums to zero; thus the following expression F must equal zero:

$$\begin{aligned}
F &= \cos x - 2\cos\left(x+\frac{w\pi}{p}\right) + 2\cos\left(x+2\frac{w\pi}{p}\right) \\
&\qquad - \ldots + 2\cos\left\{x+(p-1)\frac{w\pi}{p}\right\} - \cos(x+w\pi) \\
&= \cos x - 2\cos\left(x+\frac{w\pi}{p}\right) + 2\cos\left(x+2\frac{w\pi}{p}\right) \\
&\qquad - \ldots \pm 2\cos\left(x+\frac{p-1}{2}\cdot\frac{w\pi}{p}\right) + \cos x \\
&\qquad - 2\cos\left(x-\frac{w\pi}{p}\right) + 2\cos\left(x-2\frac{w\pi}{p}\right) \\
&\qquad - \ldots \pm 2\cos\left(x-\frac{p-1}{2}\,\frac{w\pi}{p}\right) \\
&= 2\cos x\left\{1 - 2\cos w\frac{\pi}{p} + 2\cos 2w\frac{\pi}{p} - \ldots \pm 2\cos\frac{p-1}{2}\,\frac{w\pi}{p}\right\}.
\end{aligned}$$

Multiply both sides by $\cos\frac{w\pi}{2p}$; then, applying the formula

$$2\cos x\cos y = \cos(x+y) + \cos(x-y),$$

all the terms on the right-hand side except the last cancel out, so that we get

$$F \cos \frac{w\pi}{2p} = 2 \cos x \cos \left(\frac{p-1}{2} + \frac{1}{2}\right) \frac{w\pi}{p}$$

$$= 2 \cos x \cos \frac{w\pi}{2} = 0,$$

and, since p and w have no common factor greater than unity,

$$F = 0.$$

On the other hand, if $w = p$, and we commence to divide the wave at a point where it passes through zero, so that $x = 0$, then

$$F = 2p,$$

i.e. equals p times the area of a half wave, which can also be seen directly from Fig. 174.

From this theorem we get the following rule:

A wave-line, representing graphically a semi-period of an alternating-current, can be expressed by

$$a_1 \cos x + b_1 \sin x + a_3 \cos 3x + b_3 \sin 3x + \dots.$$

In order to find the coefficient b_n of the sine terms, starting from zero, we divide the half wave-length into n equal parts and determine —by some means or other—the difference F between the sums of the even and the odd area-sections.

Then, since F equals the mean ordinate of the sine wave of amplitude b_n times τ, i.e. equals $b_n \cdot \frac{2}{\pi} \tau$, we get

$$b_n = \frac{\pi F}{2\tau},$$

where τ equals half the wave-length of the given wave.

To find the coefficients a_n of the cosine terms, we must again divide the half wave-length into n equal parts, but we must now start at a quarter wave-length from the zero of the n^{th} harmonic, i.e. at $\frac{1}{2n}$ of the interval of the given half wave. In other words, the dividing lines for the coefficients a lie midway between those for the coefficients b. Then, as above, we get from the difference F_1 of the sums,

$$a_n = -\frac{\pi F_1}{2\tau}.$$

This method is not strictly correct, since in the surfaces measured for one harmonic the surfaces of those harmonics are also included whose frequency is a multiple of that of the fundamental. This inaccuracy therefore occurs as soon as we come to the ninth harmonic.

The surfaces can be measured with a planimeter. In order, however, to obtain greater accuracy, the following device may be used: the areas of the given polygons *ABCDA* and *ABCDEA'A* (Fig. 175),

which have to be measured, are divided into equal even and odd sections, which can be omitted without further ado, so that only small surfaces remain to be measured; these are traversed in the proper sense with the planimeter, and the result can at once be read off.

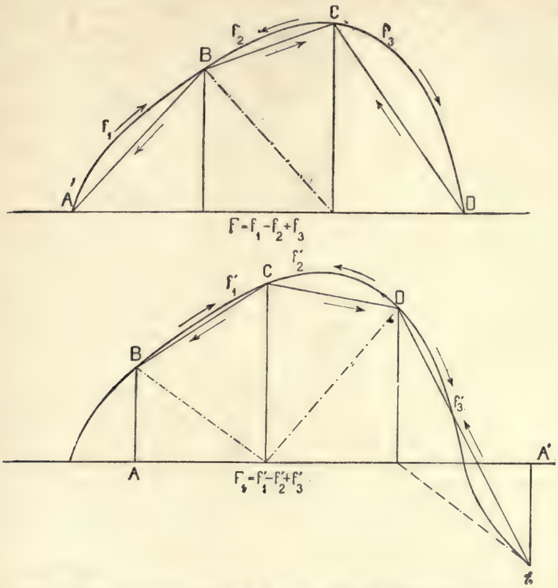


FIG. 175.—Determination of Areas for finding Third Harmonic.

In Fig. 175 the surfaces F and F_1 are obtained directly by means of a planimeter when we trace out the small areas f_1, f_2, f_3 and f_1', f_2', f_3' respectively in the direction indicated by the arrows, since

$$F=f_1-f_2+f_3 \quad \text{and} \quad F_1=f_1'-f_2'+f_3'.$$

After the coefficients a_3, a_5, a_7 ... b_3, b_5, b_7 ... of the harmonics have been found in this way, we can also determine the coefficients a_1 and b_1 of the fundamental, by taking the planimeter over the whole surface, in the one case starting from $x=0$ and in the other $x=\frac{\tau}{2}$. To obtain a_1 and b_1, however, we must not directly substitute the surfaces F and F_1 as measured in the formulae for a_n and b_n, since in addition to the area enclosed by half a wave-length of the fundamental, there is also measured the sum of the areas of all the harmonics within this half wave-length,

$$\sum_3^\infty b_n \frac{2}{\pi}\frac{\tau}{n};$$

consequently,

$$b_1=\frac{\pi F}{2\tau}-\sum_3^\infty \frac{b_n}{n}.$$

Similarly, we get $$a_1 = -\frac{\pi F_1}{2\tau} - \sum_3^\infty \frac{a_n}{n}\cos(n-1)\frac{\pi}{2}$$
$$= -\frac{\pi F_1}{2\tau} + \frac{a_3}{3} - \frac{a_5}{5} + \frac{a_7}{7} - \dots .$$

In Fig. 176 the current curve of a homopolar alternator is shewn. This curve has been analysed by both of the above methods. In the

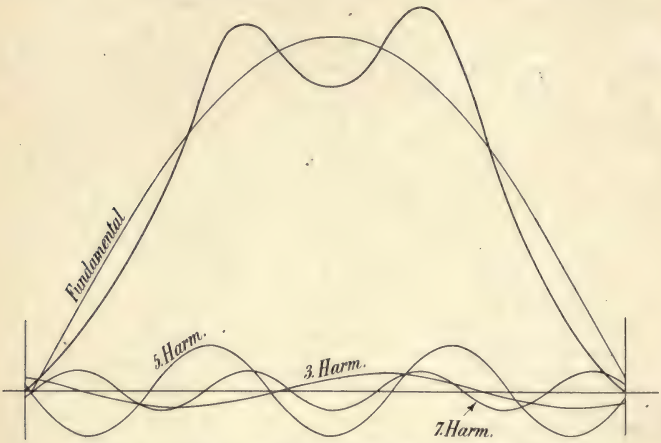


FIG. 176.—Analysis of Experimental Curve into its Harmonics.

analytical method, the distance 2τ, corresponding to 360°, has been divided into 24 parts; thus one division equals $\frac{2\pi}{2m} = 15°$. The equation found in this way is

$$i = -3{\cdot}7\cos\omega t + 99{\cdot}9\sin\omega t + 2{\cdot}96\cos 3\omega t$$
$$-3{\cdot}54\sin 3\omega t + 2{\cdot}57\cos 5\omega t - 12{\cdot}8\sin 5\omega t$$
$$-1{\cdot}73\cos 7\omega t + 5{\cdot}46\sin 7\omega t.$$

These harmonics are also shewn in Fig. 176.

The equation found by the graphical method is approximately the same, thus

$$i = -3{\cdot}82\cos\omega t + 99{\cdot}2\sin\omega t + 2{\cdot}94\cos 3\omega t - 3{\cdot}29\sin 3\omega t$$
$$+2{\cdot}38\cos 5\omega t - 13{\cdot}4\sin 5\omega t - 1{\cdot}98\cos 7\omega t + 5{\cdot}79\sin 7\omega t.$$

We thus see that the latter method is correct within one per cent. of the amplitude of the fundamental wave.

In drawing out the curve of the equation found analytically, the sine and cosine terms of each harmonic have been combined and set off in their proper position with respect to the other waves. The amplitude

i_n and the phase angle ϕ_n of such a combined wave are found as follows:

$$a_n \cos n\omega t + b_n \sin n\omega t = \sqrt{a_n^2 + b_n^2} \sin\left(n\omega t + \tan^{-1}\frac{a_n}{b_n}\right)$$

$$= i_n \sin(n\omega t + \phi_n),$$

where $$a_n = i_n \sin \phi_n$$

and $$b_n = i_n \cos \phi_n.$$

By this means, we get for the equation of the curve (Fig. 176)

$$i = 100 \sin(\omega t + 358°) + 4{\cdot}61 \sin(3\omega t + 140°)$$
$$+ 13{\cdot}05 \sin(5\omega t + 169°) + 5{\cdot}71 \sin(7\omega t + 342°{\cdot}5).$$

63. Alternating-Currents of distorted Wave-Shape. In Chapter II., we saw that when a varying pressure p acts at the terminals of a circuit containing ohmic resistance, self-induction and capacity, we have, from Kirchhoff's Second Law,

$$p = ir + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

or $$\frac{1}{L}\frac{dp}{dt} = \frac{d^2i}{dt^2} + \frac{r}{L}\frac{di}{dt} + \frac{1}{LC}.$$

Further, we saw that, with constant r, L and C, a sinusoidal pressure always produces a sinusoidal current of the same frequency.

Since the pressure equation is linear, the law of superposition can always be applied. And since the pressure always has the same frequency as the current it produces, it is obvious that each pressure harmonic of any pressure wave produces a current at its own frequency, independently of all other harmonics.

Thus, when

$$p = p_1 + p_3 + p_5 + \dots$$
$$= P_{1\,\max} \sin(\omega t + \psi_1) + P_{3\,\max} \sin(3\omega t + \psi_3) + \dots,$$

then

$$i = i_1 + i_3 + i_5 + \dots$$

$$= \frac{P_{1\,\max}}{\sqrt{r^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left\{\omega t + \psi_1 - \tan^{-1}\left(\frac{\omega L}{r} - \frac{1}{\omega Cr}\right)\right\}$$

$$+ \frac{P_{3\,\max}}{\sqrt{r^2 + \left(3\omega L - \frac{1}{3\omega C}\right)^2}} \sin\left\{3\omega t + \psi_3 - \tan^{-1}\left(\frac{3\omega L}{r} - \frac{1}{3\omega Cr}\right)\right\} + \dots$$

$$+ \frac{P_{n\,\max}}{\sqrt{r^2 + \left(n\omega L - \frac{1}{n\omega C}\right)^2}} \sin\left\{n\omega t + \psi_n - \tan^{-1}\left(\frac{n\omega L}{r} - \frac{1}{n\omega Cr}\right)\right\}.$$

Or, we can write

$$i = I_{1\,\max} \sin(\omega t + \psi_1 - \phi_1) + I_{3\,\max} \sin(3\omega t + \psi_3 - \phi_3) + \ldots + I_{n\,\max} \sin(n\omega t + \psi_n - \phi_n), \quad \ldots\ldots\ldots\ldots(112)$$

where the amplitude of the n^{th} current harmonic is

$$I_{n\,\max} = \frac{P_{n\,\max}}{\sqrt{r^2 + \left(n\omega L - \frac{1}{n\omega C}\right)^2}} \quad \ldots\ldots\ldots\ldots(113)$$

and

$$\phi_n = \tan^{-1}\left(\frac{n\omega L}{r} - \frac{1}{n\omega Cr}\right). \quad \ldots\ldots\ldots\ldots(114)$$

The phase displacement ϕ_n of the n^{th} harmonic is positive, zero or negative, according as

$$n\omega L \gtreqless \frac{1}{n\omega C}$$

or

$$n \gtreqless \frac{1}{\omega\sqrt{LC}}.$$

From this we see that each harmonic of the pressure wave produces its own current, and further, from the law of superposition, all these currents are entirely independent of each other.

The amplitudes of the currents do not all bear the same relation to the amplitudes of the pressure harmonics, since the impedance of the n^{th} harmonic

$$\frac{P_{n\,\max}}{I_{n\,\max}} = \sqrt{r^2 + \left(n\omega L - \frac{1}{n\omega C}\right)^2} = z_n \ldots\ldots\ldots\ldots(115)$$

depends on the value of n. Further, the phase displacement ϕ_n is also a function of n, so that resonance cannot occur at the same time with more than one harmonic. Since this phenomenon, however, is frequently due to the higher harmonics, it is not sufficient to consider resonance with regard to the fundamental alone, especially where capacity is present in the systems.

Since the relations between a pressure and its current are different for every harmonic both as regards magnitude and phase, the current curve is, as a rule, quite different in shape from the pressure curve. We will now shortly investigate the influence of r, L and C on the shape of the current curve.

Consider first the simplest case, when the circuit contains only ohmic resistance, then

$$I_{n\,\max} = \frac{P_{n\,\max}}{r} \quad \text{and} \quad \phi_n = 0,$$

i.e. the current curve has exactly the same shape as the pressure curve and is in phase with it. This can also be seen directly from the differential equation, since $p = ir$.

If, on the other hand, the circuit contains both resistance and self-induction, then

$$I_{n\,\max} = \frac{P_{n\,\max}}{\sqrt{r^2 + (n\omega L)^2}} \quad \text{and} \quad \phi_n = \tan^{-1}\frac{n\omega L}{r}.$$

Hence the greater the value of n, the smaller will be $\frac{I_{n\,\max}}{P_{n\,\max}}$ and the greater ϕ_n, that is to say, the higher harmonics are not so pronounced in the current curve as in the pressure curve, when the circuit contains ohmic resistance and self-induction. Thus the self-induction has the effect of making the current curve more nearly a sine wave.

On the contrary, when the circuit contains resistance and capacity, we have

$$I_{n\,\max} = \frac{P_{n\,\max}}{\sqrt{r^2 + \frac{1}{(n\omega C)^2}}} \quad \text{and} \quad \phi_n = \tan^{-1}\left(-\frac{1}{n\omega Cr}\right).$$

The higher harmonics are now more prominent in the current curve than in the pressure curve, and the current curve may become very greatly distorted, when there is sufficient capacity in the circuit.

64. Power yielded by an Alternating-Current of distorted Wave-Shape. The power of an alternating-current of any given wave-shape can be expressed by the rate at which it develops heat in a resistance, thus:

$$\frac{1}{T}\int_0^T i^2 r\,dt.$$

Putting $i = I_{1\,\max}\sin(\omega t + \psi_1 - \phi_1) + I_{3\,\max}\sin(3\omega t + \psi_3 - \phi_3) + \ldots$ and remembering that

$$\int_{-\pi}^{+\pi}\sin mx \sin nx\,dx = \begin{cases} 0 \text{ when } m \gtreqless n, \\ 0 \text{ when } m = n = 0, \\ \pi \text{ when } m = n > 0, \end{cases}$$

we see that, in the integration of $i^2 dt$, only those terms of i^2 which contain a sine squared yield a result differing from zero, and we get

$$\frac{1}{T}\int_0^T i^2 r\,dt = \frac{r}{2}\left\{I^2_{1\,\max} + I^2_{3\,\max} + I^2_{5\,\max} + \ldots\right\}.$$

Putting this power equal to $I^2 r$, as before, we get for the effective current,

$$I = \sqrt{\frac{1}{T}\int_0^T i^2 dt} = \sqrt{\frac{1}{2}(I^2_{1\,\max} + I^2_{3\,\max} + I^2_{5\,\max} + \ldots)}$$

$$= \sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots} \quad \text{.................................(116)}$$

From this it follows that each harmonic of the current curve produces its own heating in the circuit independently of the rest; that is, *the total heating losses in the circuit equal the sum of the heating losses due to the several harmonics.*

Similar to the effective current, we can express the effective pressures:

$$P=\sqrt{\frac{1}{T}\int_0^T p^2 dt}=\sqrt{\frac{1}{2}(P^2_{1\,\max}+P^2_{3\,\max}+P^2_{5\,\max}+\ldots)}$$

$$=\sqrt{P_1^2+P_3^2+P_5^2+\ldots}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(117)$$

Further, we know that the power developed by a current is

$$W=\frac{1}{T}\int_0^T pi\,dt.$$

If we substitute the values of p and i and take the product, then all terms vanish on integration except those containing a sine squared, and we get

$$W=\tfrac{1}{2}\{P_{1\,\max}I_{1\,\max}\cos\phi_1+P_{3\,\max}I_{3\,\max}\cos\phi_3+\ldots\}$$

$$=P_1I_1\cos\phi_1+P_3I_3\cos\phi_3+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(118)$$

We thus see that, with regard to power, all the harmonics are independent of one another,—each produces power for itself, whilst the current of one harmonic produces no effect with the pressure of another. *The current of any harmonic is wattless with respect to the pressures of other harmonics.*

We have now seen that all harmonics are in every respect independent of one another, and the total power is obtained by the summation of the powers of the several harmonics.

Thus, each harmonic can be treated separately by itself, and can have all the laws and graphic constructions which have already been deduced applied to it.

If we have a problem for a pressure curve of given shape, we analyse this curve into its harmonics, and treat each harmonic by itself as in the previous examples. In this way we find the current and power of the harmonics, whence we get the effective current, the total power and the efficiency. In many problems where graphic methods are used, it is possible to use with advantage separate parts of the figure for each harmonic.

65. Effect of Wave-Shape on Measurements.

I. *Measurement of Induction Coefficients.* In practice it is often required to find the coefficient of self-induction of a circuit of comparatively negligible resistance. This is usually done by sending an alternating-current through the circuit and measuring the effective pressure and current. Since, however, we have not always a sinusoidal pressure at our disposal, it is of interest to investigate whether the coefficient of self-induction can be determined from these two measurements with sufficient accuracy when the pressure curve contains harmonics.

If $$p = P_{1\max} \sin(\omega t + \psi_1) + P_{3\max} \sin(3\omega t + \psi_3) + \dots,$$

then $$i = \frac{P_{1\max}}{\omega L} \sin\left(\omega t + \psi_1 - \frac{\pi}{2}\right) + \frac{P_{3\max}}{3\omega L} \sin\left(3\omega t + \psi_3 - \frac{3\pi}{2}\right) + \dots.$$

The effective values are then

$$P = \sqrt{P_1^2 + P_3^2 + \dots}$$

and $$I = \frac{1}{\omega L} \sqrt{P_1^2 + \frac{1}{9} P_3^2 + \frac{1}{25} P_5^2 + \dots}.$$

By division, we get

$$L = \frac{P}{\omega I} \sqrt{\frac{1 + \frac{1}{9}\frac{P_3^2}{P_1^2} + \frac{1}{25}\frac{P_5^2}{P_1^2} + \dots}{1 + \frac{P_3^2}{P_1^2} + \frac{P_5^2}{P_1^2} + \dots}} \quad \text{.........(119)}$$

From this formula we see that the pressure harmonics P_3, P_5, etc., must be very large in proportion to the fundamental P_1 for the root to differ appreciably from unity. Hence, for practical purposes, it is generally sufficiently exact if we calculate the coefficient of self-induction L from the effective pressure and current as measured, thus:

$$L = \frac{P}{\omega I},$$

neglecting the shape of the pressure curve. If, for example, this curve has a third harmonic, whose amplitude equals a third of that of the fundamental wave, then the quantity under the root is 0·96. Thus the error introduced by neglecting the correcting factor is but 4 %.

If the ohmic resistance of the circuit is not negligible, for the above formula (119) we must substitute formula (124), given on p. 221.

II. *Measurement of Capacity.* (*a*) An analogous problem, namely the determination of the capacity of a circuit of low ohmic resistance by measurement of the effective pressure and current, may, on the other hand, give results which are far from exact, when the pressure curve deviates greatly from a sine wave.

If $$p = P_{1\max} \sin(\omega t + \psi_1) + P_{3\max} \sin(3\omega t + \psi_3) + \dots,$$

then $$i = C\omega P_{1\max} \sin\left(\omega t + \psi_1 + \frac{\pi}{2}\right)$$
$$+ 3C\omega P_{3\max} \sin\left(3\omega t + \psi_3 + \frac{3\pi}{2}\right) + \dots.$$

The effective values are

$$P = \sqrt{P_1^2 + P_3^2 + \dots}$$

and $$I = \omega C \sqrt{P_1^2 + 9P_3^2 + 25P_5^2 + \dots},$$

whence, by division,

$$C=\frac{I}{\omega P}\sqrt{\frac{1+\left(\frac{P_3}{P_1}\right)^2+\left(\frac{P_5}{P_1}\right)^2+\ldots}{1+9\left(\frac{P_3}{P_1}\right)^2+25\left(\frac{P_5}{P_1}\right)^2+\ldots}}. \quad \ldots\ldots\ldots\ldots(120)$$

Instead of the factors $\frac{1}{3^2}$, $\frac{1}{5^2}$, $\frac{1}{7^2}$ we have now the factors 3^2, 5^2, 7^2, ... under the root, which strongly affect the influence of the higher harmonics on the readings.

For example, if $P_3=\frac{1}{3}P_1$, then

$$C=\frac{I}{\omega P}\sqrt{0{\cdot}555}=\frac{I}{\omega P}\,0{\cdot}75 \text{ and not } \frac{I}{\omega P}.$$

In this case therefore it is not sufficient to merely know the effective values of the pressure and current, but the curve shape must also be taken into account.

(*b*) This, however, can be easily avoided in the following manner. In series with the capacity we connect an induction coil L and a large resistance R, as in Fig. 177. The induction coil must be free from iron and have sufficient stops to enable us to regulate its self-induction, so that its reactance $x_s=\omega L$ approximately equals the capacity reactance $x_c=\frac{1}{\omega C}$. We now vary the number of turns in the induction coil until the pressure P is practically zero. When minimum pressure occurs, then we know that resonance is present, whence

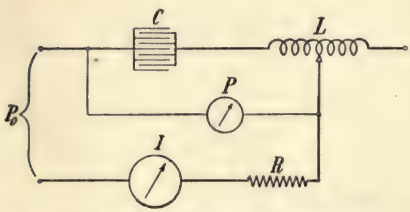


FIG. 177.—Connections for measuring Capacity.

$$\frac{1}{\omega C}=\omega L=x_s.$$

Of course, care must be taken that the resonance is due to the fundamental and not to a higher harmonic. We then measure the coefficient of self-induction L for this number of turns without the resistance and the capacity in circuit, and we get, with fair exactitude,

$$L=\frac{P_s}{\omega I_s},$$

and from this

$$C=\frac{1}{\omega^2 L}=\frac{I_s}{\omega P_s}.$$

By this means we eliminate all the disturbing influences of higher harmonics in capacity measurements.

66. Resonance with Currents of distorted Wave-Shape. If a pressure wave containing several higher harmonics acts on a circuit, partial resonance will exist under several conditions. This can be best illustrated by an example. Consider the circuit shewn in Fig. 178, which contains chiefly inductance and capacity; partial resonance will be caused by the wave of frequency c, when the self-induction is such that

$$L=\frac{1}{n^2\omega^2 C}.$$

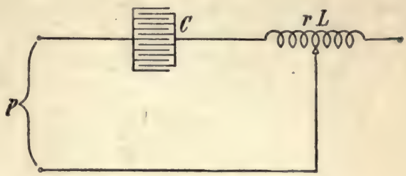


FIG. 178.

Hence, if we vary L and plot the effective current I as function of the coefficient of self-induction L, a wave-shaped curve is obtained, as in Fig. 179, which is often called the *resonance curve.* The curve shewn is drawn for a pressure curve having the equation

$$p=100\sin(\omega t+\psi_1)+30\sin(3\omega t+\psi_3)$$
$$+15\sin(5\omega t+\psi_5)+20\sin(7\omega t+\psi_7).$$

The frequency of the fundamental is $c=50$, the resistance $r=5$ ohms, the capacity $C=50$ microfarads, while the inductance L was varied

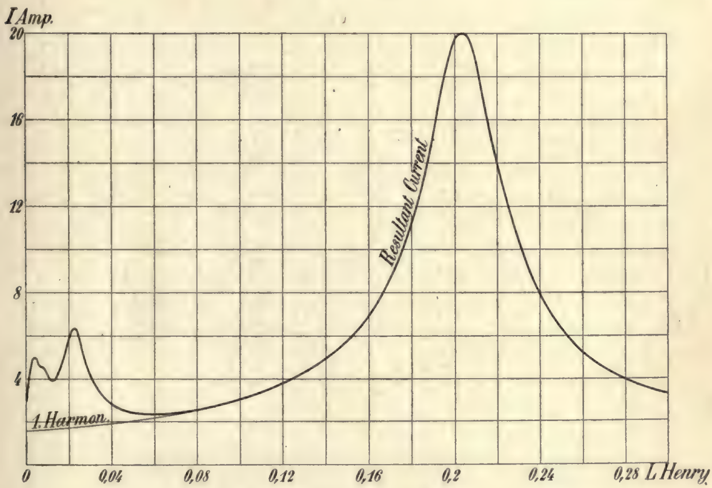


FIG. 179.—Resonance Curve.

from 0 to 0·3 henry. The maxima of the effective current occur at the different values of L for which resonance is present. The last and greatest maximum is given when resonance is due to the fundamental, the next to the third harmonic, and so on.

Fig. 180 shews the several current harmonics plotted as functions of the inductance L. In order to shew the effect of the higher harmonics more clearly, the scale has been made larger than that of Fig. 179. We see that the maxima of the several current harmonics, which occur at resonance, are related to each other in the same way as the amplitude of the pressure harmonics. The curve of *resultant current* is obtained by geometric addition of the harmonics. With a larger inductance this curve almost coincides with the fundamental. With a low inductance,

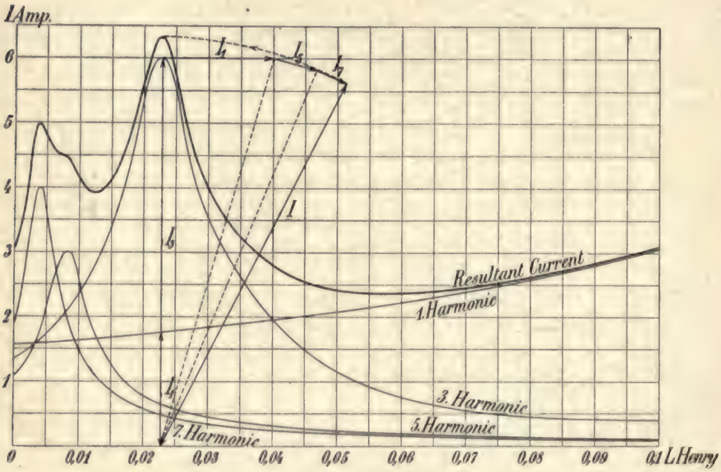


FIG. 180.—Resonance Curve.

however, it remains higher than this and also higher than the harmonics. The angles ψ, by which the harmonics are displaced from the fundamental, clearly have no effect on the resonance curve.

It is, however, also interesting to see how one current curve passes into the other as the inductance of the choking coil is altered. We shall therefore consider analytically the case when

$$n = \frac{1}{\omega\sqrt{LC}}$$

is an *even* number. This condition lies directly midway between two resonance conditions, viz. between that due to the $(n-1)^{\text{th}}$ and that due to the $(n+1)^{\text{th}}$ harmonic, for n, being even, can only represent a transient stage and not an actual harmonic. The prevailing current will therefore be

$$i_{n-1} + i_{n+1} = I_{(n-1)\max} \sin\{(n-1)\omega t + \psi_a\} \\ + I_{(n+1)\max} \sin\{(n+1)\omega t + \psi_b\}.$$

Assume $I_{n-1} = I_{n+1} = I_n$, then the current i_n can be written

$$i_n = i_{n-1} + i_{n+1} = 2I_{n\,\max} \sin\left(n\omega t + \frac{\psi_a + \psi_b}{2}\right) \cos\left(n\omega t - \frac{\psi_a - \psi_b}{2}\right).$$

This current is drawn out in Fig. 181, for $\psi_a = \psi_b = 0$ and $n = 4$.

As is seen, it forms a sine curve whose frequency is a mean of those of the two currents and whose amplitude varies after a sine wave. The higher the periodicity of the harmonic, the more periods we get

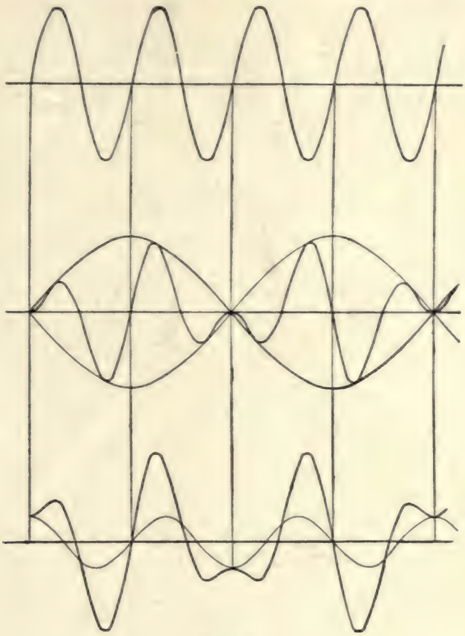


Fig. 181.

for every period of the main current. Hence, by the interference of two neighbouring harmonics, a current is produced, which possesses the same character as currents caused by surging.

If the amplitudes I_{n-1} and I_{n+1} are not equal, we still get a current whose mean periodicity is n. The amplitude of this current, however, does not vary between zero and a maximum, but only between a minimum and a maximum value, as seen from Fig. 181.

From the foregoing it is obvious that we cannot regard all pulsations, such as those represented in Fig. 181, as surging between free and forced oscillations.

B. Strasser and *J. Zenneck*,* who were the first to draw attention to

* *Annalen der Physik*, Bd. 20, p. 759.

these even harmonics, suggest that the same should be treated as individual currents. They substitute for a large number of the uneven harmonics, even harmonics which change their direction at every half period of the fundamental. Such harmonics are shewn in Fig. 182. By considering the field curve (Figs. 167*a* and *b*) of a generator on no-load and on non-inductive load, it is easy to see that the distorted part of this field—due to armature reaction—induces even harmonics in the stator winding. The armature reaction is obtained by subtracting the two curves (Fig. 167*a* and *b*), and the curve thus found is very similar to the second harmonic in Fig. 182, while the field curve in Fig. 167*b* is itself very like the curve in Fig. 182 printed with a heavy line. B. Strasser and J. Zenneck call these harmonics *phase-changing*, since they alter their phase by 180° every half period of the fundamental. Since, however, it is not easy to treat phase-changing currents and pressures analytically, we shall not pursue this method of representation further. All such phenomena can be quite well explained by means of odd higher harmonics.

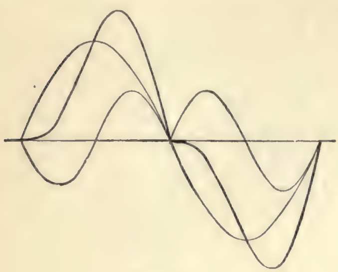


FIG. 182.

67. Form Factor, Crest Factor and Curve Factor of an Alternating-Current. Since the effective value of a periodic current or pressure

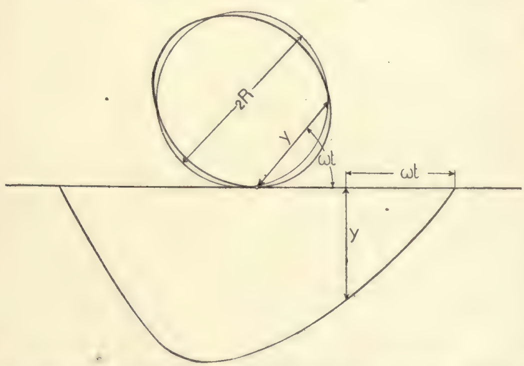


FIG. 183.—Construction for finding Effective Value of Periodic Curve (Fleming).

is often required, and it is a round-about way to first analyse the given curve into its harmonics, we shall now give a method (due to *Fleming*) by means of which the effective value of a periodic function can be determined directly.

For example, find the effective value of the curve given in Fig. 183.

Take some point on the abscissa axis as origin for the polar diagram of this curve. The area of the polar curve is then

$$\int_0^\pi \frac{y^2}{2}\,d(\omega t)=\frac{\pi}{T}\int_0^{\frac{T}{2}} y^2\,dt,$$

where y is the ordinate of the periodic curve.

Now draw a circle whose area equals that of the polar diagram, and denote the radius of this circle by R; then

$$R^2\pi=\frac{\pi}{T}\int_0^{\frac{T}{2}} y^2 dt$$

or

$$\sqrt{2}R=\sqrt{\frac{2}{T}\int_0^{\frac{T}{2}} y^2 dt}$$

= effective value of the curve.

The polar diagram of a sine wave is a circle; other periodic curves give other polar curves, which are more or less similar to circles. The circle of the same area as the polar curve can easily be estimated by the eye, when a planimeter is not available. By this means we have a simple method for approximately finding the effective value of any periodic curve.

The ratio between the effective value of a periodic curve and the mean value is often needed, and is known as the *form factor*, since it depends on the form of the curve. The more peaked the curve is, the larger is the form factor. For a pressure curve, the *form factor* is

$$f_e=\frac{\sqrt{\frac{2}{T}\int_0^{\frac{T}{2}} p^2 dt}}{\frac{2}{T}\int_0^{\frac{T}{2}} p\,dt}. \qquad (121)$$

For the pressure curves (Figs. 169, 170 and 172a) the form factors are 1·0, 1·15 and 1·11. The form factor of a sine curve is

$$\frac{1}{\sqrt{2}}\div\frac{2}{\pi}=\frac{\pi}{2\sqrt{2}}=1{\cdot}11.$$

Another characteristic factor which is met with now and again in technical literature is the *crest factor* f_s,* which denotes the ratio of the maximum to the effective value. This is only of interest for pressure curves—serving as a measure for the strain put on the insulation. The maximum value of currents and pressures of given curve-shape, on the other hand, has no direct relation to the iron and

* As suggested by Prof. G. Kapp.

copper losses in electromagnetic apparatus, and has therefore only limited importance in practice.

$$f_s=\frac{\text{maximum value}}{\text{effective value}}=\frac{P_{\max}}{\sqrt{\frac{2}{T}\int_0^{\frac{T}{2}}p^2dt}}$$

and equals $\sqrt{2}$ for sine waves.

A third factor, which is of especial importance for motors, is the *curve factor*

$$\sigma_p=\frac{\text{effective value}}{\text{amplitude of fundamental}}=\frac{P}{P_1}$$

$$=\sqrt{1+\left(\frac{P_3}{P_1}\right)^2+\left(\frac{P_5}{P_1}\right)^2+\dots}.$$

Since only the fundamental of the pressure wave causes the effective transmission of power from the stator to the rotary field, the load capacity of a motor depends chiefly on the fundamental pressure

$$P_1=\frac{P}{\sigma_p}.$$

Hence the importance of this factor.

CHAPTER XII.

GRAPHIC REPRESENTATION OF ALTERNATING-CURRENTS OF DISTORTED WAVE-SHAPE.

68. The Equivalent Sine Wave and the Power Factor. It would be possible, as already shewn, to represent graphically each one of the harmonics by itself. Since, however, such a representation is not very convenient, it is simpler to proceed as with the power diagrams and set off the apparent power PI at angle ϕ to the ordinate axis, so that the ordinate equals the power $PI\cos\phi$. $\cos\phi$ is called the *power factor*. This diagram can be drawn to any desired accuracy when the pressure, current and power are known.

In the previous load diagram (Ch. I. Sect. 12) the current and pressure waves were sinusoidal; in this case, however, the waves may have any shape whatever; thus ϕ is not the actual phase displacement, but only imaginary, being the angle between the sinusoidal pressure and current, which are equivalent to the actual pressure and the actual current with respect to effective values, and yielding, therefore, the same power. This imaginary sinusoidal wave is called the *equivalent sine wave*; and it is with this that we usually have to deal in practice. For most practical purposes this is sufficiently exact, but in exceptional cases, e.g. with condensers or with strongly-distorted pressure waves (i.e. pressure waves which deviate strongly from a sine wave), this method of calculation is inexact.

We will first examine what the actual significance of the power factor $\cos\phi$ is. The power is

$$W = PI\cos\phi = I^2 r,$$

where r is the effective resistance of the circuit; hence

$$\cos\phi = \frac{Ir}{P}$$

$$= r\frac{\sqrt{\dfrac{P_1^2}{r^2+\left(\omega L-\dfrac{1}{\omega C}\right)^2}+\dfrac{P_3^2}{r^2+\left(3\omega L-\dfrac{1}{3\omega C}\right)^2}+\dots}}{\sqrt{P_1^2+P_3^2+\dots}}$$

or
$$\cos\phi = \sqrt{\frac{\cos^2\phi_1 + \left(\frac{P_3}{P_1}\right)^2\cos^2\phi_3 + \left(\frac{P_5}{P_1}\right)^2\cos^2\phi_5 + \,..}{1 + \left(\frac{P_3}{P_1}\right)^2 + \left(\frac{P_5}{P_1}\right)^2 + \dots}}, \quad \dots\dots(122)$$

where ϕ_1, ϕ_3, ϕ_5, etc., as above, are the phase-displacement angles of the several harmonics.

Since
$$r = \frac{P_1\cos\phi_1}{I_1},$$

we can also write
$$\cos\phi = \cos\phi_1\frac{P_1 I}{P I_1}. \quad \dots\dots\dots\dots\dots\dots\dots\dots(122a)$$

Both formulae (122) and (122a) have been deduced on the assumption that the effective resistance r is independent of the frequency; this is generally true, but not always.

Let the effective resistance for the fundamental be r_1, for the third harmonic r_3, for the fifth r_5, and so on; then, in this case, we get

$$\cos\phi = \frac{I_1^2 r_1 + I_3^2 r_3 + \dots}{PI}.$$

Further, from formula (122), we have

$$\sin\phi = \sqrt{1 - \cos^2\phi}$$
$$= \sqrt{\frac{\sin^2\phi_1 + \left(\frac{P_3}{P_1}\right)^2\sin^2\phi_3 + \dots}{1 + \left(\frac{P_3}{P_1}\right)^2 + \dots}} \quad \dots\dots\dots\dots\dots\dots(123)$$

and
$$P\sin\phi = \sqrt{P_1^2\sin^2\phi_1 + P_3^2\sin^2\phi_3 + P_5^2\sin^2\phi_5 + \dots},$$

or, since
$$P_1\sin\phi_1 = xI_1,$$
$$P_3\sin\phi_3 = 3xI_3,$$
$$P_5\sin\phi_5 = 5xI_5,$$

then
$$\sin\phi = \sin\phi_1\frac{P_1}{PI_1}\sqrt{I_1^2 + 9I_3^2 + 25I_5^2 + \dots}.$$

This formula is deduced on the assumption that r remains constant for all harmonics, and that the reactance rises proportionally with the frequency.

It now remains to be seen how great is the error introduced in the experimental determination of the effective resistance and effective reactance of an *inductive* circuit by using a distorted pressure curve, when we calculate with the equivalent sine waves.

The power supplied to the circuit through which the effective current I flows is always
$$W = I^2 r,$$

when the effective resistance r is independent of the frequency; in this case, therefore, the determination of r is independent of the curve-

shape. This, however, is not the case with the effective reactance x_s; for each harmonic of the terminal pressure

$$P=\sqrt{P_1^2+P_3^2+P_5^2+\dots}$$

produces a current with its own frequency. Thus:

$$I_1=\frac{P_1}{\sqrt{r^2+x_s^2}},\quad I_3=\frac{P_3}{\sqrt{r^2+(3x_s)^2}},\dots$$

if the reactance x_s is proportional to the frequency.

$$I=\sqrt{I_1^2+I_3^2+I_5^2+\dots}$$

$$=\frac{1}{x_s}\sqrt{\frac{P_1^2x_s^2}{r^2+x_s^2}+\frac{\frac{1}{9}P_3^2\,.\,9x_s^2}{r^2+9x_s^2}+\dots}$$

$$=\frac{1}{x_s}\sqrt{P_1^2\sin^2\phi_1+\tfrac{1}{9}P_3^2\sin^2\phi_3+\dots}.$$

But, $$P\sin\phi=\sqrt{P_1^2\sin^2\phi_1+P_3^2\sin^2\phi_3+\dots}.$$

Combining these two last expressions, we get

$$x_s=\frac{P\sin\phi}{I}\sqrt{\frac{P_1^2\sin^2\phi_1+\frac{1}{9}P_3^2\sin^2\phi_3+\frac{1}{25}P_5^2\sin^2\phi_5+\dots}{P_1^2\sin^2\phi_1+P_3^2\sin^2\phi_3+P_5^2\sin^2\phi_5+\dots}}. \quad \dots.(124)$$

Generally the harmonics of the pressure curve are not known, neither are the constants r and x_s of the circuit in question; consequently we disregard the shape of the curve and calculate with the equivalent values. We then have

$$x_s=\frac{P\sin\phi}{I},$$

and introduce a small error by assuming the root equals unity. This root is always somewhat less than unity, so that the approximate formula already gives x somewhat too large. The error, however, is not large; for example, for the strongly distorted pressure curve $P_1=100$, $P_3=10$, $P_5=31{\cdot}65$, the root equals 0·943 when $\frac{x}{r}=1{\cdot}5$, and 0·948 when $\frac{x}{r}=2{\cdot}5$, i.e. the error in this case is but 5 %.

If the circuit has no inductance, but only resistance and capacity, then the capacity reactance will be

$$x_c=\frac{P\sin\phi}{I}\sqrt{\frac{P_1^2\sin^2\phi_1+9P_3^2\sin^2\phi_3+25P_5^2\sin^2\phi_5+\dots}{P_1^2\sin^2\phi_1+P_3^2\sin^2\phi_3+P_5^2\sin^2\phi_5+\dots}}, \quad \dots.(125)$$

so that the root is not approximately unity in this case.

69. The Induction Factor. In the previous load diagrams (Chap. I.), the abscissa $PI\sin\phi$ represented the so-called imaginary power. When harmonics are present, however, the matter is somewhat

different, for if we take the sum of the imaginary powers of all the harmonics, i.e.
$$W_j = P_1 I_1 \sin\phi_1 + P_3 I_3 \sin\phi_3 + \ldots,$$
this is not equal to $PI\sin\phi$, but is always smaller, as will now be shewn.

Since
$$\tan\phi_n = \frac{n\omega L}{r} - \frac{1}{n\omega Cr} = \frac{x_n}{r},$$
where x_n is the reactance of the n^{th} harmonic, then
$$\cos\phi_n = \frac{1}{\sqrt{1+\tan^2\phi_n}} = \frac{r}{\sqrt{r^2+x_n^2}} = \frac{rI_n}{P_n},$$
so that
$$W_j = \frac{1}{r}(P_1^2 \sin\phi_1 \cos\phi_1 + P_3^2 \sin\phi_3 \cos\phi_3 + \ldots).$$

From the formula for $\sin\phi$, we get
$$PI\sin\phi = \frac{1}{r}\sqrt{P_1^2\sin^2\phi_1 + P_3^2\sin^2\phi_3 + \ldots}\,.\,\sqrt{P_1^2\cos^2\phi_1 + P_3^2\cos^2\phi_3 + \ldots},$$

Hence $f = \dfrac{W_j}{PI\sin\phi}$
$$= \frac{P_1\sin\phi_1 P_1\cos\phi_1 + P_3\sin\phi_3 P_3\cos\phi_3 + \ldots}{\sqrt{P_1^2\sin^2\phi_1 + P_3^2\sin^2\phi_3 + \ldots}\,.\,\sqrt{P_1^2\cos^2\phi_1 + P_3^2\cos^2\phi_3 + \ldots}}. \quad \ldots(126)$$

Again, since $\quad P_3\sin\phi_3 = 3xI_3, \quad P_5\sin\phi_5 = 5xI_5 \ldots,$

then
$$f\sin\phi = \frac{W_j}{PI} = \frac{P_1I_1\sin\phi_1 + P_3I_3\sin\phi_3 + \ldots}{PI}$$
$$= \sin\phi_1 \frac{P_1}{PI_1} \frac{I_1^2 + 3I_3^2 + 5I_5^2 + \ldots}{I},$$
and since
$$\sin\phi = \sin\phi_1 \frac{P_1}{PI_1}\sqrt{I_1^2 + 9I_3^2 + 25I_5^2 + \ldots},$$
then f will also equal
$$f = \frac{W_j}{PI\sin\phi} = \frac{I_1^2 + 3I_3^2 + 5I_5^2 + \ldots}{I\sqrt{I_1^2 + 9I_3^2 + 25I_5^2 + \ldots}}. \quad \ldots\ldots\ldots\ldots(126a)$$

If the circuit is non-inductive and contains only resistance and capacity, the reactances of the several harmonics will be
$$x, \quad \frac{x}{3}, \quad \frac{x}{5}, \text{ etc.},$$
and we shall get in this case
$$f = \frac{W_j}{PI\sin\phi} = \frac{I_1^2 + \dfrac{I_3^2}{3} + \dfrac{I_5^2}{5} + \ldots}{I\sqrt{I_1^2 + \dfrac{I_3^2}{9} + \dfrac{I_5^2}{25} + \ldots}}. \quad \ldots\ldots\ldots\ldots\ldots(126b)$$

This factor f is always less than unity.

Consider the sum of the real powers of all the harmonics, then, by the definition of power factor, this must equal the actual power $PI \cos \phi$, which is also the case when we work the same out. We thus see that the *power factor* is

$$\cos \phi = \frac{W}{PI}, \quad \text{.............................(127)}$$

and that $$\frac{W_j}{PI} = f \sin \phi < \sin \phi. \quad \text{.........................(128)}$$

$f \sin \phi$ is a characteristic of an electric circuit, and is called the *induction factor*.

This factor, however, has only significance with sinusoidal currents in graphical representation, because in this case it equals $\sin \phi$, since $f = 1$.

70. Graphic Summation of Equivalent Sine Waves. If we have several circuits acted on by the terminal pressures P_{I}, P_{II} and P_{III} producing the effective currents I_{I}, I_{II} and I_{III}, the apparent powers $P_{\mathrm{I}}I_{\mathrm{I}}$, $P_{\mathrm{II}}I_{\mathrm{II}}$ and $P_{\mathrm{III}}I_{\mathrm{III}}$ can be set off in the power diagram at angles ϕ_{I}, ϕ_{II}, ϕ_{III} to the ordinate axis, so that the ordinates of these vectors represent the true powers W_{I}, W_{II} and W_{III}. Now arises the question: *Is it always allowable to sum up these power vectors graphically?* It will be found that it is only permissible in certain cases, as we shall now proceed to shew.

The ordinate of each vector represents the true power in its respective circuit, hence the algebraic sum W of the three ordinates

$$W_{\mathrm{I}} = P_{\mathrm{I}} I_{\mathrm{I}} \cos \phi_{\mathrm{I}},$$
$$W_{\mathrm{II}} = P_{\mathrm{II}} I_{\mathrm{II}} \cos \phi_{\mathrm{II}},$$
$$W_{\mathrm{III}} = P_{\mathrm{III}} I_{\mathrm{III}} \cos \phi_{\mathrm{III}}$$

must represent the true power in the three circuits. The same result is obtained by calculation, based on the fact that the imaginary power W_j in the three circuits equals the algebraic sum of the several imaginary powers $W_{\mathrm{I}j}$, $W_{\mathrm{II}j}$ and $W_{\mathrm{III}j}$.

We thus have

$$\begin{aligned} W &= W_{\mathrm{I}} + W_{\mathrm{II}} + W_{\mathrm{III}} \\ &= P_{\mathrm{I}} I_{\mathrm{I}} \cos \phi_{\mathrm{I}} + P_{\mathrm{II}} I_{\mathrm{II}} \cos \phi_{\mathrm{II}} + P_{\mathrm{III}} I_{\mathrm{III}} \cos \phi_{\mathrm{III}} \end{aligned}$$

and $$\begin{aligned} fPI \sin \phi = W_j &= W_{\mathrm{I}j} + W_{\mathrm{II}j} + W_{\mathrm{III}j} \\ &= f_{\mathrm{I}} P_{\mathrm{I}} I_{\mathrm{I}} \sin \phi_{\mathrm{I}} + f_{\mathrm{II}} P_{\mathrm{II}} I_{\mathrm{II}} \sin \phi_{\mathrm{II}} + f_{\mathrm{III}} P_{\mathrm{III}} I_{\mathrm{III}} \sin \phi_{\mathrm{III}}. \end{aligned}$$

If the geometric summation of power vectors is allowable, the following two relations must hold:

$$W = PI \cos \phi = P_{\mathrm{I}} I_{\mathrm{I}} \cos \phi_{\mathrm{I}} + P_{\mathrm{II}} I_{\mathrm{II}} \cos \phi_{\mathrm{II}} + P_{\mathrm{III}} I_{\mathrm{III}} \cos \phi_{\mathrm{III}}$$

and $$\frac{1}{f} W_j = PI \sin \phi = P_{\mathrm{I}} I_{\mathrm{I}} \sin \phi_{\mathrm{I}} + P_{\mathrm{II}} I_{\mathrm{II}} \sin \phi_{\mathrm{II}} + P_{\mathrm{III}} I_{\mathrm{III}} \sin \phi_{\mathrm{III}}.$$

It is at once seen, that the first of these equations is identical with the first of the two previous equations and is thus satisfied; on the other hand, the other two equations—viz. that for the imaginary powers and that for the abscissae of the power vectors—do not always agree, and we thus see that it is only allowable to add the power vectors graphically when

$$P_{\mathrm{I}}I_{\mathrm{I}}\sin\phi_{\mathrm{I}}+P_{\mathrm{II}}I_{\mathrm{II}}\sin\phi_{\mathrm{II}}+P_{\mathrm{III}}I_{\mathrm{III}}\sin\phi_{\mathrm{III}}$$
$$=PI\sin\phi=\frac{f_{\mathrm{I}}}{f}P_{\mathrm{I}}I_{\mathrm{I}}\sin\phi_{\mathrm{I}}+\frac{f_{\mathrm{II}}}{f}P_{\mathrm{II}}I_{\mathrm{II}}\sin\phi_{\mathrm{II}}+\frac{f_{\mathrm{III}}}{f}P_{\mathrm{III}}I_{\mathrm{III}}\sin\phi_{\mathrm{III}}.$$

Thus the general condition for which it is allowable to add power vectors graphically is

$$(f-f_{\mathrm{I}})P_{\mathrm{I}}I_{\mathrm{I}}\sin\phi_{\mathrm{I}}+(f-f_{\mathrm{II}})P_{\mathrm{II}}I_{\mathrm{II}}\sin\phi_{\mathrm{II}}+(f-f_{\mathrm{III}})P_{\mathrm{III}}I_{\mathrm{III}}\sin\phi_{\mathrm{III}}=0. \quad \text{......(129)}$$

The general solution of this problem has, however, less interest than the treatment of the two cases for which all the P's are equal when the three circuits are joined in parallel and all the I's are equal when the three circuits are joined in series. We then get, on the one hand, the condition for which it is allowable to geometrically add effective currents without considering the wave-shape, and on the other hand the condition for which it is allowable to geometrically add effective pressures, likewise neglecting the wave-shape. That which holds for the first case, however, does not equally well apply to the second; consequently the two cases must be treated separately.

First consider the case of circuits of any kind connected in *series*. If the current I is constant throughout the whole circuit, we can write the condition for the geometric addition of power vectors as follows:

$$(f-f_{\mathrm{I}})P_{\mathrm{I}}\sin\phi_{\mathrm{I}}+(f-f_{\mathrm{II}})P_{\mathrm{II}}\sin\phi_{\mathrm{II}}+(f-f_{\mathrm{III}})P_{\mathrm{III}}\sin\phi_{\mathrm{III}}=0.$$

This equation at the same time gives the condition for which it is permissible to graphically add pressure vectors, when the circuits on which these pressures act are in series. We shall not enter further into this general problem, but merely consider the case for which it can be directly seen that the above condition is satisfied. This is the case when

$$f=f_{\mathrm{I}}=f_{\mathrm{II}}=f_{\mathrm{III}},$$

and this is *first* the case when the ratio between r, L and C are the same for the three circuits.

Three such circuits can be called similar, since their diagrams are always similar. That it is allowable to geometrically add the vectors in this case, which make the same angle ϕ with the ordinate axis, can be seen without further demonstration.

The *second* case when $f=f_{\mathrm{I}}=f_{\mathrm{II}}=f_{\mathrm{III}}$ (Form. 126a), when the same current I flows through the whole circuit, occurs when r is independent of the frequency and the reactance x is the same function of the frequency for all the circuits. This is the case, for example, when

all the x's are proportional to the frequency or when they all vary inversely as the frequency.

A special instance of this second case, in which geometrical addition is also possible, is that for which the reactance of all parts of the circuit except one is zero; we then have, obviously, $f = f_x$, so that

$$I \sin \phi = I_x \sin \phi_x,$$

where I_x and ϕ_x relate to the x^{th} circuit.

As an example of this special case, we can take the diagram of a generator working on a non-inductive circuit. Here we have two pressures which are to be geometrically added, of which the one—the terminal pressure—is in phase with the current, whilst the pressure drop in the armature may have any desired phase. We thus get the diagram shewn in Fig. 184, where P_K is the terminal pressure and E_a the E.M.F. induced in the generator; P_i is then the pressure drop in the armature.

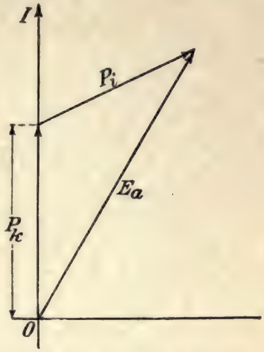


FIG. 184.—Diagram of the Effective Pressures of a Generator for $\cos \phi = 1$.

For circuits connected in *parallel* the terminal pressure will be the same for each branch.

In Equation 129, P_{I}, P_{II} and P_{III} cancel out, and the condition for the graphic summation of current vectors is then

$$(f - f_{\mathrm{I}}) I_{\mathrm{I}} \sin \phi_{\mathrm{I}} + (f - f_{\mathrm{II}}) I_{\mathrm{II}} \sin \phi_{\mathrm{II}} + (f - f_{\mathrm{III}}) I_{\mathrm{III}} \sin \phi_{\mathrm{III}} = 0.$$

This equation is satisfied when

$$f = f_{\mathrm{I}} = f_{\mathrm{II}} = f_{\mathrm{III}}.$$

This is the case, *firstly*, when the circuits in parallel are similar, i.e. when all the circuits have the same ratio between r, L and C; and, *secondly*, when the conductance g of each of the parallel branches is independent of the frequency and also the susceptance of each path is the same function of the frequency. This second case is only of mathematical interest, and has no practical importance, since g is nearly always a function of the frequency; consequently the proof will be omitted here.

A further case, where the graphical addition of the currents in parallel circuits is likewise allowable, is that in which the reactance of every circuit except one is zero; it is then easy to see that $f = f_x$, and thus

$$I \sin \phi = I_x \sin \phi_x,$$

where I_x, ϕ_x, f_x refer to the x^{th} circuit, which may possess both inductance and capacity. The proof for this is given on p. 311 in the description of the "three-ammeter method," which is more convenient for this purpose.

To shew the effect of the higher harmonics on the magnitude of the error introduced by graphically adding the currents in parallel circuits,

the values of f, $\cos\phi$ and $\cos\phi_1$ as functions of $\frac{x_{s1}}{r}$ are given in the following tables for the three pressure curves:

$$(1)\ P_1=100;\quad P_3=31{\cdot}65;\quad P_5=10.$$
$$(2)\ P_1=100;\quad P_3=22{\cdot}4;\quad P_5=22{\cdot}4.$$
$$(3)\ P_1=100;\quad P_3=10;\quad P_5=31{\cdot}65.$$

x_{s1} is the inductive reactance of the circuit with respect to the fundamental. When this ratio is given, the corresponding $\sin\phi_1$, $\cos\phi_1$, $\sin\phi_3$, $\cos\phi_3$ and so on can be easily calculated, and from them the factor f, on the assumption that x_s is proportional to the frequency.

TABLE (a).

$$\frac{x_{c1}}{r}=0.$$

$\frac{x_{s1}}{r}=$		0	0·1	0·2	0·5	1	10
	1	0·874	0·878	0·895	0·934	0·960	0·909
f	2	0·815	0·823	0·854	0·921	0·956	0·918
	3	0·766	0·776	0·802	0·898	0·945	0·909
	1	1	0·992	0·970	0·865	0·679	0·100
$\cos\phi$	2	1	0·989	0·967	0·865	0·679	0·100
	3	1	0·985	0·958	0·858	0·676	0·100
$\cos\phi_1$		1	0·995	0·981	0·894	0·707	0·100

Table (a) refers to a circuit whose capacity is zero, whilst Table (b) is drawn up for a circuit whose ratio of capacity x_{c1} to resistance r is 0·2; thus in this case,

$$\frac{x_{c1}}{r}=0{\cdot}2;\quad \frac{x_{c3}}{r}=0{\cdot}066\ldots \quad\text{and}\quad \frac{x_{c5}}{r}=0{\cdot}04.$$

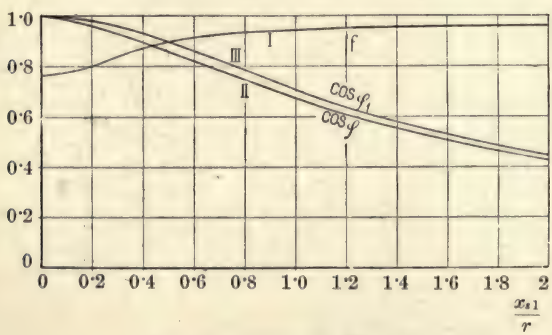


FIG. 185.—Assumption, $\frac{x_{c1}}{r}=0$.

TABLE (*b*).

$$\frac{x_{c1}}{r} = 0 \cdot 2.$$

$\frac{x_{s1}}{r} =$	0	0·1	0·2	0·5	1	10
f 1	0·945	0·521	0·235	0·838	0·943	0·909
f 2	0·948	0·434	0·237	0·817	0·938	0·918
f 3	0·946	0·322	0·273	0·780	0·922	0·909
$\cos\phi$ 1	0·984	0·992	0·988	0·928	0·748	0·101
$\cos\phi$ 2	0·984	0·989	0·985	0·926	0·748	0·101
$\cos\phi$ 3	0·984	0·985	0·978	0·918	0·745	0·101
$\cos\phi_1$	0·982	0·995	1	0·958	0·782	0·1015

In Figs. 185 and 186 the ratios f (curve I), $\cos\phi$ (curve II) and $\cos\phi_1$ (curve III) are plotted as functions of $\frac{x_{s1}}{r}$ for the pressure curve (3).

From the values for f in Table (*b*) and in curve I, Fig. 186, it is clear that there are several circuits, which are not similar, but whose

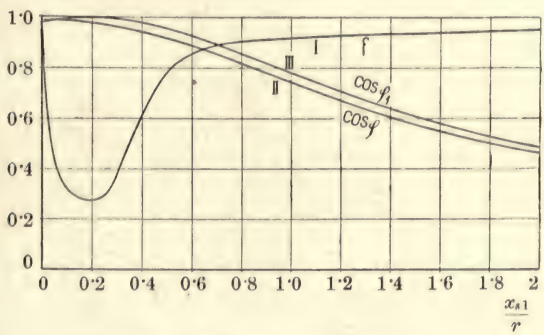


FIG. 186.—Assumption, $\frac{x_{c1}}{r} = 0 \cdot 2$.

currents can nevertheless be geometrically added without error, since the circuits have the same ratio f for the given terminal pressure.

When currents in parallel circuits are graphically added, the watt component of the resultant of all the currents always equals the sum of the watt components of the several currents; this is not the case, however, with the wattless components, and the difference between the wattless component of the resultant current and the sum of the several wattless components is

$$\Delta I_{wL} = (f_{\mathrm{I}} - f) I_{\mathrm{I}} \sin\phi_{\mathrm{I}} + (f_{\mathrm{II}} - f) I_{\mathrm{II}} \sin\phi_{\mathrm{II}} + (f_{\mathrm{III}} - f) I_{\mathrm{III}} \sin\phi_{\mathrm{III}}.$$

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$$\frac{x_{c1}}{r}=0{\cdot}2;\quad \frac{x_{c3}}{r}=0{\cdot}066\ldots \quad\text{and}\quad \frac{x_{c5}}{r}=0{\cdot}04.$$

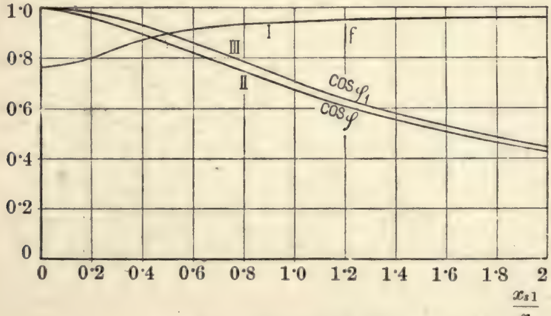

FIG. 185.—Assumption, $\frac{x_{c1}}{r}=0$.

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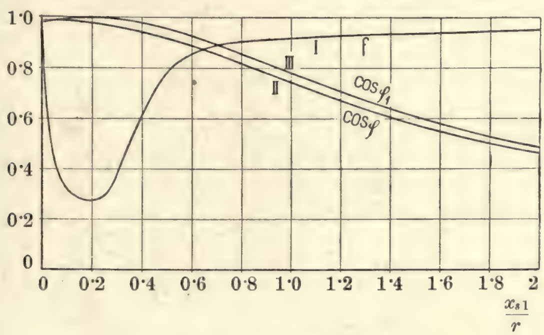


FIG. 186.—Assumption, $\frac{x_{c1}}{r} = 0{\cdot}2$.

currents can nevertheless be geometrically added without error, since the circuits have the same ratio f for the given terminal pressure.

When currents in parallel circuits are graphically added, the watt component of the resultant of all the currents always equals the sum of the watt components of the several currents; this is not the case, however, with the wattless components, and the difference between the wattless component of the resultant current and the sum of the several wattless components is

$$\Delta I_{wL} = (f_{\mathrm{I}} - f) I_{\mathrm{I}} \sin\phi_{\mathrm{I}} + (f_{\mathrm{II}} - f) I_{\mathrm{II}} \sin\phi_{\mathrm{II}} + (f_{\mathrm{III}} - f) I_{\mathrm{III}} \sin\phi_{\mathrm{III}}.$$

Example. Let pressure (3) act on three parallel circuits with the ratio $\frac{x_{c1}}{r}=0$ and $\frac{x_{s1}}{r}=0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}5$, of which the first takes the current $I_{\text{I}}=100$ amps., and each of the other two 50 amps., then $f_{\text{I}}=0{\cdot}776$, $f_{\text{II}}=0{\cdot}802$ and $f_{\text{III}}=0{\cdot}898$, whilst by calculation $f=0{\cdot}805$. Hence, in this case,

$$\Delta I_{WL}=(0{\cdot}776-0{\cdot}805)\,100\,.\,0{\cdot}173+(0{\cdot}802-0{\cdot}805)\,50\,.\,0{\cdot}286 \\ +(0{\cdot}898-0{\cdot}805)\,50\,.\,0{\cdot}526=1{\cdot}9 \text{ amps.}$$

The wattless component of the resultant current is 59·8 amps.; the percentage error in this extreme case is therefore,

$$100\,\frac{1{\cdot}9}{59{\cdot}8}=3{\cdot}17\,\%.$$

From this example and from curve I, Fig. 185, it is seen *that for all inductive circuits, whose reactances are practically proportional to the frequency, it is allowable to add the equivalent sine currents graphically.* The addition of equivalent currents of other parallel branches, where the reactances do not bear the same relation to the frequency, or whose resistances vary with the instantaneous value of the current, can lead to considerable errors. Examples of such circuits are arc lamps, condensers, polarisation cells (above the pressure for which dissociation occurs) and in high pressure mains (in which the maximum difference of pressure exceeds that for which dark discharge occurs).

In curves II and III, Figs. 185 and 186, we see the effect of the shape of the pressure curve on the *power factor* $\cos\phi$, and it is seen that this curve lies considerably lower for a distorted curve than for a sine curve. It is, therefore, not allowable to replace a terminal pressure of distorted wave-shape by its equivalent sinusoidal pressure, and with this calculate the current and power factor. In practice, however, this method is often adopted, which, in the above example for $\frac{x_{s1}}{r}=0{\cdot}5$, gives $\cos\phi_1=0{\cdot}894$ instead of $\cos\phi=0{\cdot}858$. This error, however, is too large to be neglected—and still larger errors may be introduced when we apply this method in the calculation of circuits containing capacity or apparatus with similar reactances.

71. Effect of Wave-Shape on the Working of Electric Machines and Apparatus. In the introduction to the previous chapter attention was drawn to the injurious effects of higher harmonics. We shall now illustrate this by means of examples and curves.

(*a*) *Lighting.* As already observed, the flat-shaped curve is the most suitable for this purpose, because in this case the current remains longest in the neighbourhood of its maximum value. Consequently we can work at a lower frequency with a flat curve, such as in Fig. 187, than with a peaked curve, like that shewn in Fig. 188, before variations in the intensity of the light become noticeable. The Authors found

from experiments carried out in the dark room, that the light of a 16 C.P. carbon-filament lamp for 110 volts began to fluctuate when the frequency of the current fell below 23·3, whilst this only occurred with

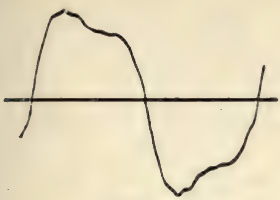


FIG. 187.

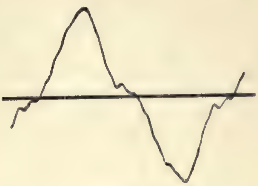


FIG. 188.

the flat-shaped curve (Fig. 187) when the periodicity fell below 20 cycles per second.

With a 25 C.P. 115 volt metal-filament lamp, the pulsations were already noticeable with the above pressure waves when the frequency fell to 28·3 and 23·7 respectively. This limit also depends on the lamp pressure—the lower the pressure, the lower the frequency at which flickering becomes noticeable.

It has often been noticed in practice that arc lamps are inclined to be somewhat noisy when the pressure curve is very peaked.

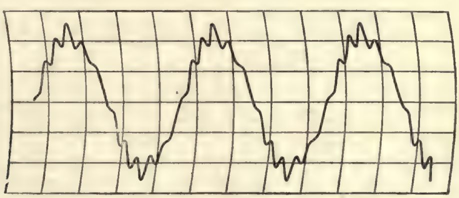


FIG. 189.

This humming noise, which is due to the pulsations set up in the arc and the surrounding air, can be sufficiently damped, at a frequency of 50 cycles, by connecting a choking coil in series to suppress the harmonics. Fig. 189 represents the pressure curve of a large three-phase central station, where—according to a report by Herr C. Zorawski (*E.T.Z.* 1906, S. 607)—the humming became so considerable that choking coils had to be connected in series. Choking coils, however, tend to lower the total power factor of the system.

(*b*) *Transformers.* Prof. G. Rössler (*E.T.Z.* 1895, S. 488) has experimentally investigated the effect of the shape of the pressure curve on the drop of pressure in a small transformer of some ½ K.W., which had comparatively high resistance and reactance. The results of his research are shewn by the curves in Fig. 190. Curve I represents the secondary pressure with non-inductive load when the peaked

(*b*) Short-circuit.

Current Curve.	Fig. 192*d*.	Fig. 192*e*.	Fig. 192*f*.
I_K=amps., - - -	10	10	10
P_K=volts, - - -	7·44	7·36	8·05
W_K=watts, - - -	46·4	44·0	45·4

(*c*) *Induction Motors.* As in the case of a transformer, the Authors have also measured the no-load losses for the curve shapes in Figs. 193*a* and *b* and short-circuit losses for those in Figs. 193*c* and *d* in a 2 H.P. three-phase motor. The results are shewn in the following

FIG. 193*a*.

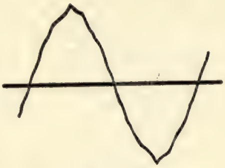


FIG. 193*b*.

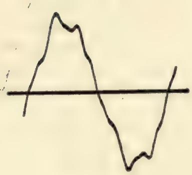


FIG. 193*c*.

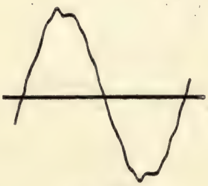


FIG. 193*d*.

table. The no-load losses remain practically the same, whilst the short-circuit losses, and still more the short-circuit reactance, for the same effective current are larger the greater the harmonics which are present.

2 H.P. THREE-PHASE MOTOR.

(*a*) No-load.

Pressure Curve.	Fig. 193*a*.	Fig. 193*b*.
P_0=volts, - - -	112	112
I_0=amps., - - -	3·7	3·65
W_0=watts, - - -	156	152

(*b*) Short-circuit.

Current Curve.	Fig. 193*c*.	Fig. 193*d*.
I_K = amps., - - -	10	10
P_K = volts, - - -	25·8	25·0
W_K = watts, - - -	204	198

Thus, the efficiency of a motor is also a maximum when the pressure curve is a sine function. The same holds for the power factor and the maximum power, for with a given applied pressure the short-circuit current is smaller, when measured whilst the rotor is just set moving. This is due to the fact that only the pressure of the fundamental $P_1 = \frac{P}{\sigma_p}$ transmits power from the stator primary to the rotor secondary. We thus get the same result as for a transformer, namely, the asynchronous motor works best with a sinusoidal pressure curve. This is also true for commutator motors; for the flat-shaped pressure curve is bad for commutation, whilst the peaked pressure curve reduces the load capacity of such a motor.

(*d*) *Synchronous Machines.* If several synchronous machines having different pressure curves work in parallel, large currents of high

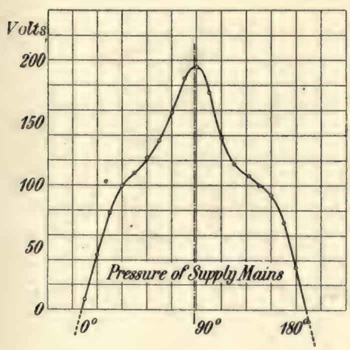


FIG. 194.

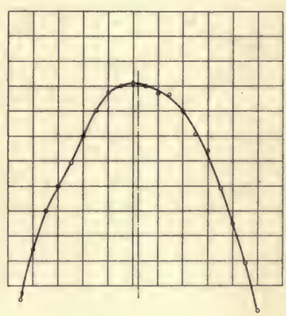


FIG. 195.

frequency will flow between them, since the pressure harmonics need not be in phase when the fundamental pressures are. If the reactances of the synchronous machines are very low, the currents due to the higher harmonics can attain such dimensions that the working may be sufficiently affected to cause the machines to fall out of step. The shape and magnitude of these currents are best illustrated by the curves in Figs. 194 to 197, taken at the Electrotechnic Institute,

Karlsruhe, by Dr. Bloch. Figs. 194 and 195 give the pressure curves of the central station and of a 5 H.P. single-phase motor, whilst the

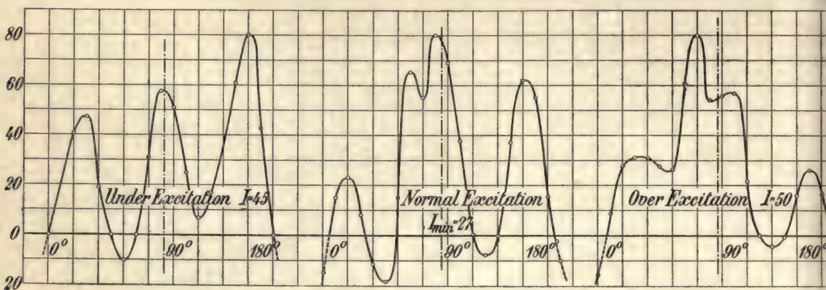


FIG. 196.

curves in Fig. 196 shew the currents in the motor. By connecting a large reactance in series, the current curves in Fig. 197 were obtained.

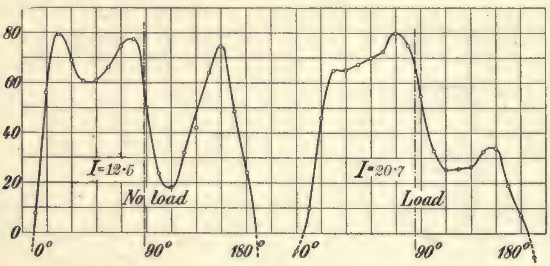


FIG. 197.

Here again the damping effect of the choking coil on the higher harmonics is clearly seen. The presence of currents of high frequencies in synchronous machines can be limited by taking care that all the synchronous machines working on the network have the same wave-shape at no-load. Since, however, the wave-shape varies with the load, it is not possible to completely avoid these internal currents. The best means for keeping them small is of course to have the pressure curves of all the machines as nearly sinusoidal as possible and to give the machines a suitable reactance.

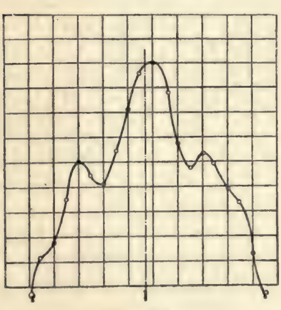


FIG. 198.

(*e*) *Cables and Conductors.* The flat-shaped pressure curve should of course place less strain on the insulators and cable-insulation, since for a given effective pressure the maximum pressure is then least. On the other hand, this requires higher harmonics, which may give rise

to resonance, under certain conditions. Since such wave forms have a disturbing effect on the pressure regulation of a system, and are more difficult to deal with analytically than pure sine waves, it is also always desirable to use sinusoidal pressures for transmission plants. The two pressure curves, Figs. 194 and 198, are for a large electricity works. The latter represents the day pressure, the former the night pressure. As is seen, the higher harmonics are more pronounced in the day curve than in the night curve, since the day load is more inductive although small.

CHAPTER XIII.

POLYPHASE CURRENTS.

72. Polyphase Systems. If three coils are arranged on the armature of a generator (Fig. 199), so that they are all displaced from one another in space, the E.M.F.'s induced in these coils will be

$$p_{\mathrm{I}} = P_{\mathrm{I\,max}} \sin \omega t,$$
$$p_{\mathrm{II}} = P_{\mathrm{II\,max}} \sin (\omega t - \alpha),$$
$$p_{\mathrm{III}} = P_{\mathrm{III\,max}} \sin (\omega t - \beta).$$

These all have the same frequency c, because all the coils rotate with the same velocity. But they are all displaced from one another in phase by the angle which the coils make with one another in space. If each of the three coils acts on its own circuit, a current will flow in each coil independent of that in the other coils. The three currents together form a three-phase current and *such a system of alternating-currents, in which several E.M.F.'s of the same frequency and displaced from one another in phase produce currents which are also displaced from one another, is known in general as a polyphase system.*

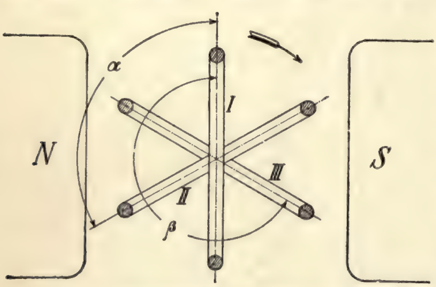


FIG. 199.—Production of a Polyphase Current.

Externally, a polyphase generator appears the same as a single-phase generator—only the stator winding is different. In Fig. 163 the stator winding of a single-phase generator is represented, and in Fig. 165 that of a three phaser.

Generally speaking, a polyphase system can be investigated by splitting up the same into its several current branches, or *phases*; the E.M.F. acting in each of these current paths produces a current in the system, which can be calculated independently of the E.M.F.'s of the other phases. The currents produced by all the E.M.F.'s must then be superposed, when the phases are electrically connected. The several systems can be classified thus:

(1) Into *symmetrical* and *unsymmetrical* systems.
(2) Into *dependent* or *interconnected* and *independent* systems.
(3) Into *balanced* and *unbalanced* systems.

The dependent or interlinked systems can be again split up into *star-connected* systems, *ring-connected* systems and systems comprising both of these two.

73. Symmetrical Polyphase Systems. If a polyphase system is formed by n pressures, whose amplitudes are equal and displaced from one another in phase by $\frac{1}{n}$ period, the system is said to be *symmetrical*,—otherwise it is unsymmetrical. Such a system can also be called a symmetrical n-phase system, since it has n phases. In the case where the pressures are sine functions of the time, the n pressures are represented by the following expressions:

$$p_{\mathrm{I}} = P \sin \omega t,$$

$$p_{\mathrm{II}} = P \sin \left(\omega t - \frac{2\pi}{n}\right),$$

$$p_{\mathrm{III}} = P \sin \left(\omega t - 2\,\frac{2\pi}{n}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_n = P \sin \left\{\omega t - (n-1)\,\frac{2\pi}{n}\right\}.$$

If we sum up the momentary values of these n pressures we obtain the well-known result that the sum of the momentary values of the pressures of a symmetrical polyphase system always equals zero.

We can now deduce the various symmetrical polyphase systems by substituting various values for n.

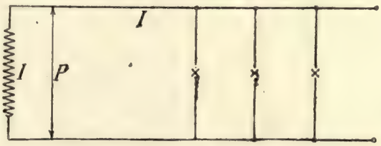


FIG. 200.—Single-phase Two-wire System.

Example 1. When $n=1$, $p_{\mathrm{I}} = P \sin \omega t$, and we get the single-phase two-wire system of Fig. 200.

When $n=2$,
$$p_{\mathrm{I}} = P \sin \omega t,$$
$$p_{\mathrm{II}} = P \sin (\omega t - 180°) = -p_{\mathrm{I}}.$$

This gives the single-phase three-wire system (Fig. 201), where the pressures are reckoned from the middle point 0. When the two halves

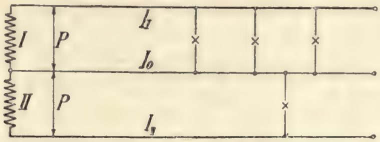


FIG. 201.—Single-phase Three-wire System.

of the generator are equally loaded, no current flows in the middle wire—consequently this wire can be made very light.

Example 2. When $n=3$,
$$p_{\mathrm{I}} = P \sin \omega t,$$
$$p_{\mathrm{II}} = P \sin \left(\omega t - \frac{2\pi}{3}\right),$$
$$p_{\mathrm{III}} = P \sin \left(\omega t - \frac{4\pi}{3}\right).$$

This is the symmetrical three-phase system, where the three pressures are displaced in phase from one another by 120°, which accordingly represents the symmetrical polyphase system having the least number of phases.

Example 3. When $n=4$, we get the symmetrical four-phase system.
$$p_{\mathrm{I}} = P \sin \omega t,$$
$$p_{\mathrm{II}} = P \sin \left(\omega t - \frac{\pi}{2}\right),$$
$$p_{\mathrm{III}} = P \sin (\omega t - \pi) = -p_{\mathrm{I}},$$
$$p_{\mathrm{IV}} = P \sin \left(\omega t - \frac{3\pi}{2}\right) = -p_{\mathrm{II}}.$$

Thus p_{I} and p_{III} occur in the same circuit, and similarly p_{II} and p_{IV}. Consequently there are only two pressures, and these are displaced 90° from each other.

74. Interconnected Polyphase Systems. In polyphase systems, each of the phases may be made to form a closed system for itself—such a polyphase system then consists of n entirely independent single-phase systems, which have only to satisfy the one condition that the frequency and the mutual phase-displacement of the E.M.F.'s of the several phases are always the same. The generators of the single-phase

currents must therefore run in perfect synchronism with one another—which is most easily attained by placing the several windings, in which the E.M.F.'s are to be induced, on the same armature. We can now go a step further, and electrically connect the windings of the several phases with one another, i.e. interconnect the phases. In this case, however, the several phases will mutually affect one another, if the system is not symmetrical both in respect to the induced E.M.F.'s and the load.

In the representation of polyphase systems it is usual to draw the windings of the several phases displaced from one another by the angle of the mutual phase-displacement.

The phases can be connected in various ways with each other; only care must be taken to have no closed circuits where the sum of the induced E.M.F.'s is not zero; for such a circuit would act as a short-circuit in which an E.M.F. is induced; consequently a heavy current would flow in the same.

The systems generally met with in practice are the *star-connected* and *ring-connected* (or mesh-connected) systems.

The *star* system is formed by joining the starting points of all the phases to a common point. This point is then termed the *neutral point*, because in a symmetrical star-connected system it generally attains the mean potential of the surroundings. This point can be connected to earth, or to another neutral point, or insulated; it is usual to regard the neutral point as having zero potential. Between the terminals of any phase, e.g. the x^{th}, we measure the *phase pressure* $P\sin\left\{\omega t-(x-1)\frac{2\pi}{n}\right\}$, whilst between the terminals of two neighbouring phases we have the *line pressure*, whose momentary value equals the difference of the momentary values of the pressures of the two phases in question. The momentary value of the line pressure between the terminals of the x^{th} and $(x+1)^{\text{th}}$ phases is thus

$$p_l = P\sin\left\{\omega t-(x-1)\frac{2\pi}{n}\right\} - P\sin\left\{\omega t - x\frac{2\pi}{n}\right\}$$

$$= 2P\sin\frac{\pi}{n}\cos\left\{\omega t-(2x-1)\frac{\pi}{n}\right\},$$

whence it follows that the effective line pressure is

$$P_l = 2\sin\frac{\pi}{n}P_p, \quad\text{..............................(130)}$$

where P_p is the effective phase pressure.

In the star-connected system, the line pressure equals the resultant pressure of two adjacent phases and the line current the phase current.

The *ring-connected* system is formed by connecting the start of one phase to the finish of the next, so that all the phases are joined in series. Accordingly, this connection can only be used when the sum of the E.M.F.'s of all the phases equals zero at every instant, which is the case with symmetrical polyphase systems having sinusoidal E.M.F.'s.

Current is taken off at the junction of each two adjacent phases, whence the number of lines equals the number of phases. Then, in accordance with Kirchhoff's First Law, the current in each line equals the difference of the currents in the two neighbouring phases. In this case, therefore, the line current does not equal the phase current, but, since the currents in two adjacent phases are displaced from one another by $\frac{2\pi}{n}$, equals

$$i_l = I \sin\left\{\omega t - (x-1)\frac{2\pi}{n}\right\} - I \sin\left\{\omega t - x\frac{2\pi}{n}\right\}$$

$$= 2I \sin\frac{\pi}{n}\cos\left\{\omega t - (2x-1)\frac{\pi}{n}\right\};$$

hence, for effective values,

$$I_l = 2\sin\frac{\pi}{n} I_p. \qquad (131)$$

The line pressure is here the same as the phase pressure.

Hence, in the ring-connected system, the line pressure equals the phase pressure and the line current the resultant current of two adjacent phases.

In the following, all magnitudes referring to the lines are denoted by the suffix l and to the phases by the suffix p.

The most usual connections for a symmetrical three-phase system are as follows:

(*a*) *Three-phase Star System.* Fig. 202 is an independent three-phase system, where the phase current equals the line current and the phase

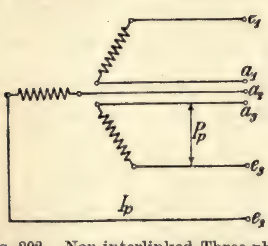


FIG. 202.—Non-interlinked Three-phase System.

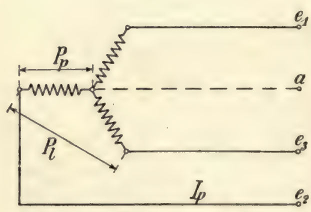


FIG. 203.—Three-phase Star System.

pressure the line pressure. By coupling the three starting points a_1, a_2, a_3 together (Fig. 203), we get the three-phase star-connected system with four wires, which can be converted into a three-wire system by omitting the middle- or neutral-wire a, which carries no current so long as the load is symmetrical. The line pressure in this system is

$$P_l = 2\sin 60^\circ P_p = \sqrt{3} P_p \qquad (132)$$

and

$$I_l = I_p. \qquad (133)$$

(*b*) *Three-phase Ring System.* Fig. 204 represents the three-phase ring system, or, as it is also termed, the *triangle-* or *delta-* (Δ) or *mesh-*connection. Here we have

$$P_l = P_p \quad\text{.......(134)}$$

and

$$I_l = 2\sin 60^\circ\, I_p = \sqrt{3}I_p. \quad\text{.......(135)}$$

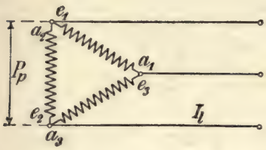


FIG. 204.—Three-phase Mesh System.

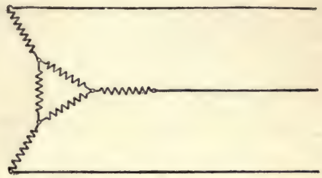


FIG. 205.—Combined System for Three-phase Current (*Dolivo von Dobrowolsky*).

Fig. 205 represents a combination due to *Dolivo von Dobrowolsky*.

When $n = 4$, we can have the following schemes:

(*c*) *Independent Four-phase System or Two-phase System.* This is represented in Fig. 206. We have

$$I_l = I_p \quad\text{and}\quad P_l = P_p.$$

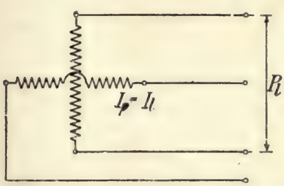


FIG. 206.—Non-interlinked Four-phase System.

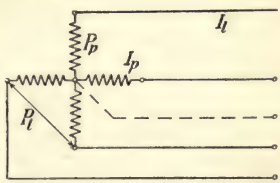


FIG. 207.—Four-phase Star System.

(*d*) *Four-phase Star System.* Fig. 207 represents the connections for this system, in which

$$I_l = I_p \quad\text{.......(136)}$$

and

$$P_l = 2\sin 45^\circ\, P_p = \sqrt{2}P_p. \quad\text{.......(137)}$$

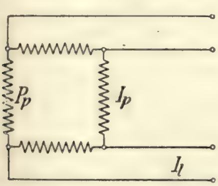


FIG. 208.—Four-phase Mesh System.

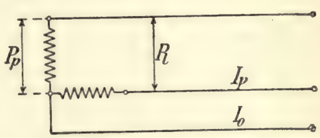


FIG. 209.—Two-phase Three-wire or Interlinked Two-phase System.

(*e*) *Four-phase Ring System.* This is shewn by Fig. 208.

$$I_l = \sqrt{2}I_p, \quad\text{.......(138)}$$

and

$$P_l = P_p. \quad\text{.......(139)}$$

(*f*) *Interconnected Two-phase System.* The scheme shewn in Fig. 207 is seldom used, but rather that shewn in Fig. 209, which is developed from the former and represents one half of an interconnected four-phase system with middle wire. This system, which is not symmetrical, is usually termed the *interconnected two-phase system* or the *two-phase three-wire system.* For this we have

$$P_l = \sqrt{2}P_p, \qquad \text{.........................(140)}$$

and

$$I_0 = \sqrt{2}I_p. \qquad \text{.........................(141)}$$

(*g*) *Scott's System.* To the interconnected polyphase systems belongs also *Scott's System*, shewn in Fig. 210. This serves for producing a three-phase current by means of a two-phase winding. If one-phase has $\sqrt{\frac{3}{4}}$ as many turns as the other and the start of this phase is connected to the middle of the second, we get a symmetrical three-phase pressure between the terminals A, B and C. Then the pressure between the terminals A and B and between A and C (Fig. 210) is $\sqrt{(\frac{1}{2})^2 + \frac{3}{4}} = 1$ times the pressure between B and C. It is thus possible to produce a symmetrical three-phase current by means of an unsymmetrical two-phase system.

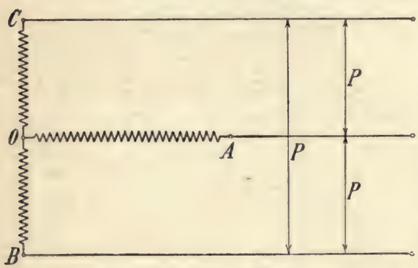


FIG. 210.—Scott's System.

The phase pressures are

$$P_A = \overline{OA} = \sqrt{\tfrac{3}{4}} \,.\, \overline{BC} = \sqrt{\tfrac{3}{4}} \,.\, P_l,$$

and

$$P_B = P_C = \overline{OB} = \overline{OC} = \tfrac{1}{2}\overline{BC} = \tfrac{1}{2}P_l,$$

whilst the phase currents equal the line currents.

(*h*) *Imperfect Polyphase Systems.* These also belong to the interconnected polyphase systems, and consist of a main phase, together with an interconnected auxiliary phase. These were all introduced in the early nineties, when it was desired to retain the simplicity of the single-phase system, and avoid its deficiencies by the use of auxiliary phases.

The simplest of the systems is the *imperfect three-phase* system (Fig. 211), which consists of two phases at 120° to one another. The auxiliary phase, which is chiefly used for starting asynchronous motors, has a phase pressure equal to the distance of the point O from the line $\overline{BC}$. The starting torque is proportional to this auxiliary pressure P_h,

$$P_h = \tfrac{1}{2}P_p.$$

When the two phases are symmetrically loaded, the currents in all three lines are equal, but displaced 60° in phase from one another. Since this system does not produce a large starting torque for motors, as just shewn, Steinmetz proposed a system, similar to Scott's system,

which is known by the unsuitable name of "monocyclic system." This is a three-phase system, and serves to produce an unsymmetrical three-phase current. Steinmetz chose the auxiliary pressure $\overline{OA}$ at the motors equal to $\sqrt{\frac{3}{4}}$ of the main pressure $\overline{BC}$, whereby the motors receive a symmetrical three-phase pressure. The auxiliary pressure $\overline{OA}$ of the generators, however, was only chosen about a fourth of the pressure of the main phase. The ratio of conversion of the transformers for the main phase is therefore $\frac{4}{\sqrt{3}}$ of that of the transformers for the auxiliary phase.

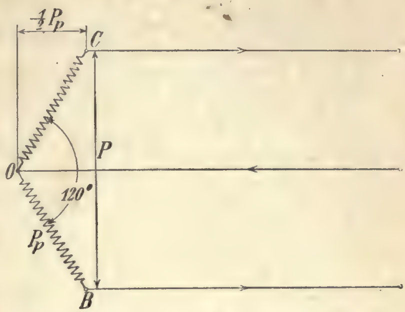


FIG. 211.—Incomplete Three-phase System.

None of these imperfect polyphase systems, however, have justified their existence, since they all need three wires, as in symmetrical three-phase system, and there is no reason why this latter should not be adopted and so completely utilise the material of both generators and motors.

75. Balanced and Unbalanced Systems. In Section 11, we have seen that the current

$$i = I\sqrt{2}\sin(\omega t - \phi),$$

produced by the pressure $p = P\sqrt{2}\sin\omega t$,

yields the momentary power

$$W = PI\{\cos\phi - \cos(2\omega t - \phi)\}.$$

Since the mean power is $W = PI\cos\phi$,

we have $$W = W\left\{1 - \frac{\cos(2\omega t + \phi)}{\cos\phi}\right\}.$$

Although this pulsation of the power of a single-phase current, which is shewn in Figs. 43 and 44 for any angle ϕ and for $\phi = 90°$, does not prevent its application for many purposes, e.g. lighting by means of glow lamps, provided the frequency is chosen sufficiently high, it is just this property of the single-phase current which makes it unsuitable for power purposes. On the other hand, a symmetrical polyphase system—as will be shewn later on—possesses the characteristic that the momentary power of the whole system is always constant; consequently such systems are used a great deal for motor purposes. Not only symmetrical systems, however, but also other polyphase systems can develop a constant power under certain conditions; thus all systems possessing this characteristic are said to be *balanced*, and all others, *unbalanced*.

The power in a polyphase system equals the sum of the powers in the several phases. If the pressures $p_{\mathrm{I}}, p_{\mathrm{II}}, p_{\mathrm{III}} \cdots$ of the several phases produce the phase currents $i_{\mathrm{I}}, i_{\mathrm{II}}, i_{\mathrm{III}} \cdots$, the momentary power will be

$$W = p_{\mathrm{I}} i_{\mathrm{I}} + p_{\mathrm{II}} i_{\mathrm{II}} + p_{\mathrm{III}} i_{\mathrm{III}} + \cdots$$

and the mean power $W = P_{\mathrm{I}} I_{\mathrm{I}} \cos \phi_{\mathrm{I}} + P_{\mathrm{II}} I_{\mathrm{II}} \cos \phi_{\mathrm{II}} + \ldots$.

If now the n-phase system is symmetrical with equally-loaded phases, we have, e.g., for the x^{th} phase,

$$p_{x+1} = P\sqrt{2} \sin \left(\omega t - 2\pi \frac{x}{n} \right)$$

and
$$i_{x+1} = I\sqrt{2} \sin \left(\omega t - \phi - 2\pi \frac{x}{n} \right),$$

where ϕ is the phase displacement of the current in a phase behind its pressure. From this it follows that the momentary power of the symmetrical n-phase system is

$$W = \sum_1^n p_x i_x = 2PI \sum_1^n \sin \left(\omega t - 2\pi \frac{x}{n} \right) \sin \left(\omega t - \phi - 2\pi \frac{x}{n} \right)$$

$$= PI \left\{ n \cos \phi - \sum_1^n \cos \left(2\omega t - \phi - 4\pi \frac{x}{n} \right) \right\} = PIn \cos \phi.$$

$$W = nPI \cos \phi. \qquad \text{.........................(142)}$$

Thus the momentary power W is constant for every symmetrical n-phase system and equals n times the mean power of a phase.

For the *three-wire two-phase* system (Fig. 209) the pressures are

$$p_{\mathrm{I}} = P_p \sqrt{2} \sin \omega t$$

and
$$p_{\mathrm{II}} = P_p \sqrt{2} \sin \left(\omega t - \frac{\pi}{2} \right).$$

If both phases are equally loaded in regard to current and phase displacement, then

$$i_{\mathrm{I}} = I_p \sqrt{2} \sin (\omega t - \phi)$$

and
$$i_{\mathrm{II}} = I_p \sqrt{2} \sin \left(\omega t - \phi - \frac{\pi}{2} \right).$$

Hence,

$$W = 2P_p I_p \left\{ \sin \omega t \sin (\omega t - \phi) + \sin \left(\omega t - \frac{\pi}{2} \right) \sin \left(\omega t - \phi - \frac{\pi}{2} \right) \right\}$$

$$= 2P_p I_p \cos \phi - P_p I_p \{ \cos (2\omega t - \phi) + \cos (2\omega t - \phi - \pi) \}$$

$$= 2P_p I_p \cos \phi = \text{const.}$$

and the mean power $\quad W = 2P_p I_p \cos \phi, \qquad$ ....(143)

or, since $\quad P_p = \dfrac{P_l}{\sqrt{2}} \quad$ and $\quad I_p = \dfrac{I_0}{\sqrt{2}},$

then $\quad W = P_l I_0 \cos \phi. \qquad$ ....(143*a*)

We thus see that the three-wire two-phase system belongs to the balanced unsymmetrical polyphase systems.

The power of a symmetrical three-phase system is, from Eq. (142),

$$W = 3P_pI_p \cos\phi,$$

or, since in a star system

$$P_p = \frac{P_l}{\sqrt{3}} \quad \text{and} \quad I_p = I_l,$$

and in a mesh system $P_p = P_l$ and $I_p = \frac{I_l}{\sqrt{3}}$,

the power in any symmetrical and interconnected three-phase system is

$$W = \sqrt{3}P_lI_l \cos\phi. \qquad (144)$$

From formulae (136) and (139) it follows similarly that the power in a symmetrical interconnected four-phase system is always

$$W = 4P_pI_p \cos\phi = 2\sqrt{2}P_lI_l \cos\phi. \qquad (145)$$

Scott's system also belongs to the balanced unsymmetrical polyphase systems.

76. Comparison of the Amount of Copper in Alternating-current Systems with that in Continuous-current Systems. To transmit a definite power over a fixed distance electrically at a given maximum pressure and efficiency, a definite amount of copper is essential. The higher the pressure and the lower the efficiency, the less the amount of copper that will be required. Since the pressure must not exceed a certain limit on account of the danger to the insulation or the employees, the pressure which enters into question here is the maximum pressure which exists between any part of the installation and earth. If the neutral point of the system is earthed, the limit is fixed by the maximum pressure between a terminal and this point. If the neutral point is not earthed, and the whole system insulated, the severity of the electric shock caused by touching a terminal depends on the pressure and the capacity of the system. If the pressures are high and the capacity considerable, as is usually the case in transmission lines, the person touching the terminal may have to pay the death penalty for his carelessness. For this reason, "live" machines and apparatus ought never to be touched unless the person has previously insulated himself against the pressure. The insulation of a non-earthed system, however, must be kept stronger than that of an earthed system, since in the former case the insulation must prevent the passage to earth of all the energy stored in the system. For this reason, earthed and non-earthed systems cannot well be compared, since the insulation of the latter must be calculated with regard to quite different pressures.

Hence, we shall only consider earthed systems for the present, and shall put the amount of copper required for a symmetrical polyphase system with earthed neutral point equal to

$$\frac{100}{\cos^2\phi}.$$

Further, we assume that the effective current density is constant in all the conductors and that the pressure curve is sinusoidal. The section of the currentless middle wire is chosen equal to half that of one of the outers. We then get the following results:

(*a*) *Symmetrical Polyphase Systems with Earthed Neutral Point.* Consider first the symmetrical three-phase system. We see that each of the three phases carries the same current I at the same maximum pressure $P_{\max}$ over the same distance l. Let the section of a conductor be q; then the copper losses per phase are

$$I^2r = I^2\frac{l\rho}{q} = Isl\rho,$$

i.e. with a given current density s they are proportional to the power transmitted per phase,

$$p_K = \frac{I^2r}{\frac{1}{\sqrt{2}}P_{\max}I\cos\phi} = \frac{\sqrt{2}sl\rho}{P_{\max}\cos\phi},$$

and the total copper volume is $3lq$.

By means of a single-phase two-wire system or any symmetrical polyphase system with n phases, the same power $3\frac{1}{\sqrt{2}}P_{\max}I\cos\phi$ could be transmitted with the same percentage losses with the same amount of copper. For in each conductor the current is $\frac{3}{n}I$ and the section of the conductor is reduced in this proportion. Thereby the current density s and also the percentage copper losses p_K remain constant, whilst the weight of copper also remains unchanged.

Hence, all symmetrical polyphase systems with earthed neutral point and the single phase two-wire system are alike with respect to the amount of copper required.

In practice, however, only the three-phase system has made headway, because this requires the fewest conductors, and consequently the least insulation of all the symmetrical polyphase systems.

(*b*) *Symmetrical Polyphase Systems with Earthed Neutral Wire.* Consider first the single-phase three-wire system with earthed middle wire, which is theoretically a symmetrical two-phase system. Since no current flows in the middle wire when the load is symmetrical, then, for the same section of outer wire as previously, the copper losses remain the same as in a single-phase two-wire system. The copper required for this system, therefore, will exceed that required for the

two-wire system by the amount required for the middle wire. If we therefore choose the cross-section of the middle wire half that of one of the outers, as mentioned above, this system will need 25 % more copper than the single-phase two-wire system, in order to transmit the same power at the same losses. The copper needed for the single-phase three-wire system is accordingly

$$\frac{100}{\cos^2\phi}\left(1+\frac{1}{2}\times\frac{1}{2}\right)=\frac{125}{\cos^2\phi}.$$

In a similar manner we find the copper required for a three-phase four-wire system is

$$\frac{100}{\cos^2\phi}\left(1+\frac{1}{3}\times\frac{1}{2}\right)=\frac{116\cdot7}{\cos^2\phi},$$

and for a four-phase five-wire system,

$$\frac{100}{\cos^2\phi}\left(1+\frac{1}{4}\times\frac{1}{2}\right)=\frac{112\cdot5}{\cos^2\phi}.$$

(*c*) *Single-phase Two-wire Systems with Earthed Outer Wire.* This system can be regarded as one phase of a polyphase system with a neutral wire of the same section as the outer wire. Consequently, this system needs the same copper and has the same losses in the earthed wire as in the outer wire. With the same section for the outer wire as the total section of all the outer wires of a polyphase system with earthed neutral point, we get double the losses in a single-phase two-wire system with earthed outer wire, when the same power is transmitted at a given maximum pressure. To reduce these losses to those in a polyphase system, we must double the section of the outer wire, and consequently also of the earthed wire. Hence, the copper required in a single-phase system with earthed outer wire is

$$\frac{100}{\cos^2\phi}(1+1)^2=\frac{400}{\cos^2\phi},$$

or, in other words, four times as much as that of a polyphase system with earthed neutral point.

(*d*) *Two-phase Three-wire System with Earthed Middle Wire.* This system can also be regarded as two phases of a polyphase system with a middle wire of $\sqrt{2}$ times the section of one of the outers. Consequently, this system requires for the middle wire

$$\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

times the copper of the two outer wires, and similarly, as in a single-phase two-wire system, the section of each outer wire must also be increased in this case in order to transmit the same power with the same losses. The increase of section of the outer wires is, of course, equal to the percentage increase of copper due to the presence of the

middle wire, i.e. proportional to $\left(1+\frac{1}{\sqrt{2}}\right)$. The copper required in a two-phase three-wire system with earthed middle wire is thus

$$\frac{100}{\cos^2\phi}\left(1+\frac{1}{\sqrt{2}}\right)^2=\frac{291{\cdot}4}{\cos^2\phi},$$

or about three times that of a polyphase system with earthed neutral point.

(*e*) *Imperfect Three-phase System with Earthed Middle Wire.* In this, the current in the middle wire equals that in each of the two outers. Then, in a similar manner to that of a two-phase three-wire system, we get the amount of copper equal to

$$\frac{100}{\cos^2\phi}(1+0{\cdot}5)^2=\frac{225}{\cos^2\phi},$$

i.e. two and a quarter times as much as in a polyphase system with earthed neutral point.

(*f*) *Continuous-current Three-wire System with Earthed Middle Wire.* In respect to the amount of copper required, this system is similar to the single-phase three-wire system. But in this case the maximum pressure P_{max} equals the working pressure P and not $\sqrt{2}$ as much, as in an alternating-current system. Further, in this case there is no phase displacement between current and pressure, thus the percentage loss is

$$p_K=\frac{sl\rho}{P_{max}}\,100,$$

i.e. with effective current density $\frac{\cos\phi}{\sqrt{2}}$ times that of a single-phase three-wire system. Since, however, in a continuous-current system, the current is $\frac{\cos\phi}{\sqrt{2}}$ times smaller, and since we can moreover choose the current density $\frac{\sqrt{2}}{\cos\phi}$ times greater than in a single-phase system, in order to obtain the same losses, we must make the copper cross-section in a continuous current system

$$\left(\frac{\cos\phi}{\sqrt{2}}\right)^2=\frac{\cos^2\phi}{2}$$

of that of a single-phase system, to obtain the same losses and to transmit the same power at the same maximum pressure. Hence the copper used in a continuous-current three-wire system is $\frac{\cos^2\phi}{2}$ times that in a single-phase three-wire system, i.e.

$$\frac{125}{\cos^2\phi}\cdot\frac{\cos^2\phi}{2}=62{\cdot}5$$

as compared with $\frac{100}{\cos^2\phi}$ in a polyphase system with earthed neutral point.

(*g*) *Continuous-current Two-wire System with Earthed Outer Wire.* This bears the same relation to the single-phase two-wire system as the continuous-current three-wire system to the single-phase three-wire system. We thus need

$$\frac{400}{\cos^2\phi} \cdot \frac{\cos^2\phi}{2} = 200,$$

or twice as much copper as a polyphase system with earthed neutral point.

Summarising the above results, we get the following table:

Continuous-current two-wire system with earthed middle point,	50
Continuous-current three-wire system with earthed middle wire,	62·5
Continuous-current two-wire system with earthed outer wire,	200
Symmetrical polyphase systems and single-phase two-wire system with earthed neutral point,	$\frac{100}{\cos^2\phi}$
Single-phase three-wire system with earthed middle wire,	$\frac{125}{\cos^2\phi}$
Three-phase four-wire system with earthed middle wire,	$\frac{116 \cdot 7}{\cos^2\phi}$
Four-phase five-wire system with earthed middle wire,	$\frac{112 \cdot 5}{\cos^2\phi}$
Single-phase two-wire system with earthed outer wire,	$\frac{400}{\cos^2\phi}$
Symmetrical three-phase system with earthed outer wire,	$\frac{300}{\cos^2\phi}$
Two-phase three-wire system with earthed middle wire,	$\frac{291 \cdot 4}{\cos^2\phi}$
Imperfect three-phase system with earthed middle wire,	$\frac{225}{\cos^2\phi}$

It is thus obvious that the systems with an earthed neutral point are the most economical; then follow the systems with earthed middle wire, which only need more copper on account of the partly ineffective middle wire; and finally, the systems with an earthed outer wire, which are very uneconomical. To this class belong the distributing systems of most modern railway installations. The advantage of a three-wire system, however, is much reduced in this case, since the rails, which serve as return, remain unused in the three-wire system. Since, moreover, the losses in the rails are very small in proportion to the losses in the overhead wire, the total losses in the line in a two-wire system are not much greater than in a three-wire system when the rails can be used as return.

CHAPTER XIV.

PRESSURES AND CURRENTS IN A POLYPHASE SYSTEM.

77. Topographic Representation of Pressures. 78. Graphic Calculation of Current in a Star System. 79. Analytic Calculation of Current in a Star System. 80. Graphic Calculation of Current in a Polyphase System. 81. Conversion of a Mesh Connection into a Star Connection. 82. Conversion of Star and Mesh Connections when E.M.F.'s are Induced in the Phases. 83. Symbolic Calculation of Current in Polyphase Systems. 84. Graphic Representation of the Momentary Power in a Polyphase System.

77. The Topographic Representation of Pressures. Whilst considering star systems, we saw that they possessed a junction—known as the neutral point. We make the assumption that this point possesses zero potential quite arbitrarily, for it is not potentials but only potential differences that we measure.

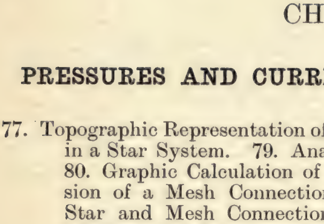


FIG. 212.—Pressure Diagram of Symmetrical Three-phase Star System.

In Fig. 212 let the three vectors $\overline{OP}_{I}$, $\overline{OP}_{II}$ and $\overline{OP}_{III}$ represent the three equal phase pressures of a symmetrical three-phase star system. Since the direction of rotation of the time-line has been chosen counter-clockwise, $\overline{OP}_{II}$ must be displaced 120° from OP_I in a counter-clockwise direction, for the E.M.F. of phase II lags 120° behind that of phase I. As shewn in Sect. 6, p. 17, a vector is determined in magnitude and direction by its two components, that is, by its extremity, and a point in the plane represents the pressure between a point in the system and the neutral point in magnitude and direction. Moreover, we have seen that the line pressure equals the difference of the two phase pressures. This difference P_l is determined by the geometrical subtraction of the two vectors $\overline{OP}_{I}$ and $\overline{OP}_{II}$, and we get

$$P_l = \overline{OP}_{I} - \overline{OP}_{II} = \overline{P_{II}O} + \overline{OP}_{I} = \overline{P_{II}P_{I}},$$

whence it follows that the distance between the ends of the two vectors gives the line pressure P_l in magnitude and direction. In

general, we have the following method of representation, as given by *Steinmetz* and *Berg* and also by *H. Görges* in the *E.T.Z.* 1898, p. 164.

If we take the potential at any point in a system as zero, the potential of a second point (i.e. the pressure between this point and the point at zero potential) is represented in magnitude and direction by a point in the plane. In this manner, each point of the system is represented by a corresponding point in the plane, and since the potential of a conductor varies from point to point along its length, the same will be represented in the plane by a curve; this has already been explained on p. 89, Sect. 29. The shape of the curve, of course, depends solely on the E.M.F.'s in the conductor. The curve may be a straight line or other curve either continuous or broken. If there is no current in the conductor, the potential at a point equals the sum of all the E.M.F.'s from the point where the potential is zero to the point considered. When no current flows in the conductor and no E.M.F.'s are present, the conductor has the same potential everywhere, and will be represented in the plane by a single point. On the other hand, if

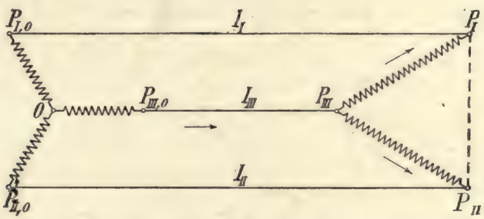


FIG. 213.—Symmetrical Three-phase System with Unbalanced Load.

the conductor carries the current I, the potential will be displaced by the distance Ir, owing to the ohmic resistance r, in the direction opposing the current, and by the distance Ix, owing to the total reactance $x = x_s - x_c$, in the direction lagging 90° behind the current. The curve of potential along the conductor can be drawn point by point in this way, when we thus start at a point with given potential.

This method of representation is well adapted for showing clearly the pressure relations in a polyphase system, whilst the distance between two points in the plane of the co-ordinates gives directly the effective pressure between the two corresponding points in the system. From Fig. 212 we see at once that the line pressure of a three-phase system equals $\sqrt{3}$ times the phase pressure; similarly, from Fig. 215, it is obvious that, in an interconnected two-phase system, the line pressure at no-load equals $\sqrt{2}$ times the pressure of a phase, and so on.

For the first example of this method of representation, we shall consider a three-phase system in which the current producer is star connected and the current consumer mesh connected. Let only two phases of the Δ system be loaded, the third being left open (Fig. 213).

If the system is unloaded, the three equidistant points P_{I0}, P_{II0}, P_{III0}

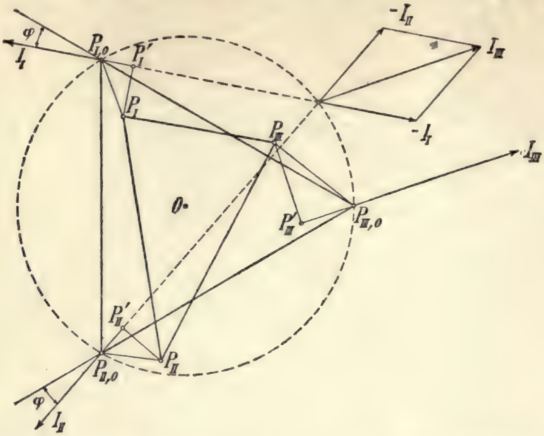


FIG. 214.—Symmetrical Three-phase System with Unbalanced Load.

(Fig. 214) represent the three potentials at the terminals of a symmetrical star system, provided that the potential of the neutral point

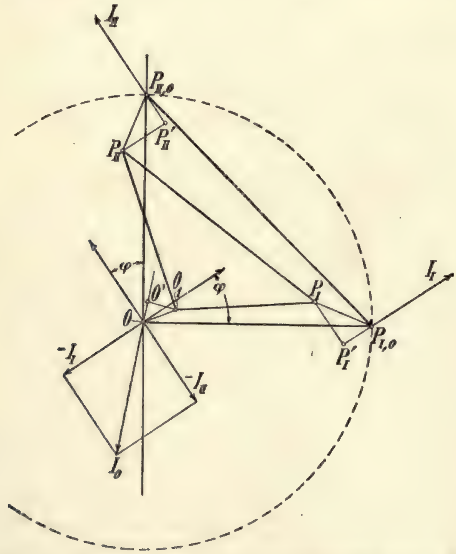


FIG. 215.—Unsymmetrical Two-phase Three-wire System with Balanced Load.

falls in the centre of the circle O. Now let the phases I and II be equally loaded; the currents I_I and $I_{II} = I_I$ are then represented by two

equal vectors making the same angle ϕ with their inducing E.M.F.'s $\overline{P_{I0}P_{III0}}$ and $\overline{P_{II0}P_{III0}}$. The current I_{III} flowing in the third phase is the geometrical sum of $-I_I$ and $-I_{II}$. On account of the currents flowing in the phases, the no load potentials at the terminals P_{I0}, P_{II0} and P_{III0} are shifted to P_I, P_{II} and P_{III}, where e.g. $\overline{P_{I0}P'_I}=I_Ir$ is in the opposite direction to I_I and $\overline{P'_IP_I}=I_Ix$ lags 90° behind the current; thus $\overline{P_{I0}P_I}$ equals I_Iz, and so on. From this we see that a symmetrical three-phase system with unsymmetrical load has no longer an equilateral pressure triangle, as $P_{I0}P_{II0}P_{III0}$ on no-load, but in this case an isosceles (*unbalanced*) triangle $P_IP_{II}P_{III}$.

As a second example, consider an unsymmetrical two-phase three-wire system with symmetrical load (Fig. 215). P_{I0}, P_{II0} and O give the terminal potentials at no-load. I_I and I_{II} are the phase currents, whilst I_0 (the current in the middle wire) is the geometrical sum of $-I_I$ and $-I_{II}$. On account of these currents, the potentials P_{I0}, P_{II0} and O are displaced to the points P_I, P_{II} and O_I. Since the pressure triangle $P_IP_{II}O_I$ is not rectangular, we see that even with symmetrical loading, the interconnected two-phase system is not exactly balanced.

78. Graphic Calculation of Current in a Star System.

Method I. In the previous section, for the sake of simplicity, we assumed that the load current of the several phases was known both in magnitude and direction. Strictly speaking, this is seldom the case. In practice, however, it is often possible to estimate the currents in the several phases with close approximation, and from these determine the pressure drops in the different phases by the above method.

If, however, we have to treat an unsymmetrically loaded system with large pressure drops in generators, mains and transformers, it is necessary, under certain conditions, to calculate these more exactly than is possible by using the above method. For this purpose we turn to the following problem:

To calculate the currents and pressures in a star system, whose generators and load admittances are all star connected. The E.M.F.'s in the several phases are known, also the resistances, reactances and load admittances.

We assume as before that the neutral point of the generator possesses zero potential. Then at no-load the terminals of the various phases have a potential corresponding to the E.M.F.'s induced in these phases. These E.M.F.'s may have any desired shape and strength. Assume, for the present, that the potential of the neutral point of the load is known; the potential difference consumed in each phase is then also known. This is, namely, equal to the potentials at the terminals of the phases at no-load, less the potential of the neutral point of the load. The current in any phase then equals the potential difference consumed in that phase divided by its total impedance. If the current is thus found in magnitude and direction, the potential at any point of the system can be easily deduced by the above method. Thus the

pressure drop from no-load to load can be simply determined for each phase.

The knowledge of the potential of the star point of the load will thus simplify the whole problem, for each phase can then be treated independently of the rest.

The determination of the potential of this neutral point, however, offers some difficulties, which can be best overcome as follows. As

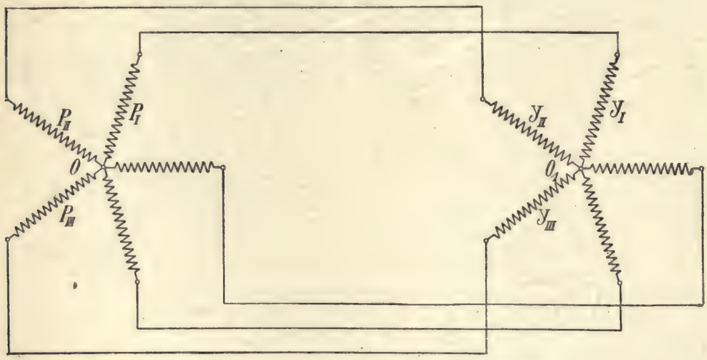


FIG. 216.—Polyphase Star-connected Generator.

example, consider the star system shewn in Fig. 216; the E.M.F.'s induced per phase can be represented by $\overline{OP}_{I0}$, $\overline{OP}_{II0}$, $\overline{OP}_{III0}$, $\overline{OP}_{IV0}$ and $\overline{OP}_{V0}$ (Fig. 217). The points P_{I0}, P_{II0}, ... P_{V0} give the no-load potentials at the terminals of the generator. The total admittances of the five phases can be represented by $g_I b_I$, $g_{II} b_{II}$, and so on. In these, the resistances and reactances of the windings of the several phases are also considered. At the ends of the pressure vectors, set off the conductances g... of the several phases parallel to the ordinate axis, and from the ends of these the susceptances b in the horizontal direction. In this way the admittances y appear as lines which are displaced from the ordinate axis by the phase-displacement angle ϕ of the several phase currents. We suppose the problem to be solved, and O_I the neutral point of the load circuit to be found; the effective E.M.F.'s of the several phases are then represented by the vectors $O_I P_{I0}$, $O_I P_{II0}$, and so on, whilst the phase currents are displaced from their respective E.M.F.'s by the angle ϕ. From Kirchhoff's First Law, the sum of the currents in all the phases at any instant must equal zero, if all in the same sense with respect to the neutral point are taken as positive.

Consider now, for example, the effective E.M.F. $\dot{P}_{III} = \overline{O_I P}_{III0}$ in phase III with the current $\dot{I}_{III}$ lagging Φ_{III} behind it. We then know that $\dot{I}_{III} = P_{III} \dot{y}_{III}$. Choose the time-line parallel to the abscissa axis; the momentary value is then

$$i_{III} = \sqrt{2} I_{III} \cos a_{III} = \sqrt{2} y_{III} P_{III} \cos a_{III}.$$

From O_{I} draw a normal on to y_{III}; this then makes an angle α_{III} with $\overline{O_{\mathrm{I}}P}_{\mathrm{III}0}$, and the shortest distance of the point O_{I} from y_{III} is $\overline{O_{\mathrm{I}}P}_{\mathrm{III}0} \cos \alpha_{\mathrm{III}}$. Imagine y_{III} to be a force; then, neglecting the factor $\sqrt{2}$, the moment of this force with regard to the pole O_{I} is represented by the momentary value i_{III} of the current I_{III}. The condition that the sum of the currents in all the phases equals zero at any instant is, therefore, the sum of the moments of all the forces y with respect to the point O_{I} must equal zero, or O_{I} must lie on the resultant of all the

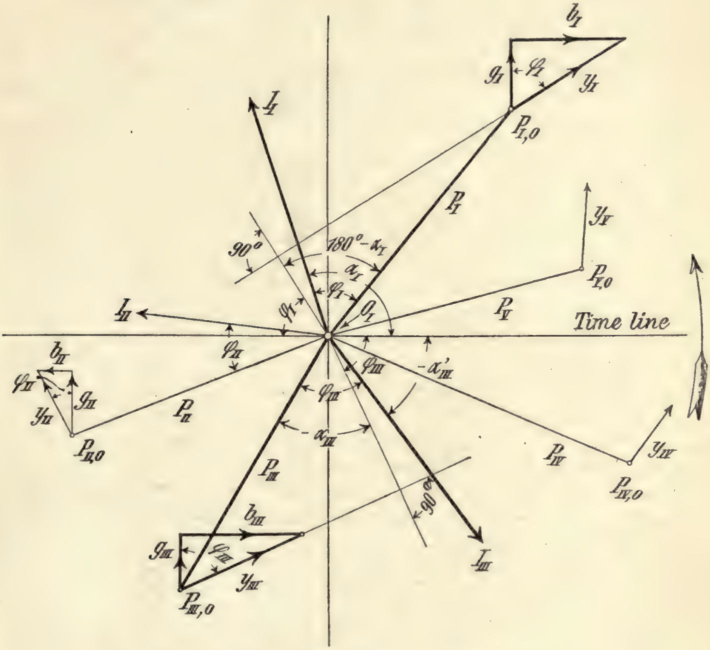


FIG. 217.—Determination of Potential of Load Star Point.

forces y. If the time-line rotates with the angular velocity ω, the forces y must also rotate with the same velocity, so that the lines g always remain normal to the time-line and the momentary values of the currents proportional to the moments of the forces y with respect to O_{I}. Imagine now that the whole diagram $O_{\mathrm{I}}P_{\mathrm{I}0}\,P_{\mathrm{II}0}\,P_{\mathrm{III}0}\,P_{\mathrm{IV}0}\,P_{\mathrm{V}0}$ as a rigid system at the terminals of which the corresponding forces y act; we know then, that if the forces be turned through equal angles about the points of application, the resultant of these forces will likewise turn through the same angle about a fixed point. This centre of the system of forces must coincide with O_{I} in order that the condition "the sum of all the moments is zero" is satisfied. From this the construction

for the point O_{I} follows at once by finding the resultant of the forces y in two directions (e.g. at 90° apart). The point of intersection of these then gives the *potential O_{I} of the load star point.*

In Fig. 217 the momentary value of the current I_{III} is positive, and the moment of the force y_{III} with respect to the middle point O_{I} of the pressure must therefore be also positive. The moment of the admittance, which represents a current, will be also called *a current moment* in what follows. The momentary value of the current I_{I} in Fig. 217 is negative, and equals

$$i_{\mathrm{I}} = \sqrt{2}I_{\mathrm{I}} \cos \alpha_{\mathrm{I}} = -\sqrt{2}I_{\mathrm{I}} \cos (180 - \alpha_{\mathrm{I}})$$
$$\doteq -\sqrt{2}y_{\mathrm{I}}P_{\mathrm{I}} \cos (180 - \alpha_{\mathrm{I}}).$$

$\sqrt{2}y_{\mathrm{I}}P_{\mathrm{I}} \cos (180 - \alpha_{\mathrm{I}})$ equals the moment of the force y_{I} with regard to O_{I}. This moment, which acts in a clockwise direction when taken negative, gives the momentary value of the current I_{I} with its corresponding sign (disregarding the factor $\sqrt{2}$). From this it follows that all current moments acting in a counter-clockwise direction are to be taken as positive, and all acting in a clockwise direction as negative. This positive sense of the current moments is due to the direction of rotation assumed for the time-line, with which the former agrees.

In Fig. 217 the currents I_{I} and I_{III} lag behind their respective pressures P_{I} and P_{III} in phase; nevertheless the susceptances b_{I} and b_{III} must be set off along the positive direction of the abscissa axis, when the conductances are set off along the positive direction of the ordinate axis, for the whole construction to be correct. The current I_{II} leads its pressure P_{II}, so that b_{II} must be set off in the negative direction of the abscissa axis. This definite direction for the admittance forces y arises from the chosen direction of rotation of the time-line.

After we have thus determined the potential of the neutral point of the load system and knowing the effective E.M.F.'s and pressures in each phase, we can find the current in each phase. The currents cause a drop of potential in the windings of the generator and in the line, which causes a displacement of the potential at the receiver terminals. This displacement equals Ir in the direction of the current and Ix normal to it, as already explained. If the E.M.F.'s and loads in the phases are not all the same, the pressures at the receiver circuit may differ considerably.

The above method for finding the neutral point was first suggested by Kennelly, *Elec. World and Engineer* 1899, p. 268.

In the special case of a symmetrical star system whose phases are symmetrically loaded, the neutral point O_{I} of the load coincides with the neutral point O of the generator, which can at once be seen from symmetry. The same current flows in each phase, and the no-load potentials, $P_{\mathrm{I}0}$, $P_{\mathrm{II}0}$, $P_{\mathrm{III}0}$, and so on, at the receiver terminals are displaced by the same amount; the system remains symmetrical and balanced.

If we have a star system with neutral wire, the neutral point O_{I} can also be determined by the above method. For this purpose it is only

necessary to introduce a force y_0 at the point O corresponding to the admittance of the neutral line, in order to consider the influence of the neutral wire on the potential of the point O_I. When y_0 is equal to zero we have the system in which no neutral wire is present,—while for the case y_0 equal to infinity, O_I and O have the same potential. The points are then short-circuited, so that the current and drop of pressure in any one phase has no effect on the loads in the other phases.

The conversion problem treated by Kennelly in the above-mentioned paper is of interest, for it also shews how, by suitably choosing the

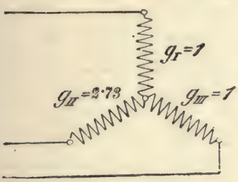


FIG. 218*a*.

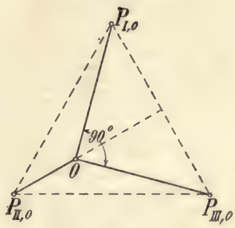


FIG. 218*b*.

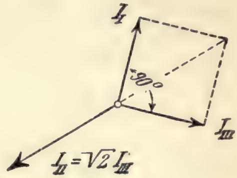


FIG. 218*c*.

FIG. 218*a–c*.—Diagram of Symmetrical Three-phase System supplying Two-phase Current.

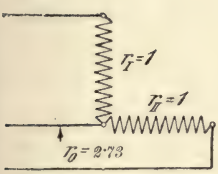


FIG. 219*a*.

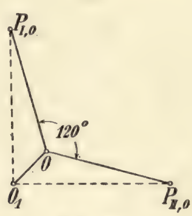


FIG. 219*b*.

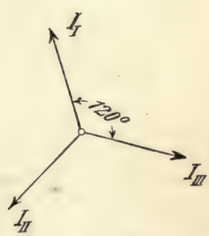


FIG. 219*c*.

FIG. 219*a–c*.—Diagram of an Interlinked Two-phase System supplying a Balanced Three-phase Current.

three load resistances of a symmetrical three-phase system, the same can be made to deliver a two-phase current. The conductances of the three load resistances (Fig. 218*a*) must bear the ratio $1:1:2·73$. Fig. 218*b* shews the pressures of the various phases, of which $\overline{OP}_{I0}$ and $\overline{OP}_{III0}$ are perpendicular to one another. Fig. 218*c* is the diagram of the currents.

Conversely, a symmetrical three-phase current can be taken from an interlinked two-phase system, by making the load resistances of the two phases equal and in the ratio $1:(1+\sqrt{3})$ to the resistance of the neutral wire (see Fig. 219*a*). Figs. 219*b* and *c* shew respectively the pressure and current diagrams for this arrangement.

79. Analytic Calculation of Current in a Star System. The graphic method described in Section 78 for the determination of the middle point O_{I} of the pressure is not always convenient, especially in the case of a star system with a neutral wire; for the latter has usually a much greater conductance than one of the loaded phases.

Further, the admittances are often nearly parallel, so that graphic summation is inconvenient and inexact, unless the resultants of the forces y are found by means of the force and vector polygon, as is customary in graphic statics.

We shall, therefore, first shew how the currents and the middle point O_{I} of the pressure of a star system, with and without neutral point, can be analytically determined.

Method I. The no-load pressures $P_{\mathrm{I}0}$, $P_{\mathrm{II}0}$, etc., of the several phases, which equal the induced E.M.F.'s, will be denoted in general by $\dot{P}_{x0}$ for a phase and the admittances of the phases by $\dot{y}$. Then

$$\Sigma(\dot{P}_{x0} - \dot{P}_0)\dot{y} = \dot{I}_0 = \dot{P}_0\dot{y}_0,$$

where $\dot{P}_0$ is the potential of the middle point O_1 of the pressure, $\dot{I}_0$ the current in and $\dot{y}_0$ the admittance of the neutral wire. From this

$$\Sigma(\dot{P}_{x0}\dot{y}) = \dot{P}_0\Sigma(\dot{y}) + \dot{P}_0\dot{y}_0,$$

where $$\Sigma(\dot{P}_{x0}\dot{y}) = \dot{I}_{0K} = \dot{I}'_{\mathrm{I}} + \dot{I}'_{\mathrm{II}} + \dot{I}'_{\mathrm{III}}, \text{ etc.}$$

$\dot{I}_{0K}$ is the current which would flow in the neutral wire if the two neutral points were connected by a wire with zero resistance, whilst $\dot{I}'_{\mathrm{I}}$, $\dot{I}'_{\mathrm{II}}$, etc., denote the currents in the phases under this assumption.

If these currents are calculated, we have

$$\dot{P}_0 = \frac{\dot{I}_{0K}}{\Sigma(\dot{y}) + \dot{y}_0} = \dot{I}_{0K}\frac{\Sigma(g) + g_0 - j\{\Sigma(b) + b_0\}}{\{\Sigma(g) + g_0\}^2 + \{\Sigma(b) + b_0\}^2}.$$

If $\dot{P}_0$ is known, we calculate

$$\dot{I}_{\mathrm{I}0} = \dot{P}_0\dot{y}_{\mathrm{I}},$$

$$\dot{I}_{\mathrm{II}0} = \dot{P}_0\dot{y}_{\mathrm{II}}, \text{ and so on.}$$

Finally, $$\dot{I}_0 = \dot{P}_0\dot{y}_0,$$

where $$\dot{I}_{\mathrm{I}0} + \dot{I}_{\mathrm{II}0} + \dot{I}_{\mathrm{III}0} + \ldots + \dot{I}_0 = \dot{I}_{0K}.$$

The phase currents are also easy to find, for

$$\dot{I}_{\mathrm{I}} = \dot{P}_{\mathrm{I}0}\dot{y}_{\mathrm{I}} - \dot{P}_0\dot{y}_{\mathrm{I}} = \dot{I}'_{\mathrm{I}} - \dot{I}_{\mathrm{I}0}.$$

Similarly, $$\dot{I}_{\mathrm{II}} = \dot{I}'_{\mathrm{II}} - \dot{I}_{\mathrm{II}0}, \text{ etc.}$$

Let us take any given star system, and supposing first that the two neutrals are connected, as in Fig. 220, calculate the current distribution —for instance, for $\dot{P}_0 = 0$. We have then

$$\dot{I}'_{\mathrm{I}} + \dot{I}'_{\mathrm{II}} + \ldots = \dot{I}_{0K}.$$

Secondly, we will suppose the current $\dot{I}_{0K}$ distributed over all the parallel conductors in the systems in proportion to their admittances,

by putting the phase pressures $\dot{P}_{I0}$, $\dot{P}_{II0}$, etc., equal to zero and calculating the currents $\dot{I}_{I0}$, $\dot{I}_{II0}$ ... $\dot{I}_0$ as if only $\dot{P}_0$ were present (see Fig. 221).

We have here, therefore,

$$\dot{I}_{I0} + \dot{I}_{II0} + \dots + \dot{I}_0 = \dot{I}_{0K}.$$

The phase currents are then obtained by superposing the two current distributions in Figs. 220 and 221.

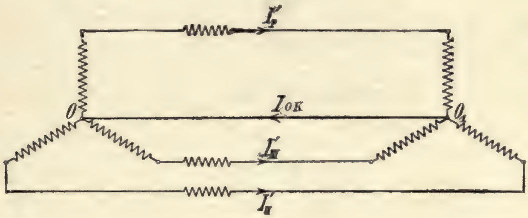


FIG. 220.

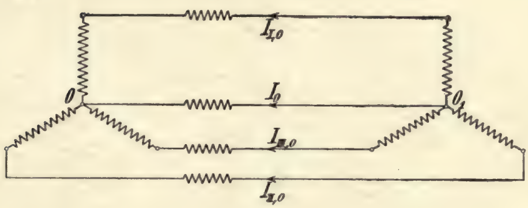


FIG. 221.

To take a practical example, we will go through the calculations for a star system. Let us take a three-phase generator, star connected,

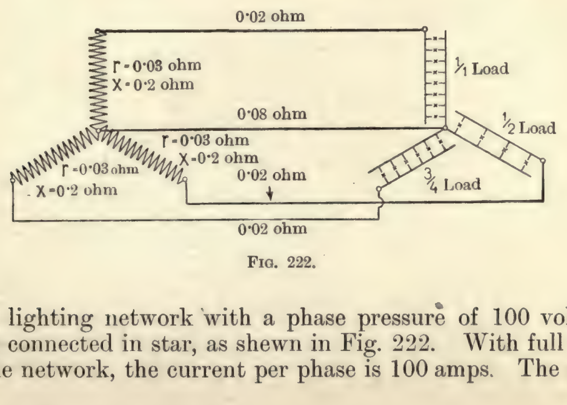


FIG. 222.

feeding a lighting network with a phase pressure of 100 volts. The lamps are connected in star, as shewn in Fig. 222. With full balanced load in the network, the current per phase is 100 amps. The armature

winding of the generator has an effective resistance of 0·03 ohm and a reactance of 0·2 ohm per phase. The mains between generator and receiver have a resistance of 0·02 ohm per phase, whilst the neutral line possesses a resistance of 0·08 ohm; the self-induction of the mains and incandescent lamps is negligibly small.

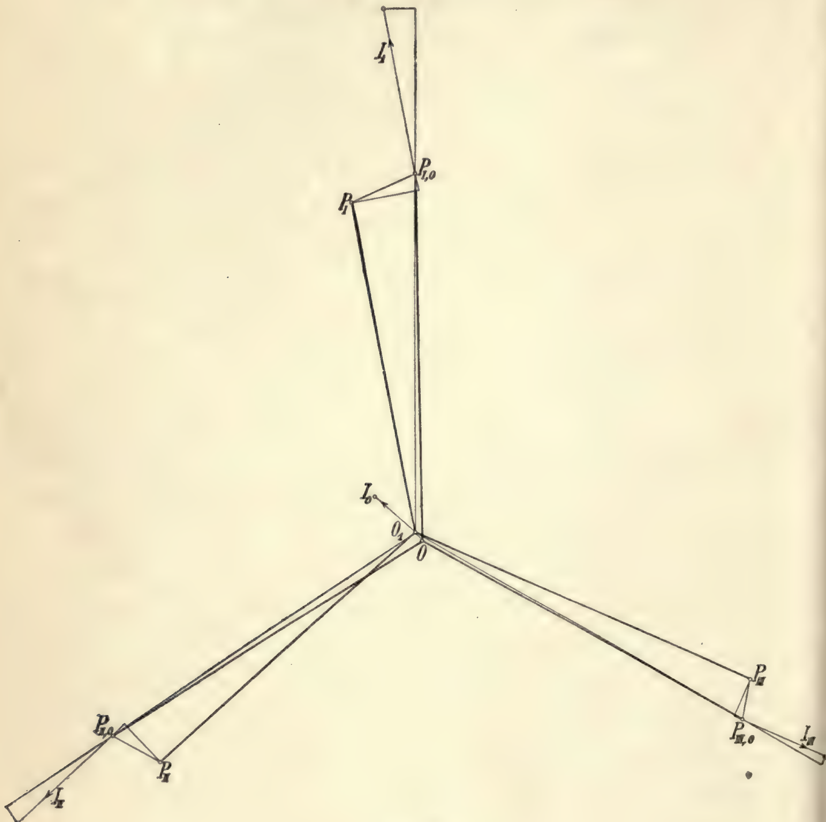


FIG. 223.

We shall determine the distribution of current and pressure in the system, assuming that the first phase is fully loaded, the second working on $\frac{3}{4}$ full-load and the third on half-load. In all the three phases of the generator, the same effective E.M.F. of 100 volts is induced; hence the no-load potentials of the four terminals of the generator are represented by the points O, P_{I0}, P_{II0} and P_{III0} (Fig. 223). The first phase of the load network has a conductance of 1 mho or a resistance of 1 ohm, the second phase $\frac{3}{4}$ mho or 1·333 ohms, and the third phase

$\frac{1}{2}$ mho or 2 ohms. To these resistances must be added the resistances of the three lines and the phases of the generator, so that we have

$$r_{\mathrm{I}}=1{\cdot}05, \quad x_{\mathrm{I}}=0{\cdot}2 \quad \text{or} \quad g_{\mathrm{I}}=0{\cdot}922, \quad b_{\mathrm{I}}=0{\cdot}1755\,;$$
$$r_{\mathrm{II}}=1{\cdot}38, \quad x_{\mathrm{II}}=0{\cdot}2 \quad \text{or} \quad g_{\mathrm{II}}=0{\cdot}710, \quad b_{\mathrm{II}}=0{\cdot}103\,;$$
$$r_{\mathrm{III}}=2{\cdot}05, \quad x_{\mathrm{III}}=0{\cdot}2 \quad \text{or} \quad g_{\mathrm{III}}=0{\cdot}484, \quad b_{\mathrm{III}}=0{\cdot}0473\,;$$

and $$r_0=0{\cdot}08 \text{ ohm} \quad \text{or} \quad g_0=12{\cdot}5 \text{ mhos}.$$

The impedance between the neutral points is

$$r-jx=\frac{g_0+g_{\mathrm{I}}+g_{\mathrm{II}}+g_{\mathrm{III}}-j(b_0+b_{\mathrm{I}}+b_{\mathrm{II}}+b_{\mathrm{III}})}{(g_0+g_{\mathrm{I}}+g_{\mathrm{II}}+g_{\mathrm{III}})^2+(b_0+b_{\mathrm{I}}+b_{\mathrm{II}}+b_{\mathrm{III}})^2}$$
$$=0{\cdot}0684-j0{\cdot}00152 \text{ ohm}.$$

We calculate now

$$\underset{\cdot}{I}'_{\mathrm{I}}=P_{\mathrm{I}0}y_{\mathrm{I}}=100(0{\cdot}922+j0{\cdot}1755)$$
$$=92{\cdot}2+j17{\cdot}55 \text{ amps.},$$
$$\underset{\cdot}{I}'_{\mathrm{II}}=P_{\mathrm{II}0}y_{\mathrm{II}}=(-50+j86{\cdot}6)(0{\cdot}710+j0{\cdot}013)$$
$$=-44{\cdot}4+j56{\cdot}4 \text{ amps.},$$
$$\underset{\cdot}{I}'_{\mathrm{III}}=P_{\mathrm{III}0}y_{\mathrm{III}}=(-50-j86{\cdot}6)(0{\cdot}484+j0{\cdot}0473)$$
$$=-20{\cdot}1-j44{\cdot}3 \text{ amps.}$$

From this we find

$$\underset{\cdot}{I}_{0K}=27{\cdot}7+j29{\cdot}7 \text{ amps.},$$
$$\underset{\cdot}{P}_0=(0{\cdot}0684-j0{\cdot}00152)(27{\cdot}7+j29{\cdot}7)$$
$$=1{\cdot}94+j1{\cdot}99 \text{ volts}.$$

This difference of potential produces the following currents:

$$\underset{\cdot}{I}_{\mathrm{I}0}=P_0y_{\mathrm{I}}=1{\cdot}44+j2{\cdot}18 \text{ amps.},$$
$$\underset{\cdot}{I}_{\mathrm{II}0}=P_0y_{\mathrm{II}}=1{\cdot}17+j1{\cdot}61 \text{ amps.},$$
$$\underset{\cdot}{I}_{\mathrm{III}0}=P_0y_{\mathrm{III}}=0{\cdot}85+j1{\cdot}06 \text{ amps.},$$
$$\underset{\cdot}{I}_0=P_0y_0=24{\cdot}22+j24{\cdot}85 \text{ amps.}$$

Finally, we get

$$\underset{\cdot}{I}_{\mathrm{I}}=\underset{\cdot}{I}'_{\mathrm{I}}-\underset{\cdot}{I}_{\mathrm{I}0}=-90{\cdot}76+j15{\cdot}37 \text{ amps.},$$
$$\underset{\cdot}{I}_{\mathrm{II}}=\underset{\cdot}{I}'_{\mathrm{II}}-\underset{\cdot}{I}_{\mathrm{II}0}=-45{\cdot}57+j54{\cdot}79 \text{ amps.},$$
$$\underset{\cdot}{I}_{\mathrm{III}}=\underset{\cdot}{I}'_{\mathrm{III}}-\underset{\cdot}{I}_{\mathrm{III}0}=-20{\cdot}95-j45{\cdot}36 \text{ amps.}$$

The absolute values of the phase currents are

$$I_{\mathrm{I}}=92 \text{ amps.}, \quad I_{\mathrm{II}}=71{\cdot}5 \text{ amps.}, \quad I_{\mathrm{III}}=49{\cdot}8 \text{ amps.}$$

The current I_{I} causes an ohmic drop in the armature winding and line $I_{\mathrm{I}}r=I_{\mathrm{I}}0{\cdot}05$ opposing the current, and an inductive drop $I_{\mathrm{I}}x=I_{\mathrm{I}}0{\cdot}2$, perpendicular to the current, as shewn in Fig. 223. Due to these two

pressure drops, the potential across the lamps in phase I is displaced from P_{I0} to P_{I}, and the lamp pressure is now $\overline{O_1P_I}$ instead of the no-load pressure $\overline{OP_{I0}}$. From Fig. 223 the lamp pressure of the three phases are

$$\overline{O_1P_I} = I_I \times 1 = 92 \text{ volts},$$
$$\overline{O_1P_{II}} = I_{II} \times 1{\cdot}33 = 95 \text{ volts},$$
$$\overline{O_1P_{III}} = I_{III} \times 2 = 99{\cdot}6 \text{ volts},$$

thus shewing the effect of the out-of-balance load.

If all phases had been equally loaded with 100 amperes, the lamp pressure would have fallen to 93 volts in each phase.

80. Graphic Calculation of Current in a Polyphase System.

Method II. As well as the analytic method in the previous section, the following simple graphic method can also be used. We will describe it in connection with a symmetrical three-phase star system with phases loaded unsymmetrically and without neutral wire.

FIG. 224.

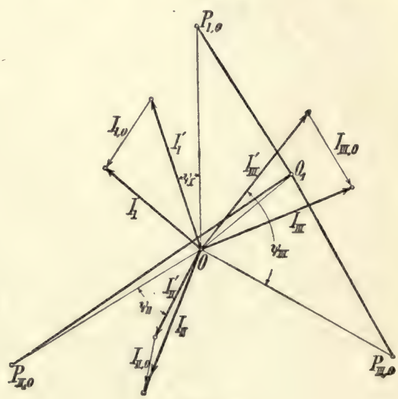


FIG. 225.

In Fig. 224, the no-load pressures P_{K0} of all the phases are drawn in the same direction, viz. along the ordinate axis. I'_I, I'_{II} and I'_{III} are the currents which these pressures would produce if the neutrals of the generator and the load were directly connected. Since all the no-load pressures are equal in magnitude in a symmetrical system, the currents I'_I, I'_{II} and I'_{III} in such a system will be proportional to the admittances y_I, y_{II} and y_{III} of the three phases. To the same scale, the vector $\overline{OA} = I'_I + I'_{II} + I'_{III}$ represents the total admittance $y = y_I + y_{II} + y_{III}$ between the neutral points.

In Fig. 225, the currents I'_I, I'_{II} and I'_{III} are drawn at their correct phase angles ψ_I, ψ_{II} and ψ_{III} to the no-load pressures P_{I0}, P_{II0} and P_{III0}.

Finally, we draw Fig. 226, in which the currents I'_I, I'_{II} and I'_{III}

are geometrically added, giving the current $\dot{I}_{0K} = \dot{I}'_{\mathrm{I}} + \dot{I}'_{\mathrm{II}} + \dot{I}'_{\mathrm{III}}$, which must flow between the neutral points, from the load to the generator. The current $\dot{I}_{0K}$ is now distributed among the several phases in proportion to their admittances by drawing on $\dot{I}_{0K}$ a polygon similar to that formed by the currents $\dot{I}'_{\mathrm{I}}$, $\dot{I}'_{\mathrm{II}}$ and $\dot{I}'_{\mathrm{III}}$ on $\overline{OA}$ in Fig. 224. $\dot{I}_{\mathrm{I}0}$, $\dot{I}_{\mathrm{II}0}$ and $\dot{I}_{\mathrm{III}0}$ are the components of $\dot{I}_{0K}$ in the different phases. By drawing parallel lines, we add $\dot{I}'_{\mathrm{I}}$ and $-\dot{I}_{0\mathrm{I}}$ together in Fig. 225, and thus obtain the resultant current $\dot{I}_{\mathrm{I}}$ in phase I. Similarly for the other phases.

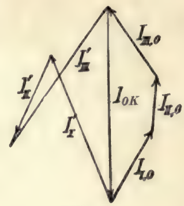


FIG. 226.

We have determined the phase currents without finding the potential of the neutral point O_1 of the load. This can now be found at once, for the potential differences must be proportional to the currents they produce. Thus, for phase I,

$$\overline{O_1P}_{\mathrm{I}0} : \overline{OP}_{\mathrm{I}0} = I'_{\mathrm{I}} : I_{\mathrm{I}},$$

$$\overline{O_1P}_{\mathrm{I}0} = \frac{I_{\mathrm{I}}}{I'_{\mathrm{I}}} P_{\mathrm{I}0}.$$

Similarly for the other phases,

$$\overline{O_1P}_{\mathrm{II}0} = \frac{I_{\mathrm{II}}}{I'_{\mathrm{II}}} P_{\mathrm{II}0},$$

$$\overline{O_1P}_{\mathrm{III}0} = \frac{I_{\mathrm{III}}}{I'_{\mathrm{III}}} P_{\mathrm{III}0}.$$

If we strike off arcs about the points $P_{\mathrm{I}0}$, $P_{\mathrm{II}0}$ and $P_{\mathrm{III}0}$ with these radii in Fig. 225, they will all cut in the point O_1. This is the middle point of pressure in the load. For each phase we get a pressure triangle similar to the current triangle for the same phase.

The direct determination of the point O_1 is most easily done by the construction in Fig. 227. Here again

$$\dot{P}_{\mathrm{I}0}(\dot{y}_{\mathrm{I}} + \dot{y}_{\mathrm{II}} + \cdots) = \dot{P}_{\mathrm{I}0}\dot{y} = \dot{I}$$

is represented by the vector $\overline{OA}$. Further,

$$\dot{P}_{\mathrm{I}0}\dot{y}_{\mathrm{I}} + \dot{P}_{\mathrm{II}0}\dot{y}_{\mathrm{II}} + \cdots = \dot{I}'_{\mathrm{I}} + \dot{I}'_{\mathrm{II}} + \cdots = \dot{I}_{0K}.$$

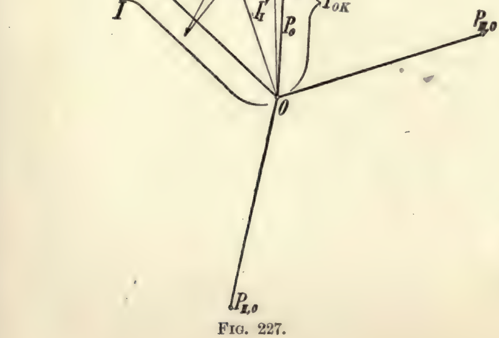


FIG. 227.

On the other hand,

$$\dot{I}_{0K} = \dot{P}_0 y = \frac{\dot{I}}{\dot{P}_{10}} \dot{P}_0 \quad \text{or} \quad \frac{\dot{I}_{0K}}{\dot{P}_0} = \frac{\dot{I}}{\dot{P}_{10}},$$

i.e. if we rotate the co-ordinate system of the pressures so that the direction of $\dot{P}_{10}$ coincides with that of $\dot{I}$, then $\dot{P}_0$ lies in the direction of $\dot{I}_{0K}$, and it is only necessary to construct the fourth proportional in Fig. 227 to find the point O_1.

81. Conversion of a Mesh Connection into a Star Connection. Of the different ring-connected systems, mesh connection is almost the only one which has found favour in practice; consequently we must study this connection more especially.

In the previous section was shewn how the neutral point of a star connection can be easily determined, and the calculation of the currents in a star system thus reduced to the treatment of simple conductors. In order to obtain the same simplicity for a mesh connection, the following method due to Kennelly (*Electrical World*, vol. 34, p. 413) for converting a mesh connection into an equivalent star connection—with respect to the outside circuit—may be used.

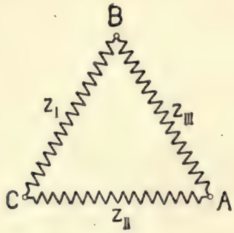


FIG. 228*a*.

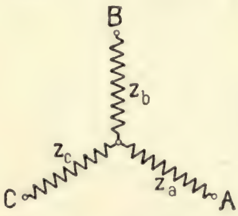


FIG. 228*b*.

FIG. 228.—Mesh System and its Equivalent Star System.

Fig. 228*a* represents a mesh system with the impedances z_{I}, z_{II}, z_{III} in the several branches. Let the equivalent star connection (Fig. 228*b*) have the impedances z_a, z_b and z_c. Now, in order that the mesh can be replaced by the star without altering the conditions in the external circuit, the impedances between the three terminals A, B and C of the star must equal the impedances between the angles A, B and C of the mesh. We have thus the following symbolic expressions for the impedances:

$$z_a + z_b = \frac{z_{\text{III}}(z_{\text{I}} + z_{\text{II}})}{z_{\text{I}} + z_{\text{II}} + z_{\text{III}}},$$

$$z_b + z_c = \frac{z_{\text{I}}(z_{\text{II}} + z_{\text{III}})}{z_{\text{I}} + z_{\text{II}} + z_{\text{III}}},$$

$$z_c + z_a = \frac{z_{\text{II}}(z_{\text{III}} + z_{\text{I}})}{z_{\text{I}} + z_{\text{II}} + z_{\text{III}}}.$$

Multiplying each of these equations in turn by -1 and adding, we obtain

$$\left.\begin{aligned} z_a &= \frac{z_{\mathrm{II}} z_{\mathrm{III}}}{z_{\mathrm{I}} + z_{\mathrm{II}} + z_{\mathrm{III}}}, \\ z_b &= \frac{z_{\mathrm{III}} z_{\mathrm{I}}}{z_{\mathrm{I}} + z_{\mathrm{II}} + z_{\mathrm{III}}}, \\ z_c &= \frac{z_{\mathrm{I}} z_{\mathrm{II}}}{z_{\mathrm{I}} + z_{\mathrm{II}} + z_{\mathrm{III}}}. \end{aligned}\right\} \qquad (146)$$

Substituting the complex quantities for z_{I}, z_{II} and z_{III} in these symbolic formulae and splitting up the expressions z_a, z_b and z_c into their real and imaginary components, we get the resistances and reactances of the equivalent star connection expressed in terms of those in the mesh connection.

$$\begin{aligned} r_a - jx_a &= \frac{(r_{\mathrm{II}} - jx_{\mathrm{II}})(r_{\mathrm{III}} - jx_{\mathrm{III}})}{r_{\mathrm{I}} + r_{\mathrm{II}} + r_{\mathrm{III}} - j(x_{\mathrm{I}} + x_{\mathrm{II}} + x_{\mathrm{III}})} \\ &= \frac{(r_{\mathrm{II}} - jx_{\mathrm{II}})(r_{\mathrm{III}} - jx_{\mathrm{III}})}{r - jx} \\ &= \frac{(r_{\mathrm{II}} - jx_{\mathrm{II}})(r_{\mathrm{III}} - jx_{\mathrm{III}})(r + jx)}{r^2 + x^2} \\ &= \frac{r(r_{\mathrm{II}} r_{\mathrm{III}} - x_{\mathrm{II}} x_{\mathrm{III}}) + x(r_{\mathrm{II}} x_{\mathrm{III}} + r_{\mathrm{III}} x_{\mathrm{II}})}{z^2} \\ &\quad - j\frac{r(r_{\mathrm{II}} x_{\mathrm{III}} + r_{\mathrm{III}} x_{\mathrm{II}}) - x(r_{\mathrm{II}} r_{\mathrm{III}} - x_{\mathrm{II}} x_{\mathrm{III}})}{z^2}. \end{aligned}$$

In a similar manner, we get also

$$\begin{aligned} r_b - jx_b &= \frac{r(r_{\mathrm{III}} r_{\mathrm{I}} - x_{\mathrm{III}} x_{\mathrm{I}}) + x(r_{\mathrm{III}} x_{\mathrm{I}} + r_{\mathrm{I}} x_{\mathrm{III}})}{z^2} \\ &\quad - j\frac{r(r_{\mathrm{III}} x_{\mathrm{I}} + r_{\mathrm{I}} x_{\mathrm{III}}) - x(r_{\mathrm{III}} r_{\mathrm{I}} - x_{\mathrm{III}} x_{\mathrm{I}})}{z^2}, \\ r_c - jx_c &= \frac{r(r_{\mathrm{I}} r_{\mathrm{II}} - x_{\mathrm{I}} x_{\mathrm{II}}) + x(r_{\mathrm{I}} x_{\mathrm{II}} + r_{\mathrm{II}} x_{\mathrm{I}})}{z^2} \\ &\quad - j\frac{r(r_{\mathrm{I}} x_{\mathrm{II}} + r_{\mathrm{II}} x_{\mathrm{I}}) - x(r_{\mathrm{I}} r_{\mathrm{II}} - x_{\mathrm{I}} x_{\mathrm{II}})}{z^2}. \end{aligned}$$

Conversely, if a star connection is given, we can substitute for this a mesh connection.

In this case we assume that the admittances of the star are known, whilst the admittances of the mesh are to be determined.

If the two systems in Figs. 228a and b are equivalent, they will still be equivalent if we connect like circuits between two like terminals in

both. They will therefore be equivalent when we connect a circuit of impedance $z=0$ between A and B in both, i.e. if we short-circuit A and B. In this case we have

$$y_{\mathrm{I}}+y_{\mathrm{II}}=\frac{y_c(y_a+y_b)}{y_a+y_b+y_c}.$$

If we connect B, C and C, A in turn in the same way, we also get

$$y_{\mathrm{II}}+y_{\mathrm{III}}=\frac{y_a(y_b+y_c)}{y_a+y_b+y_c},$$

$$y_{\mathrm{III}}+y_{\mathrm{I}}=\frac{y_b(y_c+y_a)}{y_a+y_b+y_c}.$$

From these three equations, we then get

$$\left.\begin{aligned} y_{\mathrm{I}}&=\frac{y_b y_c}{y_a+y_b+y_c},\\ y_{\mathrm{II}}&=\frac{y_c y_a}{y_a+y_b+y_c},\\ y_{\mathrm{III}}&=\frac{y_a y_b}{y_a+y_b+y_c}.\end{aligned}\right\} \qquad (147)$$

From the last three expressions, which can also be expressed as complex quantities, the equivalent mesh connection of any star connection can be calculated.

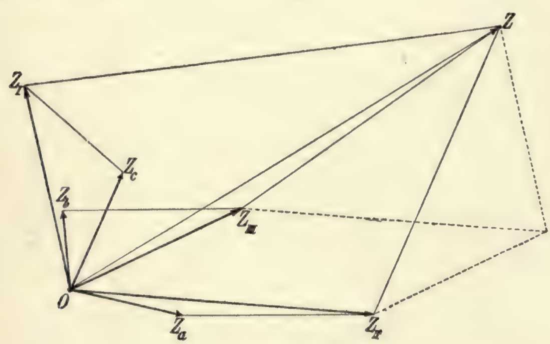


FIG. 229.—Graphical Transformation.

This problem of conversion can also be solved graphically. In Fig. 229, $\overline{OZ}_{\mathrm{I}}$, $\overline{OZ}_{\mathrm{II}}$ and $\overline{OZ}_{\mathrm{III}}$ represent the impedances z_{I}, z_{II} and z_{III} of a mesh connection in magnitude and direction.

To determine now the impedances z_a, z_b and z_c of the equivalent star connection, we first draw the vector $\overline{OZ}$ to represent the resultant impedance $z=z_{\mathrm{I}}+z_{\mathrm{II}}+z_{\mathrm{III}}$, and then construct the triangle OZ_aZ_{II}

similar to $OZ_{III}Z$. Then $\overline{OZ_a}$ is the required impedance z_a in magnitude and direction, for the following geometric relation is fulfilled:

$$\frac{z_a}{z_{II}} = \frac{z_{III}}{z_I + z_{II} + z_{III}},$$

thus satisfying Eq. 146. The construction for z_b and z_c is exactly similar.

The graphic determination of the admittances of the equivalent mesh from the admittances of the star is quite similar to the construction of Fig. 229, as shewn by Eq. 147.

Example. Let $g_I = 1$, $g_{II} = \frac{3}{4}$ and $g_{III} = \frac{1}{2}$ mho, or $r_I = 1$, $r_{II} = 1{\cdot}333$ and $r_{III} = 2$ ohms.

Find $r_a = z_a$, $r_b = z_b$ and $r_c = z_c$.

$$r_a = \frac{r_{II} r_{III}}{r_I + r_{II} + r_{III}} = \frac{1{\cdot}33 \, . \, 2}{1 + 1{\cdot}33 + 2} = \frac{2{\cdot}66}{4{\cdot}33} = 0{\cdot}614 \text{ ohm},$$

or $\quad g_a = 1{\cdot}63$ mhos;

$$r_b = \frac{r_{III} r_I}{r_I + r_{II} + r_{III}} = \frac{2}{4{\cdot}33} = 0{\cdot}462 \text{ ohm},$$

or $\quad g_b = 2{\cdot}16$ mhos;

and
$$r_c = \frac{r_I r_{II}}{r_I + r_{II} + r_{III}} = \frac{1{\cdot}33}{4{\cdot}33} = 0{\cdot}308 \text{ ohm},$$

or $\quad g_c = 3{\cdot}28$ mhos.

Thus a mesh connection whose phase loads g_I, g_{II} and g_{III} are in the ratio of 4 : 3 : 2 is equivalent to a star connection whose phase loads g_b, g_c and g_a bear the ratio of 4 : 3 : 2, whence it follows that the influence of unsymmetrical loading is no greater in a star system than in a mesh system.

82. Conversion of Star and Mesh Connections when E.M.F.'s are Induced in the Phases. Until now it has been assumed that no E.M.F.'s are induced in the phases which have to be transformed from mesh to star and vice versa. If such E.M.F.'s are present, we have to proceed precisely the same as before; considering, e.g., Fig. 230, where the paths of the mesh connection possess both E.M.F.'s and the impedances z_I, z_{II} and z_{III}, we can first imagine a condition where no current at all flows, on account of the E.M.F.'s in the star system maintaining equilibrium in the former—as can actually occur with generators working in parallel.

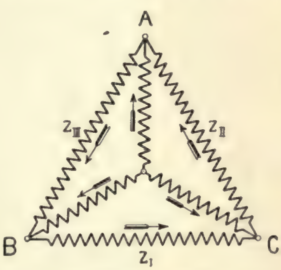


FIG. 230.—Transformation of a Mesh System where E.M.F.'s are induced in the Phases.

If the E.M.F. in one or more of the phases of the star connection is now altered, currents at once begin to flow, and these currents will depend only on the impedance of the

whole system and on the amount by which the E.M.F.'s in the star system are varied, since it is quite immaterial which E.M.F.'s maintain the equilibrium. Hence it follows that the impedances of the star system which is equivalent to the mesh system remain the same

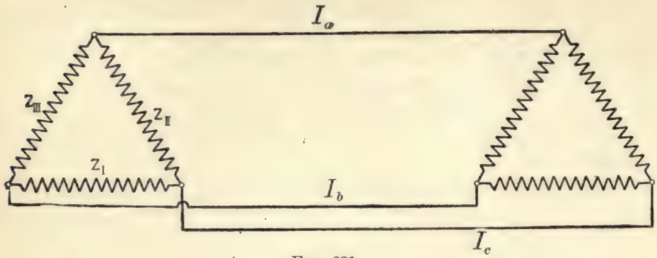


FIG. 231.

whether E.M.F.'s are present in the branches or not. As regards the conversion of star connections it is therefore immaterial whether E.M.F.'s are present or not.

As an example illustrating the complete procedure, we can take a system in which both the generator and the load are mesh con-

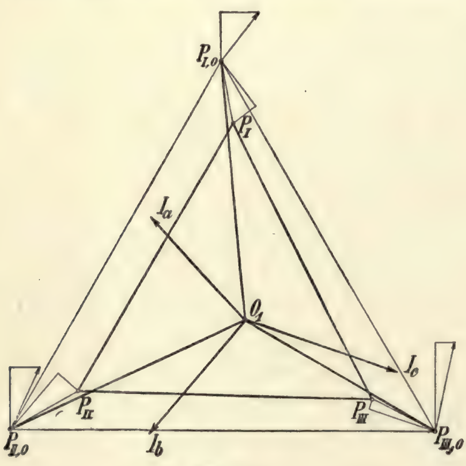


FIG. 232.

nected, as shewn in Fig. 231. We first calculate the impedances of the equivalent star connections, and then find the sum of the admittances in each phase and draw the pressure triangle for the generator on no-load (Fig. 232). At each corner of this triangle, we then set off the admittance of the corresponding phase as a force. The centre of these forces is then the neutral point O_1 of the load, and

the distances of this point from the angles of the pressure triangle give the E.M.F.'s of the phases. These, multiplied by the respective phase admittances, give the line currents (equal to the phase currents), which make angles $\tan^{-1}\frac{b}{g}$ with $\overline{O_1P_0}$. These currents cause a displacement of the potentials from the angles of the pressure triangle which is drawn for the terminal pressures at the generator on no-load. The

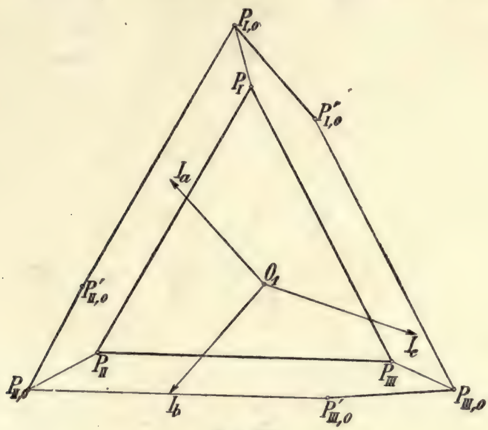


FIG. 233.

displacement of each angle is equal to the corresponding phase impedance of the equivalent star connection for the generator multiplied by the line current. The displacement Ir opposes the current in direction, whilst Ix lags behind the same by 90°. By this means, we get the three new angular points P_{I}, P_{II}, P_{III}, giving the pressure triangle of the generator on load (see Fig. 232).

In Fig. 233 the three lines I_a, I_b and I_c represent the three line currents. It is often useful, however, to know the currents in the network, i.e. in the branches of the mesh. These can be found for the generator by taking the geometrical difference $\overline{P_{\mathrm{I}0}P'_{\mathrm{I}0}}$ of $\overline{P_{\mathrm{I}}P_{\mathrm{III}}}$ and $\overline{P_{\mathrm{I}0}P_{\mathrm{III}0}}$, and dividing this difference by the impedance z_{II} of the branch connecting them (Fig. 234). If the currents of the load triangle are required, we must first construct the pressure triangle for the pressures at the terminals of the receiver. The sides of this triangle are the phase pressures, and each such side divided by the impedance of the respective branch gives the current in that part of the load triangle.

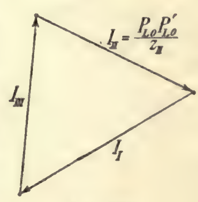


FIG. 234.

We have thus completely solved the given problem without knowing the potential of the neutral point of the equivalent star system for the generator—this point is unnecessary for the construction.

Example I. The load admittances of the mesh system are all alike in every respect.

Then, since $$z_a = \frac{z_{III} z_{II}}{z_I + z_{II} + z_{III}}$$

and $$z_I = z_{II} = z_{III} = z,$$

we have $$z_a = z_b = z_c = \tfrac{1}{3} z,$$

i.e. a mesh connection with equal impedances in all the branches can be replaced by a star connection whose phase impedance equals *one-third* of the phase impedance of the mesh connection. That this is so is clear, for with star connection, the pressure per phase is $\sqrt{3}$ times smaller and the current $\sqrt{3}$ times greater than in the equivalent mesh connection; consequently the star impedance must be $\frac{1}{(\sqrt{3})^2} = \frac{1}{3}$ that of the mesh impedance.

Example II. In three-phase systems several star connections are often joined in parallel. Since the admittances of the several branches of the stars cannot be directly added when the load is unsymmetrical, each star must first be replaced by its equivalent mesh. The admittances of the various meshes are simply added for each branch, which is allowable, since these admittances are all in parallel between the same two terminals. Consequently we get one resultant admittance for every path, and the resultant admittances of the three paths form a single triangle, which is equivalent to all the equivalent parallel connected stars. This triangle can further be replaced by an equivalent star, whereby it is seen that several different star connections have been reduced to a single equivalent star. In a similar manner it is possible to treat any desired load on a three-phase system.

83. Symbolic Calculation of Current in Polyphase Systems. In a symmetrical polyphase system with n phases, the E.M.F. p_x induced in the x^{th} phase is

$$p_x = \sqrt{2} P \sin\left\{\omega t - (x-1)\frac{2\pi}{n}\right\}$$
$$= \sqrt{2} P \left\{\sin \omega t \cos (x-1)\frac{2\pi}{n} - \cos \omega t \sin (x-1)\frac{2\pi}{n}\right\},$$

or, symbolically,

$$\dot{P}_x = P\left\{\cos (x-1)\frac{2\pi}{n} + j \sin (x-1)\frac{2\pi}{n}\right\}$$
$$= P\epsilon^{j(x-1)\frac{2\pi}{n}}.$$

Since $p_I = \sqrt{2} P \sin \omega t$, i.e. symbolically $\dot{P}_1 = \dot{P}$, and since also

$$\epsilon^{j\frac{2\pi}{n}} = \cos\frac{2\pi}{n} + j \sin\frac{2\pi}{n} = \sqrt[n]{1} = e,$$

we can write for the E.M.F.'s induced in the several phases,

$$P_{\mathrm{I}} = P,$$
$$P_{\mathrm{II}} = Pe,$$
$$P_x = Pe^{x-1},$$
$$P_n = Pe^{n-1}.$$

Consider first the interconnected four-phase system (Fig. 235), whose generator is star connected, whilst the load admittances form a quadrilateral. In this case it is best to start with Kirchhoff's Laws, which state that the sum of all the currents at any junction is zero, and that

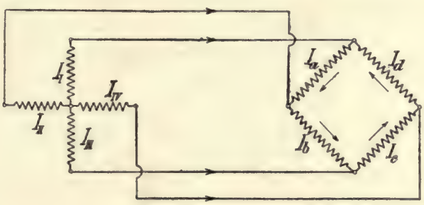


FIG. 235.

the sum of all the E.M.F.'s in a closed circuit must be zero. Up to the present there is no graphical solution for such a system; consequently the symbolic method is used for treating this particular case—which seldom finds practical application. Applying Kirchhoff's First Law for the five junctions in the system, we have:

$$I_{\mathrm{I}} + I_d - I_a \qquad = 0,$$
$$I_{\mathrm{II}} + I_a - I_b \qquad = 0,$$
$$I_{\mathrm{III}} + I_b - I_c \qquad = 0,$$
$$I_{\mathrm{IV}} + I_c - I_d \qquad = 0,$$

and

$$I_{\mathrm{I}} + I_{\mathrm{II}} + I_{\mathrm{III}} + I_{\mathrm{IV}} = 0.$$

Since the last equation can also be obtained by addition of the other four, we need not consider it further.

Similarly, applying Kirchhoff's Second Law for the five closed circuits in the system, we have:

$$P_{\mathrm{I}} - P_{\mathrm{II}} - I_{\mathrm{I}} z_{\mathrm{I}} - I_a z_a + I_{\mathrm{II}} z_{\mathrm{II}} \qquad = 0,$$
$$P_{\mathrm{II}} - P_{\mathrm{III}} - I_{\mathrm{II}} z_{\mathrm{II}} - I_b z_b + I_{\mathrm{III}} z_{\mathrm{III}} \quad = 0,$$
$$P_{\mathrm{III}} - P_{\mathrm{IV}} - I_{\mathrm{III}} z_{\mathrm{III}} - I_c z_c + I_{\mathrm{IV}} z_{\mathrm{IV}} = 0,$$
$$P_{\mathrm{IV}} - P_{\mathrm{I}} - I_{\mathrm{IV}} z_{\mathrm{IV}} - I_d z_d + I_{\mathrm{I}} z_{\mathrm{I}} \qquad = 0,$$

and

$$I_a z_a + I_b z_b + I_c z_c + I_d z_d \qquad = 0.$$

The last equation can likewise be obtained by adding the other four, and can therefore be omitted.

If, in the pressure equations, we now replace the phase currents I_{I}, I_{II}, I_{III} and I_{IV} by the line currents I_a, I_b, I_c and I_d, we get the following four linear equations with the four unknown currents I_a, I_b, I_c and I_d.

$$\begin{aligned}
P_{\mathrm{I}} - P_{\mathrm{II}} - I_a(z_{\mathrm{I}} + z_a + z_{\mathrm{II}}) + I_b z_{\mathrm{II}} + I_d z_{\mathrm{I}} &= 0,\\
P_{\mathrm{II}} - P_{\mathrm{III}} - I_b(z_{\mathrm{II}} + z_b + z_{\mathrm{III}}) + I_c z_{\mathrm{III}} + I_a z_{\mathrm{II}} &= 0,\\
P_{\mathrm{III}} - P_{\mathrm{IV}} - I_c(z_{\mathrm{III}} + z_c + z_{\mathrm{IV}}) + I_d z_{\mathrm{IV}} + I_b z_{\mathrm{III}} &= 0,\\
P_{\mathrm{IV}} - P_{\mathrm{I}} - I_d(z_{\mathrm{IV}} + z_d + z_{\mathrm{I}}) + I_a z_{\mathrm{I}} + I_c z_{\mathrm{IV}} &= 0.
\end{aligned}$$

From these we find the currents,

$$I_a = \frac{D_1}{D}, \quad I_b = \frac{D_2}{D}, \quad I_c = \frac{D_3}{D} \quad \text{and} \quad I_d = \frac{D_4}{D},$$

where

$$D = \begin{vmatrix}
-(z_{\mathrm{I}} + z_a + z_{\mathrm{II}}), & z_{\mathrm{II}}, & 0, & z_{\mathrm{I}}\\
z_{\mathrm{II}}, & -(z_{\mathrm{II}} + z_b + z_{\mathrm{III}}), & z_{\mathrm{III}}, & 0\\
0, & z_{\mathrm{III}}, & -(z_{\mathrm{III}} + z_c + z_{\mathrm{IV}}), & z_{\mathrm{IV}}\\
z_{\mathrm{I}}, & 0, & z_{\mathrm{IV}}, & -(z_{\mathrm{IV}} + z_d + z_{\mathrm{I}})
\end{vmatrix}$$

$$= \begin{vmatrix}
0, & z_{\mathrm{III}}, & -(z_{\mathrm{III}} + z_c + z_{\mathrm{IV}}), & z_{\mathrm{IV}}\\
z_{\mathrm{I}}, & 0, & z_{\mathrm{IV}}, & -(z_{\mathrm{IV}} + z_d + z_{\mathrm{I}})\\
-(z_{\mathrm{I}} + z_a + z_{\mathrm{II}}), & z_{\mathrm{II}}, & 0, & z_{\mathrm{I}}\\
z_{\mathrm{II}}, & -(z_{\mathrm{II}} + z_b + z_{\mathrm{III}}), & z_{\mathrm{III}}, & 0
\end{vmatrix}$$

is the determinant of the above four equations, whilst D_1, D_2, D_3 and D_4 can be found from D when the coefficients of the unknowns I_a, I_b, I_c and I_d are respectively interchanged with regard to the constant terms,

$$P_{\mathrm{I}} - P_{\mathrm{II}}, \quad P_{\mathrm{II}} - P_{\mathrm{III}}, \quad P_{\mathrm{III}} - P_{\mathrm{IV}} \quad \text{and} \quad P_{\mathrm{IV}} - P_{\mathrm{I}}.$$

When the four currents I_a, I_b, I_c and I_d have thus been determined, the four terminal pressures,

$$I_a z_a, \quad I_b z_b, \quad I_c z_c, \quad I_d z_d,$$

and the four phase currents,

$$I_{\mathrm{I}}, \quad I_{\mathrm{II}}, \quad I_{\mathrm{III}}, \quad I_{\mathrm{IV}},$$

can be easily found.

The problem is accordingly solved, and for the solution we have only used the simplest means. This method of symbolic treatment, however, yields a result which has very little meaning until we work out the determinants, and then from the symbolic expressions come back to the complex. The final result is thus always long and complicated.

In practice we usually meet with the independent two- or four-phase system and the two-phase three-wire system. The former can be calculated both graphically and analytically in the same way

as a single-phase system. The two-phase three-wire system can be best analytically and graphically treated by calculating the neutral point of the pressure. We shall, however, treat this case here symbolically, and by means of an example explain the operations with complex quantities somewhat more fully. In Fig. 215 a two-phase three-wire system, with equal currents in the two phases, was graphically investigated, and it was found that the drop of pressure in the two phases was unequal. If we consider the same system on the assumption that both the phases have equal load admittances, we find that in this case also the pressure drops are different. Thus the two-phase three-wire system is always unsymmetrical with respect to pressures and currents, even with symmetrical loading.

In order to shew this, let

$P_{I0} = P =$ E.M.F. induced in phase I of generator,
$P_{II0} = jP =$ E.M.F. induced in phase II of generator,
I = current in phases I and II,
I_0 = current in neutral line,
z = impedance in phase lines,
z_0 = impedance in neutral line,
y = load admittance of the two phases.

P_I and P_{II} = terminal pressures between the phase terminals and the middle wire.

We have, then, $$I_I + I_{II} = -I_0,$$

where all currents leaving the neutral point are taken as positive.

$$I_I = yP_I \quad \text{and} \quad I_{II} = yP_{II},$$

$$P_I = P_{I0} - I_I z + I_0 z_0 = P - I_I(z + z_0) - I_{II} z_0$$

and $$P_{II} = P_{II0} - I_{II} z + I_0 z_0 = jP - I_{II}(z + z_0) - I_I z_0,$$

or $$P_I\{1 + y(z + z_0)\} + yz_0 P_{II} = P,$$

$$P_I y z_0 + \{1 + y(z + z_0)\} P_{II} = jP,$$

whence $$P_I = \frac{1 + y(z + z_0) - jyz_0}{\{1 + y(z + z_0)\}^2 - (yz_0)^2} P, \quad \text{.................... (148)}$$

$$P_{II} = \frac{yz_0 - j\{1 + y(z + z_0)\}}{\{1 + y(z + z_0)\}^2 - (yz_0)^2} P. \quad \text{.................... (149)}$$

Take, for example, $z = z_0\sqrt{2}$, then

$$P_I = \frac{1 + (1{\cdot}707 - 0{\cdot}707j)yz}{1 + 3{\cdot}414yz + 2{\cdot}414y^2z^2} P$$

and $$P_{II} = \frac{1 + (1{\cdot}707 + 0{\cdot}707j)yz}{1 + 3{\cdot}414yz + 2{\cdot}414y^2z^2} jP.$$

For further similar calculations, see *Steinmetz* and *Berg*.

The dissymmetry of currents and pressures is due to the fact that the reaction of first leading phase in such a system on the second lagging phase differs from that of the second upon the first. Hence, such a system is not to be recommended for current distribution—rather it is preferable to use the independent two-phase system, whose pressure regulation is just as simple as that of an ordinary single-phase system. For power transmission, however, the interconnected two-phase system is often used, since it necessitates only three wires, one of which can be earthed. In this case it is customary to use two concentric cables with uninsulated outers.

84. Graphic Representation of the Momentary Power in a Polyphase System. In Fig. 45, p. 36, the momentary value of the power,

$$pi = PI\left\{\cos(\phi_1 - \phi_2) + \sin\left[2\omega t + \left(\phi_1 + \phi_2 - \frac{\pi}{2}\right)\right]\right\},$$

of an alternating current is graphically illustrated. This method of representation, however, is not suitable for polyphase currents. We therefore set off the momentary power as a vector, at an angle $\left(\omega t - \frac{\psi}{2}\right)$ to the abscissa axis.*

Putting $$PI\cos(\phi_1 - \phi_2) = PI\cos\phi = W$$

and $$\frac{\pi}{2} - (\phi_1 + \phi_2) = \psi,$$

then $$w = W\left(1 + \frac{\sin(2\omega t - \psi)}{\cos\phi}\right)$$

will be represented by a closed symmetrical curve, the so-called power curve, whose centre is a point of the fourth degree. Since the power of each phase in a polyphase system varies with double the frequency of the current, the total power in a polyphase system can also be expressed by an expression of the following form:

$$w = W\{1 + e\sin(2\omega t - \psi)\}.$$

eW is here the amplitude of the double-frequency power. Returning now to the rectangular co-ordinates x and y by putting

$$w = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\left(\omega t - \frac{\psi}{2}\right) = \frac{y}{x},$$

we get the following equation for the power curve:

$$(x^2 + y^2)^3 - W^2(x^2 + y^2 + 2exy)^2 = 0,$$

which is a curve of the sixth degree.

*See Steinmetz and Berg, *Alternating-Current Phenomena.*

In this equation, put

$$w_{\max} = (1+e)W = \text{maximum power},$$
$$w_{\min} = (1-e)W = \text{minimum power},$$

then
$$W = \frac{w_{\max} + w_{\min}}{2} \quad \text{and} \quad e = \frac{w_{\max} - w_{\min}}{w_{\max} + w_{\min}}.$$

Inserting this in the above equation of the power curve, we get

$$(x^2+y^2)^3 - \tfrac{1}{4}\{w_{\max}(x+y)^2 + w_{\min}(x-y)^2\}^2 = 0$$

as the final form of the equation for this curve, whose main power axes are $w_{\max}$ and $w_{\min}$. The ratio $w_{\max} : w_{\min}$ is often referred to as the *balance factor* of the system. In Figs. 236 to 239 the power curves of the most important alternating-current systems are given.

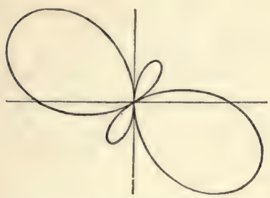


FIG. 236.—Single-phase System on Inductive Load, $\phi=60°$.

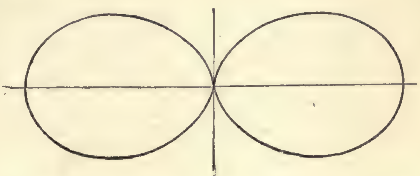


FIG. 237.—Single-phase System on Non-inductive Load, $\phi=0$.

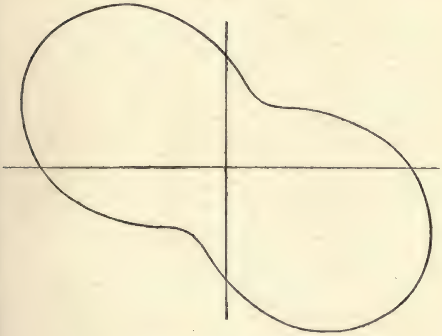


FIG. 238.—Inverted Three-phase System on Non-inductive Load.

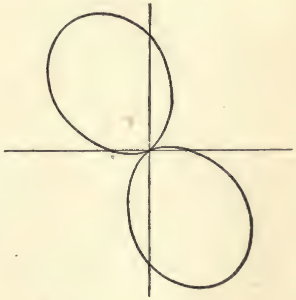


FIG. 239.—Inverted Three-phase System on Inductive Load, $\phi=60°$.

The single-phase system with non-inductive load (i.e. $\cos\phi = 1$) has the following power equation:

$$w = W\{1 + \sin(2\omega t - \psi)\},$$

or, since $w_{\max} = 2W$, $w_{\min} = 0$ and $e = 1$, we get, in the rectangular co-ordinate system,
$$(x^2+y^2)^3 - W^2(x+y)^4 = 0.$$

The power curve is shewn in Fig. 237.

As is obvious from the above figures, the power in an alternating-current system is completely characterised by the two main power axes w_{max} and w_{min}. All symmetrical polyphase systems with $n \geqq 3$ give circles for the power curves when symmetrically loaded. These systems therefore transmit the power quite uniformly, and for this reason have almost completely ousted all other unbalanced alternating-current systems for power purposes.

CHAPTER XV.

NO-LOAD, SHORT-CIRCUIT AND LOAD DIAGRAM OF A POLYPHASE CURRENT.

85. No-load Diagram. 86. Short-circuit Diagram. 87. Load Diagram.

85. No-load Diagram. (Percentage Current Variation.) When a symmetrical polyphase system is uniformly loaded, each phase behaves

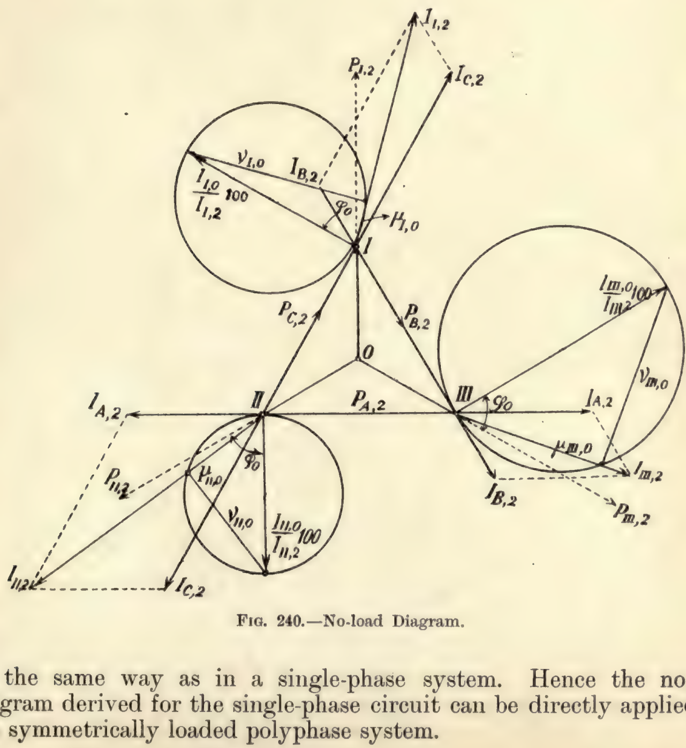


FIG. 240.—No-load Diagram.

in the same way as in a single-phase system. Hence the no-load diagram derived for the single-phase circuit can be directly applied for the symmetrically loaded polyphase system.

In practice, polyphase systems are almost exclusively met with, the chief amongst these being the three-phase. We shall therefore now derive the no-load diagram for a symmetrical three-phase star system with unsymmetrical load and with the no-load currents in the three phases equal.

The no-load diagram enables us to determine the percentage change of current from the receiver terminals to the supply terminals. This

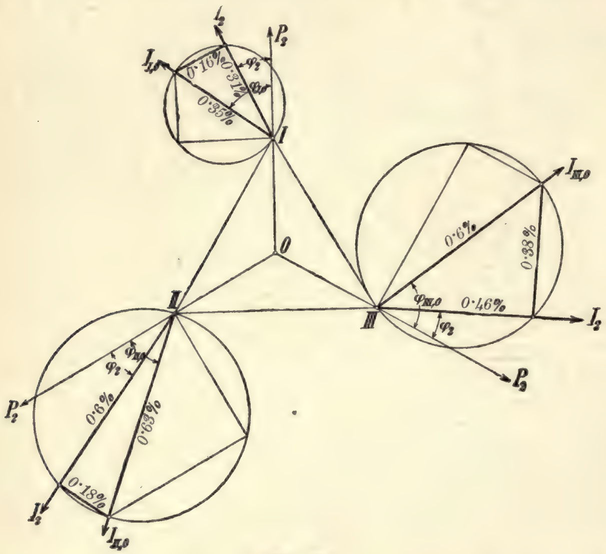


FIG. 241.

percentage current variation is nearly equal to the current variation at the receiver terminals from short-circuit to load when the current in the supply circuit is maintained constant.

If the system is unsymmetrically loaded, we must first find the line currents I_{I2}, I_{II2} and I_{III2} by geometrically adding the three load currents I_{A2}, I_{B2} and I_{C2}. The pressure triangle (Fig. 240) is then drawn for the pressures at the receiver terminals, as an equilateral triangle—this is not quite correct—and the no-load currents $\frac{I_{I0}}{I_{II0}}100$, and so on, are set off as percentages of the line currents at an angle ϕ_0 to the phase pressures P_{I2}, and so on. With the no-load currents as diameters, we describe circles and obtain the variations of the three line currents as we pass from the receiver terminals to the supply terminals, thus,

$$i_{I}\% = \pm \mu_{I0} + \frac{v_{I0}^2}{200},$$

$$i_{II}\% = \pm \mu_{II0} + \frac{v_{II0}^2}{200}$$

and $$i_{\mathrm{III}}\% = \pm \mu_{\mathrm{III}0} + \frac{\nu^2_{\mathrm{III}0}}{200}.$$

In Fig. 241, in the same way, the no-load diagram is represented for a three-phase network, to which several unsymmetrical transformers are connected of the kind shewn in Fig. 242. The load is symmetrical and inductive, with a power factor of 0·9. Since the no-load currents in the several phases of the unsymmetrical transformers vary considerably, we get large differences in the diameters of the circles (see Fig. 241).

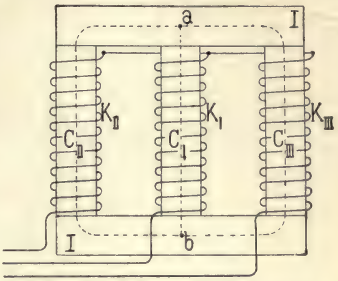


Fig. 242.—Three-phase Transformer.

86. Short-circuit Diagram. (Percentage Pressure Variation.) The short-circuit diagram enables us to determine the percentage change of the supply pressures when the pressures at the receiver terminals are kept constant from no-load to full load. This percentage variation nearly equals the change of pressure which takes place at the receiver terminals when the pressures at the supply terminals are maintained constant.

When a symmetrical polyphase system is uniformly loaded, each phase behaves as in a single-phase system. Hence the short-circuit diagram of a symmetrical three-phase system can be found directly from that of a single-phase.

We have here, however, three pressures at the receiver terminals, whose directions are represented by the three sides P_{A2}, P_{B2}, P_{C2} of an equilateral triangle. When the load is uniform, the line currents $I_{\mathrm{I}2}$, $I_{\mathrm{II}2}$ and $I_{\mathrm{III}2}$ will all be equal and make the same angle ϕ_2 with the phase pressures $P_{\mathrm{I}2}$, $P_{\mathrm{II}2}$ and $P_{\mathrm{III}2}$. Each of these line currents causes a displacement of the potential at the supply terminals by the amount $I_2 z_K$ in passing from no-load to full load. Hence we set off the impedance pressures

$$\frac{I_2 z_K}{P_2} 100$$

at angle ϕ_K to the line currents, as a percentage of the pressure P_2 at the receiver terminals. On this, as diameter, we describe a circle and find the distances μ_K and ν_K, given by the three terminal pressures in these circles. $I_2 z_K$ is here the short-circuit pressure per phase, and consequently equals P_{AK} when the load is uniform, where P_{AK} denotes the terminal pressure at short-circuit. The direction of each terminal pressure cuts out lengths μ_K and ν_K from two circles. We thus get the percentage change of pressure at the supply terminals, on passing from no-load to load:

$$e_A\% = e_B\% = e_C\% = \pm \mu_K \pm \mu'_K + \frac{(\nu_K + \nu'_K)^2}{200}.$$

If the three-phase system is unsymmetrically loaded, we first determine the line currents I_{I2}, I_{II2} and I_{III2}, as shewn in Fig. 243, by geometrically adding the load currents I_{A2}, I_{B2} and I_{C2}.

The short-circuit diagram, Fig. 243, is drawn for an unsymmetrical non-inductive load. In this figure, therefore, the load currents coincide in direction with their respective terminal pressures P_{A2}, P_{B2} and P_{C2}.

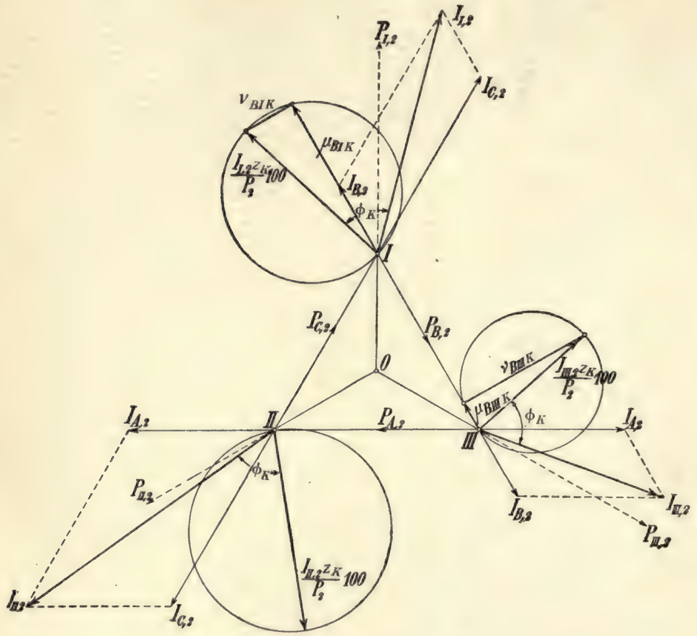


FIG. 243.—Short-circuit Diagram of a Three-phase System.

The impedance pressures $\frac{I_{I2}z_K}{P_2}100$ are then set off at an angle ϕ_K to the line currents, as a percentage of the pressure at the receiver terminals. On this as diameter, we describe a circle, and so obtain the percentage variation of the pressure at the supply terminals. For phase B this variation is

$$e_B\% = \mu_{BIK} - \mu_{BIIIK} + \frac{(\nu_{BIK} + \nu_{BIIIK})^2}{200},$$

and similarly for the other two phases.

87. Load Diagram. With uniform loading, each phase of a polyphase system acts just as a single-phase system. Hence we can apply the load diagram for single-phase currents directly for polyphase currents, if we carry out the calculations for each phase and afterwards multiply the power per phase by the number of phases.

The relations, however, are not so simple when we come to deal with unsymmetrical systems or systems unsymmetrically loaded, since the currents in the different phases mutually affect one another, but not all in the same way. Since systems with a considerable want of symmetry, or with very unsymmetrical loading, seldom occur in practice, we shall not treat such systems exhaustively, but rather satisfy ourselves by shewing how the load diagrams for such systems can be constructed.

(*a*) In *star systems*, it is best to find the neutral point of the pressure for different loads. If this point does not alter much with the load, the pressures between the terminals and the neutral point can be regarded as constant; and the load diagram for each phase is constructed in the usual manner and the several powers summed up. If all the diagrams have the same conductance, they can be replaced by an equivalent diagram, whose pressure P equals the root of the sum of the squares of all the phase pressures P_{I}, P_{II}, P_{III}, and so on, i.e.

$$P=\sqrt{P_{\mathrm{I}}^2+P_{\mathrm{II}}^2+P_{\mathrm{III}}^2+\dots},$$

and the current I in the equivalent diagram bears the same relation to the phase currents:

$$I=\sqrt{I_{\mathrm{I}}^2+I_{\mathrm{II}}^2+I_{\mathrm{III}}^2+\dots}.$$

An interlinked four-phase system, where one double-phase is displaced 90° in phase from the other double-phase, but is of different magnitude, yields an equivalent diagram, for example, if both phases feed circuits of equal conductance y. The equivalent pressure is then

$$P=\sqrt{P_{\mathrm{I}}^2+P_{\mathrm{II}}^2}$$

and the equivalent current

$$I=\sqrt{I_{\mathrm{I}}^2+I_{\mathrm{II}}^2}.$$

If the two phases supply circuits, however, whose conductances are different, but with similar diagrams, the equivalent diagram can also be found for this case, if we take the pressure

$$P=\sqrt{P_{\mathrm{I}}^2\frac{y_{\mathrm{I}}}{y}+P_{\mathrm{II}}^2\frac{y_{\mathrm{II}}}{y}}$$

and the current

$$I=\sqrt{I_{\mathrm{I}}^2\frac{y}{y_{\mathrm{I}}}+I_{\mathrm{II}}^2\frac{y}{y_{\mathrm{II}}}},$$

where the conductance y of the equivalent diagram equals the root of the product of the two phase admittances y_{I} and y_{II}, i.e.

$$y=\sqrt{y_{\mathrm{I}}y_{\mathrm{II}}}.$$

The same also holds for an m-phase system, if we write for the equivalent admittance

$$y=\sqrt[m]{y_{\mathrm{I}}y_{\mathrm{II}}\cdots y_m}.$$

The equivalent pressure is then

$$P=\sqrt{P_{\mathrm{I}}^2\frac{y_{\mathrm{I}}}{y}+P_{\mathrm{II}}^2\frac{y_{\mathrm{II}}}{y}+\ldots+P_m^2\frac{y_m}{y}},$$

and the equivalent current,

$$I=\sqrt{I_{\mathrm{I}}^2\frac{y}{y_{\mathrm{I}}}+I_{\mathrm{II}}^2\frac{y}{y_{\mathrm{II}}}+\ldots+I_m^2\frac{y}{y_m}}.$$

Since these ratios of conductances are the same for all loads, we can calculate them for any desired load—e.g. no-load—and substitute them in the formulae

(*b*) In *ring systems*, the phase pressures generally remain constant for all loads, and on this account the formulae that have been deduced for star systems may also be applied for ring systems.

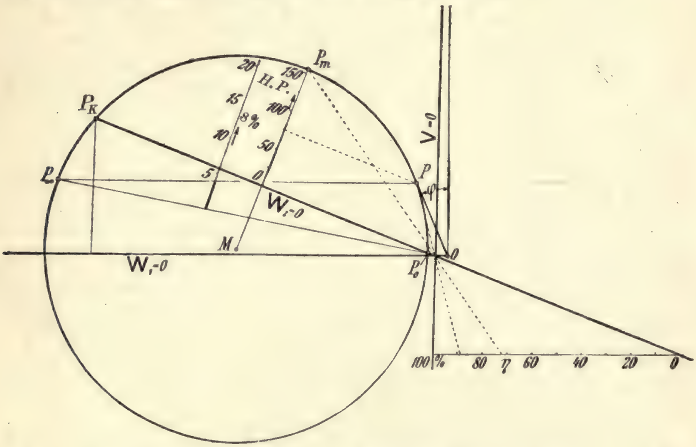


Fig. 244.—Load Diagram of a Three-phase Induction Motor.

(*c*) As an *example* of a symmetrically loaded three-phase system, we will consider the load diagram for a 75 H.P. three-phase asynchronous motor at 580 r.p.m. and 50 cycles. Measurements were taken at no-load and short-circuit, and the following mean values were obtained for each of the three phases:

No-load:

$$P_1=289 \text{ volts},\quad I_0=21 \text{ amps.},\quad W_0=1{\cdot}0 \text{ K.W.}$$

Short-circuit:

$$P_K=61 \text{ volts},\quad I_{1K}=80 \text{ amps.},\quad W_K=1{\cdot}72 \text{ K.W.}$$

From this we get

$$\cos\phi_0=\frac{W_0}{P_1I_0}=0{\cdot}165,\quad \phi_0=80^\circ\ 30'.$$

The short-circuit current at full phase-pressure is

$$I_K = I_{1K}\frac{P_1}{P_K} = 379 \text{ amps.},$$

and $$\cos\phi_K = \frac{W_K}{P_K I_{1K}} = 0{\cdot}352, \quad \phi_K = 69^\circ\ 20'.$$

From these the load diagram per phase is drawn in Fig. 244 to a scale of 1 cm = 75 amps., together with the power and loss-lines, in accordance with the constructions given in Sect. 58.

For the maximum power (P_m), the diagram gives:

$$\text{Supplied power } W_1 = 53{\cdot}3 \text{ K.W.},$$
$$\text{Efficiency } \eta = 72\ \%,$$

from which $$W_{2\,max} = 0{\cdot}72\,.\,3\,.\,53{\cdot}3 = 115 \text{ K.W.}$$

for all three phases, or $$\frac{115}{0{\cdot}746} = 154 \text{ H.P.}$$

With this scale, we find for the full-load power of 75 H.P. (point P) $I = 80$ amps., $\eta = 89\ \%$, $\cos\phi = 0{\cdot}9$, $s = 3{\cdot}9$.

The maximum power for $\phi_2 = 0$, $\Delta\psi \simeq 0$ is, from Formula 104,

$$W_{2\,max} = m\frac{P_1\{I_K - I_0\cos(\phi_0 - \phi_K)\}}{2(1+\cos\phi_K)}$$
$$= 3\frac{289(379 - 21\,.\,0{\cdot}981)}{2(1+0{\cdot}352)}$$
$$= 115 \text{ K.W. or } 154 \text{ H.P.}$$

CHAPTER XVI.

POLYPHASE CURRENTS OF ANY WAVE-SHAPE.

88. Higher Harmonics of Current and Pressure in Polyphase Systems. As with a single-phase current, so also with polyphase currents, each harmonic (fundamental and higher harmonics) can be treated separately, and just as the resultant E.M.F. of the fundamental waves of two phases is found by geometric addition, so also harmonics of the same frequency can be summed up, only the angle at which they act is different for the several harmonics. The harmonics of the same frequency in an n-phase system form a pressure polygon of n sides, and the laws deduced for this will apply quite generally. The effective pressure between two points and the effective current in a conductor are likewise found, as before, by taking the square root of the sum of the squares of the effective pressures or currents of the several frequencies. The total power of the system is the algebraic sum of powers of the several harmonics.

In an unsymmetrical system, there are such manifold variations that it is preferable to treat the harmonics of symmetrical systems only. Particular unsymmetrical cases can then be studied for themselves.

As an example of a symmetrical n-phase system, we shall examine that which most frequently occurs in practice, viz. the three-phase system.

The phase pressures in the three phases are as follows:

$$\begin{aligned} p_{\mathrm{I}} = {} & P_{p1}\sqrt{2}\sin(\omega t+\psi_1) \\ & + P_{p3}\sqrt{2}\sin(3\omega t+\psi_3) \\ & + P_{p5}\sqrt{2}\sin(5\omega t+\psi_5)+\ldots, \end{aligned}$$

$$\begin{aligned} p_{\mathrm{II}} = {} & P_{p1}\sqrt{2}\sin(\omega t+\psi_1-120^\circ) \\ & + P_{p3}\sqrt{2}\sin(3\omega t+\psi_3-3\,.\,120^\circ) \\ & + P_{p5}\sqrt{2}\sin(5\omega t+\psi_5-5\,.\,120^\circ)+\ldots, \end{aligned}$$

$$p_{\mathrm{III}} = P_{p1}\sqrt{2}\sin(\omega t + \psi_1 - 240°)$$
$$+ P_{p3}\sqrt{2}\sin(3\omega t + \psi_3 - 3\,.\,240°)$$
$$+ P_{p5}\sqrt{2}\sin(5\omega t + \psi_5 - 5\,.\,240°) + \dots;$$

or, working these out:

$$p_{\mathrm{I}} = P_{p1}\sqrt{2}\sin(\omega t + \psi_1)$$
$$+ P_{p3}\sqrt{2}\sin(3\omega t + \psi_3)$$
$$+ P_{p5}\sqrt{2}\sin(5\omega t + \psi_5) + \dots,$$

$$p_{\mathrm{II}} = P_{p1}\sqrt{2}\sin(\omega t + \psi_1 - 120°)$$
$$+ P_{p3}\sqrt{2}\sin(3\omega t + \psi_3)$$
$$+ P_{p5}\sqrt{2}\sin(5\omega t + \psi_5 - 240°) + \dots,$$

$$p_{\mathrm{III}} = P_{p1}\sqrt{2}\sin(\omega t + \psi_1 - 240°)$$
$$+ P_{p3}\sqrt{2}\sin(3\omega t + \psi_3)$$
$$+ P_{p5}\sqrt{2}\sin(5\omega t + \psi_5 - 120°) + \dots.$$

From this it is seen that every harmonic whose frequency is a multiple of the third harmonic is equal in all the phases, i.e. at any instant these E.M.F.'s have the same magnitude and the same direction with regard to the neutral point, whilst all the other harmonics of the three phases are displaced at 120° to one another, and can therefore be treated as ordinary symmetrical three-phase currents. It must be observed, however, that the order in which the phases follow one another is not always the same as that of the fundamental; e.g. for the fifth harmonic the order is 1, 3, 2, where 1, 2, 3 is the order of the fundamental.

From the momentary values p_{I}, p_{II} and p_{III} of the E.M.F.'s induced in the three phases, the momentary values p_a, p_b and p_c of the line pressures of a star system can be found. Thus

$$p_c = p_{\mathrm{I}} - p_{\mathrm{II}}$$
$$= \sqrt{3}P_{p1}\sqrt{2}\sin(\omega t + \psi_1 + 30°)$$
$$+ \sqrt{3}P_{p5}\sqrt{2}\sin(5\omega t + \psi_5 - 30°) + \dots,$$

$$p_a = p_{\mathrm{II}} - p_{\mathrm{III}}$$
$$= \sqrt{3}P_{p1}\sqrt{2}\sin(\omega t + \psi_1 - 90°)$$
$$+ \sqrt{3}P_{p5}\sqrt{2}\sin(5\omega t + \psi_5 + 90°) + \dots$$

and

$$p_b = p_{\mathrm{III}} - p_{\mathrm{I}}$$
$$= \sqrt{3}P_{p1}\sqrt{2}\sin(\omega t + \psi_1 - 210°)$$
$$+ \sqrt{3}P_{p5}\sqrt{2}\sin(5\omega t + \psi_5 - 150°) + \dots.$$

If the time t is reckoned from another instant, e.g. $\omega t' = \omega t + 30°$ we get

$$\begin{aligned} p_c &= \sqrt{3}P_{p1}\sqrt{2}\sin(\omega t' + \psi_1) \\ &\quad - \sqrt{3}P_{p5}\sqrt{2}\sin(5\omega t' + \psi_5) \\ &\quad - \sqrt{3}P_{p7}\sqrt{2}\sin(7\omega t' + \psi_7) + \dots, \end{aligned}$$

$$\begin{aligned} p_a &= \sqrt{3}P_{p1}\sqrt{2}\sin(\omega t' + \psi_1 - 120°) \\ &\quad - \sqrt{3}P_{p5}\sqrt{2}\sin(5\omega t' + \psi_5 - 240°) \\ &\quad - \sqrt{3}P_{p7}\sqrt{2}\sin(7\omega t' + \psi_7 - 120°) + \dots \end{aligned}$$

and

$$\begin{aligned} p_b &= \sqrt{3}P_{p1}\sqrt{2}\sin(\omega t' + \psi_1 - 240°) \\ &\quad - \sqrt{3}P_{p5}\sqrt{2}\sin(5\omega t' + \psi_5 - 120°) \\ &\quad - \sqrt{3}P_{p7}\sqrt{2}\sin(7\omega t' + \psi_7 - 240°) + \dots. \end{aligned}$$

This way of expressing instantaneous values of the line pressures agrees with that of the phase pressures, except that instead of P_{p1} we have $\sqrt{3}P_{p1}$; instead of P_{p3}, 0; instead of P_{p5} and P_{p7}, $-\sqrt{3}P_{p5}$ and $\sqrt{3}P_{p7}$, and so on. Hence, if we reckon from the time t', where

$$\omega t' = \omega t + 30°,$$

in a three-phase system, we get the following expressions for the effective line pressures of the several harmonics in the system,

$$\left.\begin{aligned} P_{l1} &= \sqrt{3}P_{p1}; \quad & P_{l3} &= 0; \quad & P_{l5} &= -\sqrt{3}P_{p5}, \\ P_{l7} &= -\sqrt{3}P_{p7}; \quad & P_{l9} &= 0; \quad & P_{l11} &= +\sqrt{3}P_{p11}. \end{aligned}\right\} \quad \dots\dots\dots\dots(150)$$

A star system with the phase pressures P_{p1}, P_{p3}, P_{p5}, etc., is equivalent to a mesh system with the phase pressures P_{l1}, P_{l3}, P_{l5}, etc., if the star system is regarded as lagging 30° behind the mesh system.

The harmonics of the third order have no effect on the pressure between the terminals (in a star system), for these have the same direction in the several phases and neutralise one another in respect of the outside terminals. Hence, the effective terminal pressure will be

$$\begin{aligned} P_l &= \sqrt{P_{l1}^2 + P_{l5}^2 + P_{l7}^2 + \dots} \\ &= \sqrt{3(P_{p1}^2 + P_{p5}^2 + P_{p7}^2 + \dots)}, \end{aligned}$$

whilst the phase pressure is

$$P_p = \sqrt{P_{p1}^2 + P_{p3}^2 + P_{p5}^2 + P_{p7}^2 + \dots},$$

whence we get the ratio

$$\frac{P_l}{P_p} = \sqrt{3}\sqrt{\frac{1 + \left(\frac{P_{p5}}{P_{p1}}\right)^2 + \left(\frac{P_{p7}}{P_{p1}}\right)^2 + \dots}{1 + \left(\frac{P_{p3}}{P_{p1}}\right)^2 + \left(\frac{P_{p5}}{P_{p1}}\right)^2 + \dots}} \quad \dots\dots\dots\dots(151)$$

For example, if $P_{p1}=100$, $P_{p3}=31{\cdot}65$ and $P_{p5}=10$;

then $$\frac{P_l}{P_p}=\sqrt{3}\sqrt{\frac{1+(0{\cdot}1)^2}{1+(0{\cdot}3165)^2+(0{\cdot}1)^2}}=\sqrt{3}\,.\,0{\cdot}954=1{\cdot}655.$$

If P_{p1}, P_{p3}, P_{p5}, etc., are the effective values of the several harmonics in the phase pressure of an interconnected two- or four-phase system, we get the effective values of the line pressures in a similar way to the above:

$$\left.\begin{array}{lll} P_{l1}=\sqrt{2}P_{p1}; & P_{l3}=-\sqrt{2}P_{p3}; & P_{l5}=-\sqrt{2}P_{p5}, \\ P_{l7}=+\sqrt{2}P_{p7}; & P_{l9}=\sqrt{2}P_{p9}; & P_{l11}=-\sqrt{2}P_{p11}, \end{array}\right\} \quad \text{......(152)}$$

whence $$P_l=\sqrt{2}P_p. \quad \text{......(153)}$$

Further, the momentary value of one phase pressure is

$$\begin{aligned} p_p &= P_{p1}\sqrt{2}\sin(\omega t+\psi_1) \\ &\quad + P_{p3}\sqrt{2}\sin(3\omega t+\psi_3) \\ &\quad + P_{p5}\sqrt{2}\sin(5\omega t+\psi_5)+\ldots, \end{aligned}$$

whence the momentary value of one line pressure is

$$\begin{aligned} p_l &= P_{l1}\sqrt{2}\sin(\omega t'+\psi_1) \\ &\quad + P_{l3}\sqrt{2}\sin(3\omega t'+\psi_3) \\ &\quad + P_{l5}\sqrt{2}\sin(5\omega t'+\psi_5)+\ldots, \end{aligned}$$

where $$\omega t'=\omega t+45°.$$

From this it is easy to find the momentary values of the remaining phase and line pressures.

To find the currents due to the several harmonics in a three-phase star system, the pressure triangle can be drawn for each harmonic, and the pressure of the load star point found for each triangle. The triangles of the third, ninth, and so on, harmonics come together at a point which is also the star point of the load, and is displaced from the neutral point of the plane of the respective harmonics by an amount equal to the phase pressure. Hence, in a symmetrical three-phase star system, there is a difference of potential between the star point of the generator and that of the load equal to the effective E.M.F. of the harmonics of the third order. This potential difference can only produce a current when these two neutral points are connected, whereby this P.D. can equalise itself along the neutral wire. Consequently, in a three-phase system without a neutral wire, only currents of the first, fifth, seventh, etc., order can flow, and only pressures of these frequencies will exist at the terminals. On the other hand, in a symmetrical three-phase system with harmonics of the third order, currents of these frequencies will flow when the neutral points are connected (Fig. 245).

We have thus the general rule: A symmetrical n-phase star system without a neutral line acts like a system on no-load with respect to all harmonics of the n^{th} order; for currents of these frequencies cannot flow in the outer wires nor can their corresponding pressures act between the same. If n is a prime number, or only divisible by some

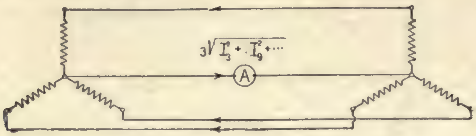


FIG. 245.

power of 2, it will be found that all the other harmonics in the n-phase star system act like the fundamental, if we disregard the order in which they occur. When n is not a prime number, the phase E.M.F.'s of the harmonics, whose order have a common factor with n, will partly coincide. For example, with $n=9$, we shall only get three different triple harmonics, since the nine-sided polygon reduces to a triangle.

If the three phases of a symmetrical three-phase system are mesh connected, the sum of the three momentary E.M.F.'s will not equal zero, but

$$p_{\text{I}}+p_{\text{II}}+p_{\text{III}}=3P_3\sqrt{2}\sin(3\omega t+\psi_3)+3P_9\sqrt{2}\sin(9\omega t+\psi_9)+\ldots.$$

Such a system, therefore, with harmonics of the third order, does not satisfy the above requirement, that the sum of the E.M.F.'s of the phases connected in a closed circuit equals zero. These E.M.F.'s of the third, ninth, etc., harmonics will always produce a current in the mesh (i.e. even on no-load) and only in the mesh. Under certain conditions this current may reach a considerable value. The mesh connection acts like a short-circuited generator with respect to these harmonics, and just as the terminal pressure in such a case is zero, so also these harmonics cannot have any effect on the pressure between the outside terminals. If the mesh is opened at any point and a voltmeter is inserted (Fig. 246), the effective pressure

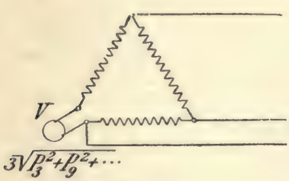


FIG. 246.

$$3\sqrt{P_3^2+P_9^2+\ldots}$$

will be measured, which may be denoted as the *internal pressure.*

In this connection an internal current is produced which can be measured by inserting an ammeter in the mesh. With a star connection the internal pressure produces no current. Thus the harmonics of the third order do not send any currents through the outer wires and exert no pressures at the terminals. This holds generally for the harmonics of the n^{th} order in a symmetrical n-phase system.

89. Polycyclic Systems. In an alternating-current installation which has to provide simultaneously light and power, the selection of a suitable number of phases and frequency often presents considerable difficulties. One condition for the proper working of all known means of electric lighting is a high frequency. On the other hand, both single- and polyphase motors, together with rotary converters, work better, and have a greater overload capacity, with low frequencies.

For a pure power supply, a polyphase system is preferable, whilst for lighting—on account of the better pressure regulation and simpler installation—single-phase currents are more suitable.

Moreover, with regard to the pressures, the conditions for power are different from those for lighting. The lighting pressure, on which the cost of the network mains depends, must be chosen low to meet the requirements of the lamps used at the present day; the pressures for motors, however, can with advantage be chosen much greater than those commonly met with for lighting.

On account of the sensitiveness of electric lamps to variations in the network pressure, it is advisable to keep the pressure drop in the network and the generator much smaller in installations giving both light and power simultaneously, than is necessary with one giving power only. Consequently, in the former case the amount of copper used is greater, and therefore the cost of the network and the generator is increased.

The object of the polycyclic system, therefore, is to *simultaneously* transmit electrical energy by means of currents at different pressures and frequencies *through one and the same conductor*, and to distribute the same without their affecting one another. For this to be possible, it is of course necessary that the currents of different frequencies should have no mutual effect on one another.

Consider a symmetrical three-phase system (Fig. 247); then, assuming sinusoidal currents of equal amplitude, no pressure will exist between the neutral points O and O_1. Hence, considering such a star system

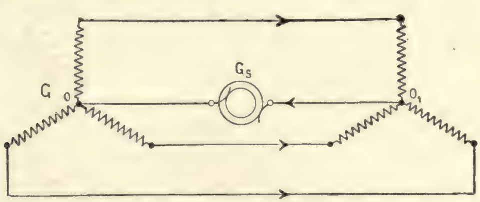


FIG. 247.

(main system) as a whole, we can use the same as *one* conductor for conveying other currents between its neutral points, by connecting, for example, a source of supply G_s in the conductor OO_1. These currents, which flow through the phases of the main system in the same sense and phase, and superpose themselves on the currents already existing in the main system (main currents), produce no detectable motor or

inductive effects in the generators, motors or transformers in the main system. This superposed current may be an alternating-current of any frequency or a continuous current. The two currents, the three-phase current and the superposed single-phase current produced in generator G_s (Fig. 247), are entirely independent of one another, and the superposed single-phase current will flow along the conductors of the main system in the direction shewn by the arrows (Fig. 247), just as if the three-phase currents were not present.

Instead of a three-phase system, a single-phase system might have been used as the main system, as shewn by Fig. 248; for a single-phase system can always be regarded as a two-phase system with its phases displaced at 180°.

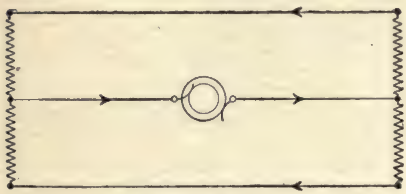


Fig. 248.

Dr. F. Bedell has shewn how currents—especially direct current—can be introduced and drawn out at points having the same potential in a power-transmission scheme without affecting the currents which already exist.

It is, however, easy to see that the superposed alternating-currents—introduced at the neutral point—must cause a large inductive drop of pressure in the generator and transformer windings, and, for this reason, Bedell's arrangement for introducing and withdrawing the superposed current has not met with practical success.

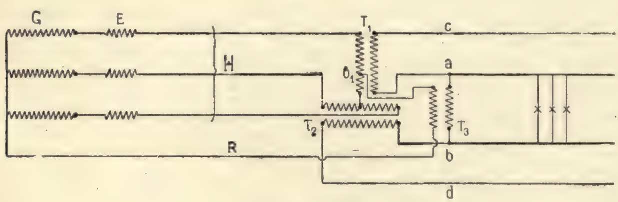


Fig. 249.

The Authors, together with Prof. E. Arnold, however, have overcome these disadvantages in Bedell's arrangement and worked out a polycyclic system. This system is based on the application of bifilarly-wound choking coils, and on the introduction and withdrawal of the superposed current by means of special transformers and generators. Owing to the apparently complicated scheme of connections, however, this system has never been used in practice.

As an illustration of the complete arrangement of an installation for transmitting and distributing polycyclic currents, the scheme shewn in Fig. 249 can be used. In the double generator G and E having one armature and two pole systems arranged in the relative

positions shewn in Fig. 250 to one another—the three-phase current and the superposed single-phase current are simultaneously produced. The single-phase current, which is the third harmonic of the three-phase current, is superposed on the main current in such a way that the maximum momentary pressure between the return R and the remaining conductors of the transmission line is as small as possible. At the receiver station, the three-phase current is transformed into two-phase current by means of two single-phase transformers connected as in Scott's arrangement, this being better for a polycyclic supply network than a three-phase current on account of symmetry.

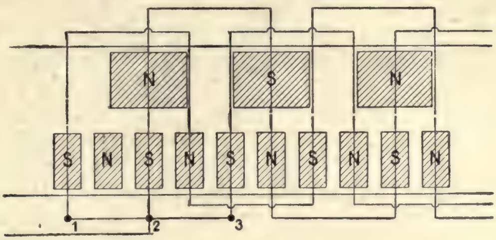


FIG. 250.

The superposed alternating-current produces no flux in the two transformers, and can therefore be withdrawn at the point O_1 in the primary of the transformer T_1. In the transformer T_3, the superposed single-phase current is transformed, and since the secondary winding is connected between the two conductors a and b of the two-phase system, incandescent lamps can be connected directly between the two wires.

Taking an uninterlinked two-phase system as the main system, the weight of copper is 66·7 % of that required by a single-phase system, when the same total power is transmitted over the same distance with the same effective pressure between the conductors and the same percentage loss, if we take the power of the single-phase current as 50 % of that of the three-phase.

The polycyclic system, therefore, may become important in cases where power and light have to be distributed by the same network and the lighting load is the less of the two. We then combine in the one network all the advantages of independent networks with different frequencies, without introducing any complications whatever into the scheme.

CHAPTER XVII.

MEASUREMENT OF ELECTRIC CURRENTS.

90. Systems of Units and Standards. 91. Measuring Instruments. 92. Electrostatic Instruments (the Electrometer). 93. Electromagnetic Instruments. 94. Electrodynamic Instruments. 95. Hot-wire Instruments. 96. Wattmeters. 97. Direct Measurement of the Effective Values of the Several Harmonics. 98. Measurement of Power by Means of Three Voltmeters or Three Ammeters. 99. Measurement of Power in a Polyphase Circuit. 100. Measurement of the Wattless Component of an Alternating-Current. 101. Determination of Wave Shape of a Pressure or Current by Means of Contact Apparatus and Galvanometer. 102. The Oscillograph. 103. Braun's Tube. 104. Measurement of Frequency of an Alternating-Current. 105. Instrument Transformers. 106. Electricity Meters. 107. Calibration of Alternating-current Instruments.

90. Systems of Units and Standards. On the basis of the work of *Gauss* and *Weber* (1833–1852), the Committee of the British Association on Electrical Standards was able, in 1869, to draw up a practical system of electrical units which could be derived from the absolute system of magnetic units. At the International Congress held in Paris in 1881, these units were designated as the ohm, the volt, the ampere, the coulomb and the farad.

Since these practical units can only be derived from the fundamental units of length, mass and time of the C.G.S. system by means of very elaborate and expensive measurements, which distinctly belong to the region of physics, the need arose for *standards* of the above electric units which would remain practically constant and could be easily reproduced. As such standards, approximating as closely as possible to the units derived from the absolute C.G.S. system, and suitable for use both in practice and at law, we have:

The International Ohm equal to the resistance of a column of mercury 106·3 cm long and 1 sq. mm section at 0° C. and weighing 14·4521 gm.

The International Ampere equal to the constant current which, when passed through a silver voltameter, deposits silver at the rate of 1·118 mg per second.

The remaining units can be then found from these two. The

following two *units of electric pressure* (so-called standard cells) are also used, however:

The Clark Cell. The positive electrode is mercury and the negative amalgamated zinc. The electrolyte consists of a concentrated solution of zinc sulphate and mercurous sulphate. The pressure between the terminals of this cell, on open-circuit, at t° C., is

$$1{\cdot}4292 - 0{\cdot}00123(t-18) - 0{\cdot}000007(t-18)^2 \text{ volts}$$

between 0° and 30° C.

The Weston or Cadmium Cell. This cell differs from the above only in having cadmium and cadmium sulphate instead of zinc and zinc sulphate. With a saturated solution of $CdSO_4$, the pressure between 10° and 30° C. is, at t° C.,

$$1{\cdot}0187 - 0{\cdot}000035(t-18) - 0{\cdot}00000065(t-18)^2 \text{ volts.}$$

The Weston Co. make a cell in which the $CdSO_4$ solution is saturated at 4° C. Such a cell has a pressure of 1·0190 volts, almost independently of the temperature.

91. Measuring Instruments. The standards described in the last section do not, as a rule, admit of direct use in practice, the methods of measurement being somewhat roundabout. For practical purposes, therefore, special instruments are used, which permit of measurements being made directly by noting the position taken up by a pointer capable of moving over a scale. Such instruments must of course be first calibrated or standardized by comparison with the above standards.

Generally speaking, these instruments have a movable system which carries the pointer, and a fixed system to which the scale is fastened. The electric measurement, then, depends on the mechanical force set up between the two systems. For the measurement of continuous currents and pressures, the fixed system may consist of a permanent magnet and the movable system of a coil through which the current flows; but for alternating-currents and pressures both the fixed and movable system must consist of coils. In the older *torsion instruments* (e.g. Siemens and Halske's Torsion Galvanometer and Torsion Dynamometer) the action of this force is always measured for one and the same position of the movable system, the latter being brought into its zero position by means of a spiral spring, the force then varying directly as the angle of torsion. In the current balance also (Kelvin balance), the movable system is kept in its original position, the magnitude of the force being determined by weighing.

In general, for one and the same relative position of the two systems, the force varies either directly (when the fixed system consists of a magnet) or as the square of the electric magnitudes being measured. Let a, therefore, denote the angle through which the spiral spring of the torsion instrument must be turned, or the static moment of the counter-weight in the current balance, the electric magnitude x to be measured is either given by

$$x = k_1 a \quad \text{or} \quad x = k_2\sqrt{a}.$$

The advantage of these instruments lies in the fact that their reduction factor k_1 or k_2 can be determined once for all by a single measurement (calibration), and remains constant. A disadvantage of this arrangement is the necessary hand-adjustment of the torsion spring or weight, which makes it impossible to take such measurements rapidly, whilst for the measurement of quickly varying currents such instruments are out of the question. For this reason, the instruments used in practice at the present day are so arranged that the movable system with the pointer moves away from the zero position, and takes up a position corresponding to the magnitude of the electric quantity being measured. In such instruments, even when the controlling force (which tends to bring the needle back into its zero position) is proportional to the deviation of the needle from the zero position (as can easily be obtained by using springs), the readings nevertheless no longer follow the simple or the quadratic law, because the force between the two systems changes with their relative position. Such instruments, therefore, must be calibrated at as many points on the scale as possible, whilst intermediate points can be obtained by interpolation (graduation).

For measuring alternating-currents, only instruments can be used which obey the law of squares, for it is only in such instruments that the direction of movement does not alter with the change in current direction. Provided, then, that the mass of the moving parts is sufficiently large and the frequency sufficiently great, the deflection of the instrument will remain practically steady in a position corresponding to the *mean* turning moment acting on the movable system.

92. Electrostatic Instruments (The Electrometer). As first pointed out by Lord Kelvin, electrostatic instruments can be made for absolute measurements, but in practice only those graduated by comparison with standards are used, and these chiefly for measuring pressures. In principle a static voltmeter can be considered as a small air-condenser, of which one part is fixed, and consists of one or more plates, whilst the other—the needle—is movable, and also consists of plates and carries a pointer. The fixed part of the instrument is made up of one or two systems of plates insulated from each other, called the *quadrants*. If there is only one fixed system of plates in the instrument, one terminal is connected to it and the other to the needle. The force exerted between the plates and the needle is proportional to the square of the pressure existing between the charges, and therefore to the pressure at the terminals, whatever the wave-shape and frequency. If the instrument has two fixed sets of plates, one terminal is connected to one of these and the other terminal to the needle and the other set of plates, so that the force acting on the needle is approximately double that in the former case.

Electrostatic instruments are well adapted for measuring high pressures, because they only need an extremely small current. The capacity of such instruments is of the order 0·00001 microfarad.

Fig. 251 shews an instrument for 60 to 120 volts, made by Hartmann

and Braun. In order to obtain sufficient force in the case of this low pressure, several needles and pairs of quadrants are used (multi-cellular instrument). For the purpose of damping, the movable axis carries a metal disc at the bottom, which turns between the poles of a horse-shoe magnet.

For pressures of more than about 10,000 volts, the plates with the opposite charge to the needle are completely embedded in rubber, to prevent sparking from one to the other. In instruments for pressures under 10,000 volts, a separate spark-gap is provided, of which the

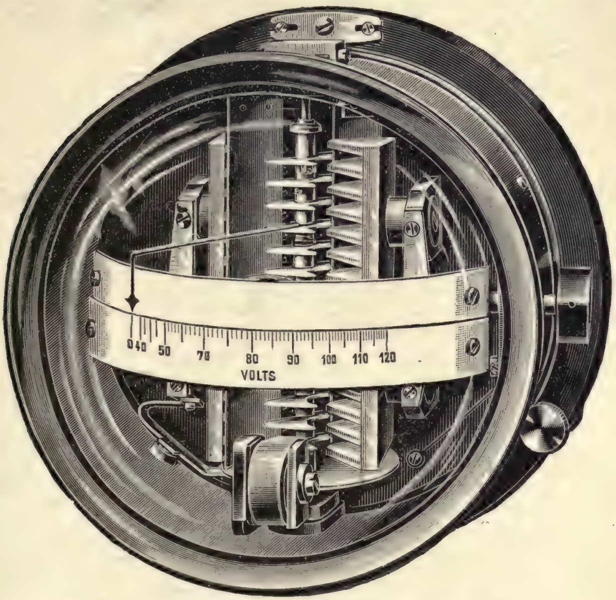


FIG. 251.—Multi-cellular Voltmeter (Hartmann and Braun).

contacts are at a smaller distance from each other than the smallest space between needle and plate, so that all sparks are kept away from the needle. In order that the quantity of electricity passing shall not be too great, double-pole high resistances are connected in series in the form of tubes filled with liquid.

Static voltmeters can also be used for different ranges of measurement by connecting in series two or more condensers, and placing the voltmeter in parallel with one of these. If the condensers are similar, the reading on the scale must be multiplied by the number of condensers. The *tuning* of the condensers, however, is so elaborate, that the scales are usually calibrated separately. The dielectric of these condensers is micanite. This arrangement can be used with good results up to 40,000 volts. Dividing resistances are also employed in a similar manner.

For laboratory purposes the instruments are provided with horizontal scales; for switchboards, on the other hand, vertical-scale instruments are more generally employed.

Recently, electrostatic wattmeters have also been introduced, which are very useful in the laboratory. The chief advantages of these are as follows:

1. Accurate readings can be obtained even with low power-factors.
2. They are especially suited to high pressures, because they do not possess any high non-inductive resistances.
3. There is not so much danger of overloading the instrument as with an ordinary wattmeter.
4. The construction is cheap and simple.

The arrangement of the instrument is exactly the same as the quadrant voltmeter.

Denoting the potential of the needle by P_0,
" " " first quadrant by P_1,
and " " " second quadrant by P_2,
the deflection α of the needle is given by

$$k\alpha = (P_1 - P_2)\left(P_0 - \frac{P_1 + P_2}{2}\right),$$

where k is a constant.

Putting $$P_0 - P_1 = P, \quad P_0 - P_2 = P + \Delta P,$$

in accordance with Fig. 252, for an alternating-current we must substitute the momentary values $P\sqrt{2}\sin\omega t$ and $\Delta P\sqrt{2}\sin(\omega t + \phi)$ for P and ΔP, where ϕ is the phase displacement between current and pressure. Hence we obtain for the mean of the deflection α, by integrating over half a period,

$$k\alpha = (2P\Delta P\cos\phi + \overline{\Delta P^2}).$$

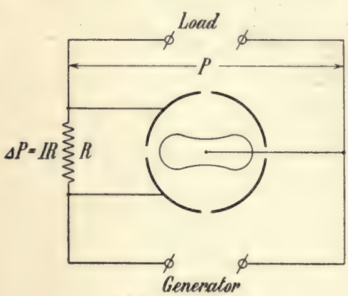


FIG. 252.

Since $\overline{\Delta P^2}$ is negligibly small compared with the first term, and ΔP is proportional to the current flowing through the non-inductive resistance R, α is clearly proportional to $2PIR\cos\phi$, that is, to the power $PI\cos\phi$.

93. Electromagnetic Instruments. These instruments depend on the action between a coil carrying an electric current and a magnet.

In instruments for measuring pressure (*voltmeters*) the coil is connected in series with a non-inductive resistance across the terminals of the pressure to be measured; whilst in those for measuring currents (*ammeters*) the current to be measured, or a proportional part of it, flows through the coil. Since it is not good to allow heavy currents to

pass through the moving coil, it becomes necessary to use calibrated resistances (*shunts*) in parallel with the ammeter.

(*a*) If the magnet is permanent and its strength is not appreciably influenced by the current in the conductor, the force in a given relative position of coil to magnet will be directly proportional to the current. Consequently, such instruments are only suitable for continuous currents. Usually the magnet is the fixed part and the current-carrying coil the movable (e.g. Weston and Deprez-d'Arsonval instruments).

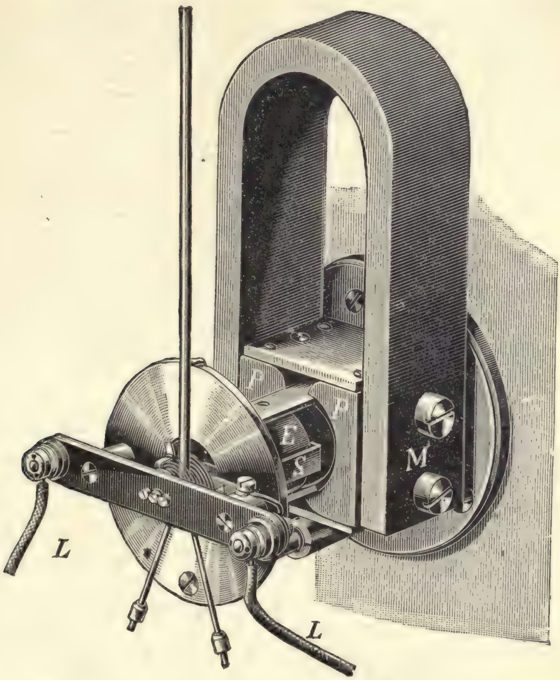


FIG. 253.—Moving-coil Instrument (Hartmann and Braun).

Fig. 253 shews the internal arrangement of such a moving-coil instrument by Hartmann and Braun. M is a horse-shoe magnet with two pole-shoes P turned cylindrically. A solid soft-iron cylinder E of smaller diameter than the bore of the shoes is placed between them concentrically, and in the space between E and P the rectangular coil S rotates, to which the current is brought through two spiral springs, which provide at the same time a retarding force. The iron core and coil can be pulled out bodily, and they are shewn in this position in the figure. Since the field in the gap is practically constant, the scale divisions are nearly uniform. A heavy damping effect is obtained by making the frame, on which the coil is wound, of metal.

(*b*) In some electromagnetic instruments (known as soft-iron instruments) a small moving soft-iron magnet is employed, magnetised by the current in a fixed coil. In such instruments the quadratic law only holds approximately, because the magnetism in the iron is not exactly proportional to the current in the coil, and also because of the screening effect of the eddy currents set up in the iron, which vary with the frequency. These instruments, therefore, read less with alternating-currents than with direct, and cannot be calibrated directly by means of continuous current. Such an instrument must be graduated by comparing it with another alternating-current instrument, which can be calibrated or graduated with direct current, the comparison being made when connected to the actual system.

In spite of these inconveniences, such instruments are nevertheless often used in practice on account of their cheapness and simplicity. Moreover, they can be made very sensitive, that is, to consume very little power.

94. Electrodynamic Instruments. The principle on which these instruments are based is the action between two coils carrying electric currents. In electrodynamic instruments for measuring pressure and current, the two coils—the fixed and the movable—are generally connected in series. Fig. 254 shews a *Torsion dynamometer* by Siemens and Halske. The movable coil consists of a rectangular copper frame of one turn, and is perpendicular to the fixed coil. It is suspended by means of a thread and a spiral spring from the torsion head at the top of the instrument. One pointer is carried by the head and one by the coil, and both of these pointers must stand at zero when no current flows through the instrument. The current is led to the movable coil through mercury contacts. The instrument shewn has two fixed coils, the number and section of the turns on each being different, thus increasing the range of the instrument. When in use, the movable coil is held in its zero position by rotating the torsion head. Since in this constant position, the torque is proportional to the square of the current, the angle through which the head is rotated is a measure of the square of the current. Hence the instrument is suitable for both continuous and alternating-currents,

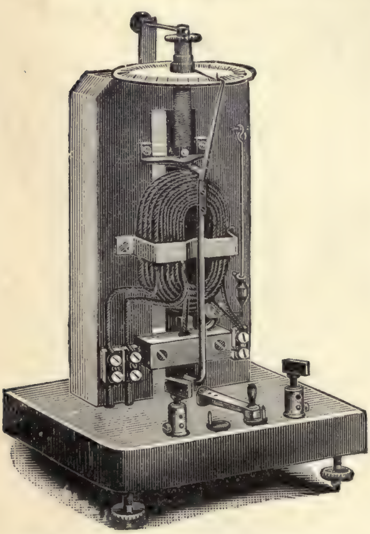


FIG. 254.—Torsion Dynamometer (Siemens and Halske).

and in the latter case measures effective values independent of wave-shape or frequency.

For measuring pressures, the two coils are made of several turns of fine wire. A variable non-inductive resistance is placed in series with the instrument, which can therefore be used over a wide range. If the self-induction of such an instrument is negligible compared with the ohmic resistance, the current will equal the pressure divided by the

FIG. 255.—Direct-reading Electrodynamic Voltmeter (Weston).

resistance. Hence the instrument can be used directly to measure pressures. If there is a self-induction L present, the resistances for alternating and continuous currents will have the ratio

$$\frac{\sqrt{r^2 + \omega^2 L^2}}{r},$$

where r is the total ohmic resistance in the circuit (coils + resistances). Hence, if the instrument has been calibrated for direct current, the readings must be multiplied by the above correcting factor when alternating-current is measured. The readings in this case depend on the wave-shape and frequency, since ω occurs in the correcting factor.

(*a*) The newer electrodynamic instruments for measuring pressure and current are made direct reading by reading off the position of the pointer fixed to the moving coil. Since the action between the two coils under these conditions obeys no simple law, the scale must be graduated by comparison with a direct-current instrument. Fig. 255 depicts such a direct-reading instrument by Weston for measuring *pressure*.

The rotation of the moving coil due to the action of the current is always such that the total self-induction L of the two coils (in series)

is increased. Hence, in this case, the correcting factor $\frac{\sqrt{r^2 + \omega^2 L^2}}{r}$ is not quite constant. For practical measurements, however, this source of error in the pressure dynamometer is quite negligible.

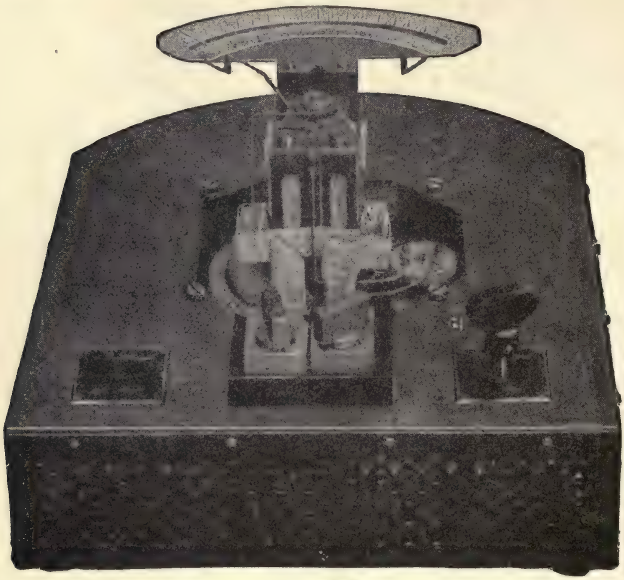


FIG. 256.—Electrodynamic Ammeter (Siemens and Halske).

In Fig 256 an electrodynamic instrument by Siemens and Halske for *measuring currents* is shewn. The movable coil is mounted on pivots and controlled by spiral springs, which also serve to convey the current to and from the coil, as in the pressure dynamometer and the electromagnetic Weston instrument. Since only a very small current can be conducted through the springs, the fixed and movable coils in these instruments are connected in parallel. Fig. 257 shews the diagram of connections for such an instrument.

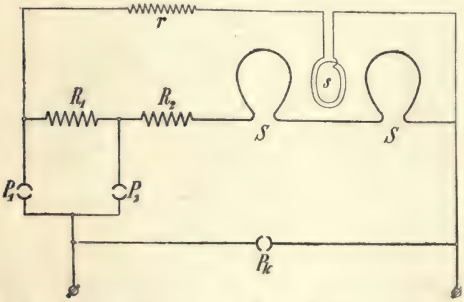


FIG. 257.—Connections of an Electro-dynamic Ammeter (Siemens and Halske).

SS denotes the fixed and s the movable coil. P_k is a plug for short-circuiting the instrument. The two plugs P_1 and P_2 serve to vary

the range of the instrument, thus with P_2 plugged, the range of the instrument may be double that when P_1 is plugged. The current must be distributed in constant ratio between the two parallel branches, independently of the heating. This is achieved by making the resistances R_1, R_2 and r of material whose temperature coefficient is very small. In order that the instrument can be graduated with direct current, the distribution of the current between the two coils must be the same with alternating-current as with continuous. Consequently the time constants, or the ratio

$$\frac{\text{ohmic resistance}}{\text{apparent self-induction}}$$

in the two branches should be the same. The apparent self-induction of a coil equals the pure self-induction of the same plus its mutual induction relative to the second coil; hence, for the fixed coil,

$$L_s = L + M,$$

and for the movable coil

$$l_s = l + M.$$

In order that the time constants may be equal, we must have therefore

$$\frac{R}{r} = \frac{L_s}{l_s} = \frac{L+M}{l+M},$$

where R and r are the ohmic resistances in the two branches.

In this case, however, M is variable, since the relative position of the coils varies. In the neighbourhood of the zero position on the scale, M is negative; when the coils are perpendicular to one another, M equals zero; and for larger deflections M is positive. Hence this condition can only be approximately fulfilled by making M small—this, however, cannot be carried too far for mechanical reasons, for the change of M corresponds to the energy expended in the movement of the pointer. Another means is to make L_s and l_s small in comparison with R and r, in which case these magnitudes, and consequently any change in the same, have but little influence on the current distribution.

$$\frac{\sqrt{R^2 + \omega^2 L_s^2}}{\sqrt{r^2 + \omega^2 l_s^2}} \simeq \frac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{r^2 + \omega^2 l^2}} \simeq \frac{R}{r}.$$

This is the means usually employed, and although such ammeters have comparatively large losses, they are, nevertheless, very valuable for accurate laboratory work owing to their exact and convenient readings.

(*b*) A special class of electrodynamic instruments is known by the name of *Induction instruments.* Currents are produced in the movable system by the electromagnetic induction of the fixed system. Fig. 258 shews the arrangement of a Siemens and Halske induction instrument. It is based on the principle, due to Ferraris, of producing

a rotary field by splitting up a single-phase current into two perpendicular components. The laminated iron ring a carries the poles ee and ff. Between the latter, there is the laminated iron cylinder c. In the gap there is a movable aluminium drum b, to which the pointer of the instrument is connected, and this drum tries to follow the rotary field. If the instrument is to be used for measuring pressures, sufficient non-inductive resistance is connected in series with the winding on the pole ee, to bring the current approximately into phase with the pressure being measured. The winding of the pole ff forms the branch SS of the bridge arrangement shewn in Fig. 259. The pressure to be measured acts between A and C, whilst between C and B a choking coil of impedance Z_d is connected. By suitably adjusting the two equal resistances rr and the resistance r_b of the bridge, it can be arranged that the two equal currents in the paths SS are displaced 90° in phase from

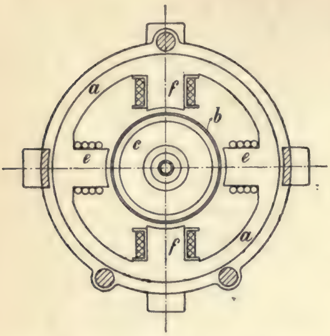


FIG. 258.—Induction Instrument.

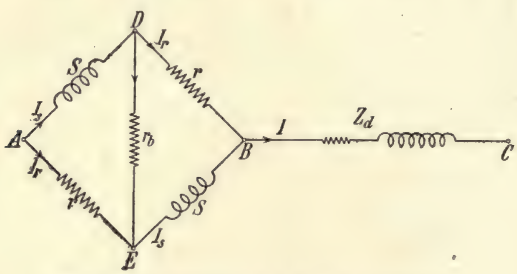


FIG. 259.—Connections of Induction Instrument.

the pressure acting across AC. In Fig. 260 the vector diagram of the scheme is shewn. The total current I produces the pressure drop $\overline{BC}$ in the choking coil. Pressure $\overline{AB}$ is made up of $\overline{AD}$ and $\overline{DB}$ on the one side and of $\overline{AE}$ and $\overline{EB}$ on the other. Since the pressures across diagonal paths of the bridge are equal and similarly directed, their vectors form a parallelogram. This is also the case with the currents in the four paths. Further, we have I_s perpendicular to $\overline{AC}$. Since branches rr and r_b are non-inductive, we have also $I_r \parallel \overline{AE} \parallel \overline{DB}$ and $I_b \parallel \overline{DE}$. ϕ_s is the phase-displacement of the current in the coils SS of the instrument. The diagram only holds for one frequency, and only for this frequency will the instrument read correctly. For the same reason, the readings also depend on the wave-shape, and

the instrument must be calibrated with an alternating-current having the same wave-shape as that which has to be measured.

Induction instruments made by several firms are based on the production of a rotary field having a very local and very unsymmetrical distribution. Fig. 261 shews the arrangement of such an instrument. An aluminium disc *S*, carrying the pointer of the instrument, is capable of moving between the poles of the horse-shoe magnet *M*. The current to be measured is sent through the winding *W*. The pole surfaces of the magnet are slotted, to take the coils *w*. In the latter, currents are induced which react on the resultant field between the pole surfaces,

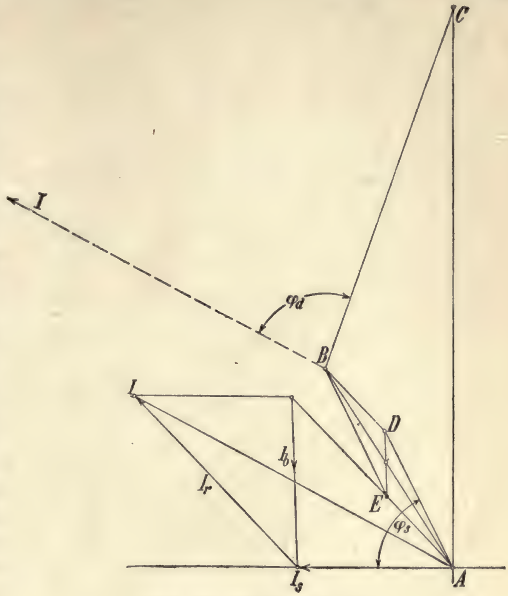


Fig. 260.—Vector Diagram.

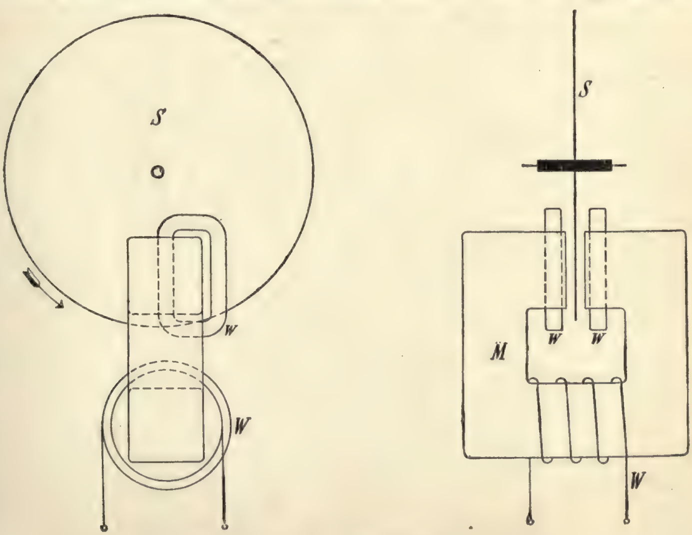


Fig. 261.—Arrangement of Induction Instrument.

so that at the right pole tip (hence inside the coils w) the field is lagging with respect to the field in the left pole tip. We thus get a local rotary field moving over from left to right, so that the disc S tends to turn in the same sense.

To the category of electrodynamic instruments also belong the *wattmeters* in general use for measuring power. These, however, will be dealt with in a separate section.

95. Hot-wire Instruments. The heating of a wire by a current is proportional to the square of the effective value of the latter, and is independent of the frequency or wave shape. Hot-wire instruments—in which the heating of a wire is measured by its extension—were first introduced by *Cardew.*

Fig. 262 represents such an instrument, as made by Hartmann and Braun. The extension of the comparatively short wire h causes the pointer to move over the scale, as the figure shews. The axis of the pointer is provided with an aluminium disc, which moves between the poles of a strong permanent magnet, thus preventing the instrument from oscillating. The system is mounted on a plate, made up of brass and iron in such a way that it has the same coefficient of expansion as the wire. In this way it becomes entirely independent of the temperature variations of the surroundings. An adjusting screw is connected to one end of the wire, for the purpose of bringing the pointer to zero when no current is passing.

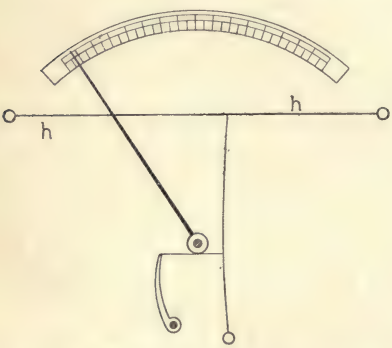


FIG. 262.—Hot-wire Instrument (Hartmann and Braun).

These instruments are made both as volt- and ammeters. As voltmeter, a current of about 0·22 amp. flows through the hot-wire to give the maximum deflection, which corresponds to a pressure drop of 3 volts.

For higher pressures a resistance made of constantin wire is connected in series, which, up to a range of 400 volts, is made part of the instrument, and for still higher voltages is contained in a separate box.

A pressure drop of 3 volts is much too high for ammeters, and consequently thicker hot-wires are used and several connected in parallel in such instruments, so that the drop is reduced to about 0·26 volt. The wires would become too thick, however, for currents above 4–5 amps., so that in this case a shunt of constantin strip is placed across the hot wire. For currents up to 100 amps. these shunts are made part of the instrument, but above this range they are kept separate.

In spite of the disadvantage of a high current consumption, the hot-wire instrument possesses many advantages. Firstly, the heat produced is independent of the wave shape or frequency, and secondly, external magnetic variations have no effect, because there is no magnetic field or solenoid present. They can therefore be used for either continuous or alternating-currents, and can be calibrated by means of continuous current.

Voltmeters for over 10 volts can be protected by fuses renewable from the outside, but for lower pressures such protection is impracticable on account of the high resistance. Ammeters can be protected from injury in a simple way by an automatic short-circuiting switch.

The hot-wire wattmeter has not yet been made practicable. It is based on the formula

$$(i+i')^2-(i-i')^2=4ii',$$

where i is proportional to the current to be measured and i' to the pressure. By arranging two hot wires in such a way that the added current $(i+i')$ flows through one and the subtracted current $(i-i')$ through the other, with the pointer to indicate the difference of the heating of the two wires, an instrument for measuring the power of a circuit is obtained.

96. Wattmeters. All wattmeters—*i.e.* instruments for measuring power—used in practice are based on the electrodynamic principle.

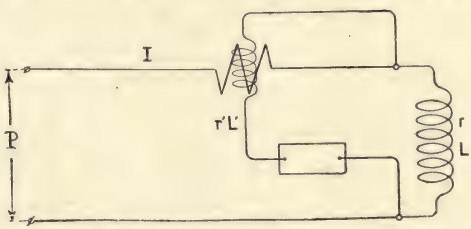


FIG. 263.—Wattmeter Connections for Low Pressures and Large Currents.

Of the two coils of the wattmeter, the fixed one is connected in series with the circuit, and is thus traversed by the main current; whilst the movable coil is connected in parallel with the circuit whose power has to be measured. The connections are shewn in Fig. 263.

Suppose, for the time being, that the terminal pressure p follows a sine wave, thus

$$p=P_{max}\sin\omega t \quad \text{and} \quad P=\frac{P_{max}}{\sqrt{2}},$$

and the main current $$i=I_{max}\sin(\omega t-\phi),$$

where $$I_{max}=\frac{P_{max}}{\sqrt{r^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$$

and $$\phi=\tan^{-1}\left(\frac{\omega L}{r}-\frac{1}{\omega Cr}\right).$$

Similarly, the current in the shunt coil is

$$i' = I'_{\max} \sin (\omega t - \phi'),$$

where
$$I'_{\max} = \frac{P_{\max}}{\sqrt{r'^2 + \omega^2 L'^2}}$$

and
$$\phi' = \tan^{-1} \frac{\omega L'}{r'}.$$

The torque acting on the movable coil is proportional to the product of i and i', assuming that the coil is always held in the same position by a torsion spring. The reading a, which is proportional to the torsion of the spring, is therefore proportional to the mean torque.

Then, if k_1 is a constant,

$$\begin{aligned} k_1 a &= \frac{1}{T}\int_0^T ii'\,dt \\ &= II' \cos (\phi - \phi') \\ &= I \frac{P}{\sqrt{r'^2 + \omega^2 L'^2}} \cos (\phi - \phi') \\ &= I \frac{P}{r'} \cos (\phi - \phi') \cos \phi'. \end{aligned}$$

The power to be measured is, however,

$$W = \frac{1}{T}\int_0^T pi\,dt = PI \cos \phi.$$

Substituting PI from the first equation, we get

$$\begin{aligned} W &= k_1 a r' \frac{\cos \phi}{\cos (\phi - \phi') \cos \phi'} \\ &= k_1 a r' \frac{1 + \tan^2 \phi'}{1 + \tan \phi \tan \phi'}. \end{aligned}$$

By suitably choosing and arranging r' we can make $\tan \phi' = \frac{\omega L'}{r'}$ very small, so that

$$W = k_1 r' a = \text{constant} \times \text{reading}.$$

When we have a terminal pressure whose wave is not sinusoidal, but

$$p = P_{1\max} \sin (\omega t + \psi_1) + P_{3\max} \sin (3\omega t + \psi_3) + \dots,$$

we get, as shewn by *Prof. H. F. Weber*, in the official report of the Frankfort Exhibition, 1891,

$$W = k_1 a r' \frac{1 + \tan^2 \phi'}{1 + \tan \phi \tan \phi'} \cdot \frac{1 + \dfrac{P_3 I_3 \cos \phi_3}{P_1 I_1 \cos \phi_1} + \dfrac{P_5 I_5 \cos \phi_5}{P_1 I_1 \cos \phi_1} + \dots}{1 + \dfrac{P_3 I_3 \cos \phi_3 \cos^2 \phi'_3}{P_1 I_1 \cos \phi_1 \cos^2 \phi'_1} \cdot \dfrac{1 + \tan \phi_3 \tan \phi'_3}{1 + \tan \phi_1 \tan \phi'_1} + \dots}.$$

The phase displacements ϕ and ϕ' apply to the current circuit and pressure circuit respectively.

The first correcting factor is

$$\frac{1+\tan^2\phi'}{1+\tan\phi\tan\phi'} \simeq \frac{1}{1-\tan\phi\tan\phi'} \gtreqless 1 \begin{cases} \text{for } \tan\phi > 0, \\ \text{,,} \quad \tan\phi = 0, \\ \text{,,} \quad \tan\phi < 0. \end{cases}$$

The second correcting factor is always greater than unity, but even for very distorted wave shapes is only about $\frac{1}{10000}$ greater, and can therefore be always put equal to 1. Hence, for any wave,

$$W = k_1 ar' \frac{1}{1-\tan\phi\tan\phi'}. \qquad \text{(154)}$$

The measured power W is not exactly equal to the power given to the circuit, but somewhat greater, since the heating loss $\frac{P^2}{r'}$ in the pressure coil of the wattmeter is measured with it. Hence the true power is $W - \frac{P^2}{r'}$, where r' is the resistance of the pressure coil.

At any stated voltage, it is a very simple matter to determine the error $\frac{P^2}{r'}$ experimentally, by noting the wattmeter reading when the load circuit is opened so that the true power is zero.

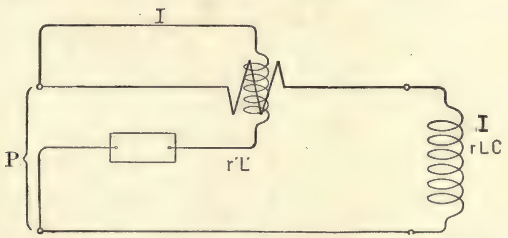


FIG. 264.—Wattmeter Connections for High Pressures and Small Currents.

The wattmeter can also be connected as shewn in Fig. 264. Here likewise the power measured is too great by the amount I^2r'', lost in the current coil of resistance r''.

If in the above circuits we have power produced and not power consumed, the above losses I^2r'' and $\frac{P^2}{r'}$ must be added to the measured power W in order to find the power produced in the circuit.

To obtain minimum error, the former scheme of connections should be used for small currents and large pressures, and the latter for low pressures and large currents. When powers at high pressures are measured, a resistance must be placed in series with the shunt circuit, to keep the potential difference between the two coils of the wattmeter as small as possible, as in Figs. 263 and 264.

In addition to the earlier wattmeters with torsion springs, several firms, *e.g. Weston* and *Siemens & Halske*, make more convenient instruments in which the movable coil (together with the pointer) changes

its position relative to the fixed coil. From this it follows that these direct-reading instruments have not a uniform scale, and must consequently be calibrated by experiment.

The Weston instruments for smaller powers have a compensating coil wound over the current coil and carrying the current flowing through the pressure coil, so that the currents oppose one another in the two fixed coils. The number of turns of the compensating coil is chosen so that the power is measured directly.

For direct connection in high pressure circuits, wattmeters of the type due to Lord Kelvin are specially suitable.

97. Direct Measurement of the Effective Values of the Several Harmonics. The wattmeter, however, can also be used for other purposes than the measurement of power. For example, with two

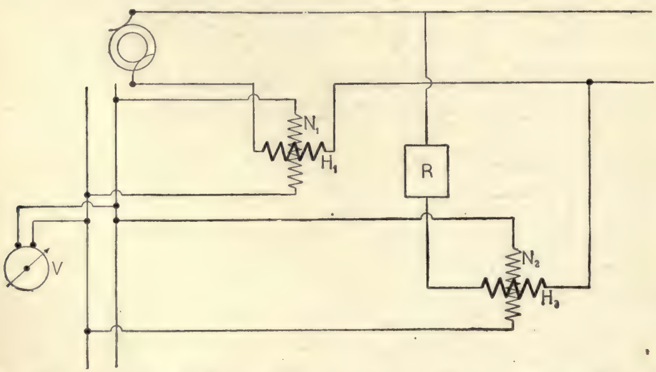


FIG. 265.—Connections for Direct Measurement of the Effective Values of the Several Harmonics in a Circuit.

wattmeters the effective values of the pressures and currents of the several harmonics in any wave can be measured directly. For this purpose we must have auxiliary sinusoidal pressures at our disposal at frequencies of the first, third, fifth and seventh harmonics.

The current under investigation is sent through the current coil of one of the wattmeters, whilst the current coil of the other wattmeter (which must be made for small currents and high pressures) is connected in shunt. The pressure coils of both wattmeters are connected to the circuit in which the sine-wave pressure is produced.

In Fig. 265 H_1 and H_2 represent the current coils and N_1, N_2 the pressure coils. The voltmeter V measures the sinusoidal pressure P_h in the auxiliary circuit whose frequency can be adjusted to that of the first, third, fifth or seventh harmonic.

From Section 64 we know that only currents of the same frequency can act on one another electrodynamically, and that this action is a maximum when the two currents are in phase. If we then wish to measure the magnitude of the fundamental, we induce the auxiliary

current at this frequency and vary its phase until it is in phase with the main current; the reading on the wattmeter is then a maximum. Let W_1 watts denote this maximum reading and P_{h1} the value of the auxiliary pressure read on voltmeter V; the effective value of the fundamental of the current is then

$$I_1 = \frac{W_1}{P_{h1}}.$$

To determine the effective pressure P_1 of the fundamental current, the phase of the auxiliary current is varied until the pointer of the second wattmeter shews a maximum. Denoting this maximum reading in watts by V_1 and the pressure of the auxiliary circuit again by P_{h1}, the effective value P_1 of the fundamental of pressure will be

$$P_1 = k\frac{V_1}{P_{h1}},$$

where k is a constant depending on the resistance R.

We can also measure the phase displacement ϕ_1 between the fundamental pressure P_1 and the fundamental current I_1. This is best done by adjusting the phase of the auxiliary current until the needle of the first wattmeter shews no deflection. Starting from this position, the angle through which the phase of the auxiliary current must be altered to bring the deflection of the pointer of the second instrument to zero then gives directly the phase angle ϕ_1 between P_1 and I_1.

If we arrange the auxiliary pressure to have the frequency of the third harmonic, we get in a similar manner the effective pressure and current of the third harmonic, viz.

$$P_3 = k\frac{V_3}{P_{h3}} \quad \text{and} \quad I_3 = \frac{W_3}{P_{h3}},$$

where W_3 is the maximum power on the first wattmeter and V_3 on the second, whilst P_{h3} is the effective pressure of the auxiliary current at this periodicity. ϕ_3 is found in the same way as ϕ_1.

By this means, the effective values of the currents and pressures, and also their phase displacements, for the several harmonics can be found directly, and an insight is obtained into the action of the same.

In most machines, the magnitude of the several harmonics is of more interest than their phase displacement, and in such cases the above method is sufficient for their investigation. In other cases, *e.g.* arc lamps, insulation testing, transformers on no-load, where the shape of the pressure curve and not the magnitude of the several harmonics, is the important part, the above determination of the harmonics one by one is not sufficient. For this purpose, the oscillograph can be resorted to—for this instrument shews the complete curve at a glance.

98. Measurement of Power by Means of Three Voltmeters or Three Ammeters. In addition to the measurement of power by wattmeters,

two other methods may be mentioned, viz. the *three-voltmeter method of Ayrton, Swinburne and Sumpner* and the *three-ammeter method of Fleming.*

The former can be carried out as follows (see Fig. 266). r is a non-inductive resistance in series with the circuit whose power W is to be measured. Since the pressure P_{I} is in phase with the current I, P_{I} and P_{II} can be geometrically

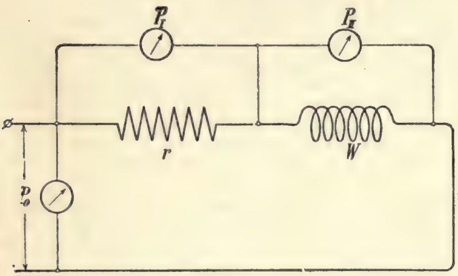


Fig. 266.—Connections for the Three-Voltmeter Method.

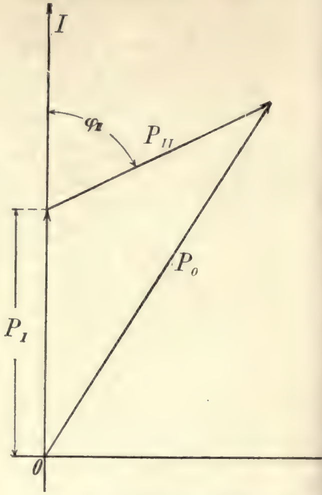


Fig. 267.—Pressure Diagram of Three-Voltmeter Method.

added, independently of their wave shape. Fig. 267 is the vector-diagram of this arrangement, where P_0 is the resultant of P_{I} and P_{II}.

The power due to P_{II} is:

$$W = P_{\mathrm{II}} I \cos\phi_{\mathrm{II}}$$

$$= P_{\mathrm{II}} \frac{P_{\mathrm{I}}}{r} \cos\phi_{\mathrm{II}}$$

$$= \frac{1}{2r}(P_0^2 - P_{\mathrm{I}}^2 - P_{\mathrm{II}}^2). \quad \text{.....................(155)}$$

This method is of no practical use because, unless the power consumed in the inserted resistance is fairly large, the results are very inaccurate.

The second method, the three ammeter method, is also of little importance, but nevertheless is preferable to the above, since the full pressure is applied to the load circuit, the non-inductive resistance being placed in parallel with the latter (see Fig. 268). The diagram is shewn in Fig. 269, and the proof is as follows:

From the diagram (Fig. 269) we have firstly

$$W = PI_{\mathrm{II}} \cos\phi_{\mathrm{II}} = rI_{\mathrm{I}} I_{\mathrm{II}} \cos\phi_{\mathrm{II}}$$

$$= \frac{r}{2}(I_0^2 - I_{\mathrm{I}}^2 - I_{\mathrm{II}}^2). \quad \text{...........................(156)}$$

Secondly, denoting the momentary values of the pressure and currents by p, i_0, i_{I} and i_{II}, we get, independently of the wave shape,

$$i_0 = i_{\mathrm{I}} + i_{\mathrm{II}}\ ; \quad i_{\mathrm{I}} = \frac{p}{r}.$$

The momentary power in branch II. is

$$w = p i_{\mathrm{II}} = i_{\mathrm{I}} i_{\mathrm{II}} r,$$

and since $\quad i_0^2 = i_{\mathrm{I}}^2 + i_{\mathrm{II}}^2 + 2 i_{\mathrm{I}} i_{\mathrm{II}},$

then $\quad w = \frac{r}{2}\left(i_0^2 - i_{\mathrm{I}}^2 - i_{\mathrm{II}}^2\right);$

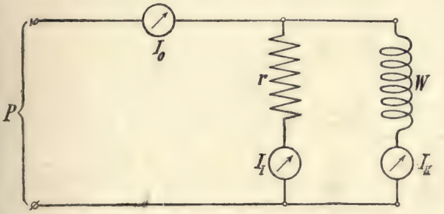


FIG. 268.—Connections for the Three-Ammeter Method.

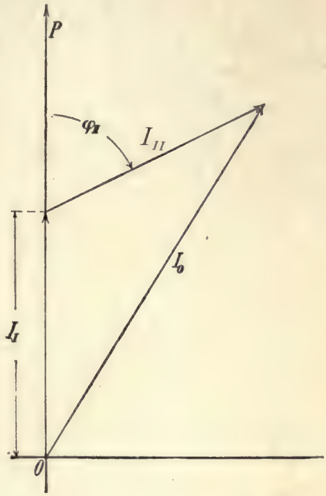


FIG. 269.—Current Diagram of Three-Ammeter Method.

hence the mean power is

$$W = \frac{1}{T}\int_0^T w\,dt = \frac{r}{2}\left(I_0^2 - I_{\mathrm{I}}^2 - I_{\mathrm{II}}^2\right).$$

This is the same as the previous result, from which we see that the diagram in Fig. 269 is correct.

From this it follows in general that the graphical addition of the current vectors of parallel circuits is allowable if all these circuits except one have zero reactance.

A method for experimentally examining the admissibility of geometrically adding effective E.M.F.'s of any wave shape has been given by *Bedell* in the *Elec. World*, Vol. 28, No. 3.

Let P_{I} and P_{II} be two pressures of any wave shape, and let P_s be their measured sum, whilst P_d is their measured difference (see Fig. 270).

Then we must have

$$P_s^2 = \frac{1}{T}\int_0^T (p_{\mathrm{I}} + p_{\mathrm{II}})^2\,dt$$

and

$$P_d^2 = \frac{1}{T}\int_0^T (p_{\mathrm{I}} - p_{\mathrm{II}})^2\,dt,$$

whence, by addition,

$$P_s^2 + P_d^2 = 2P_{\mathrm{I}}^2 + 2P_{\mathrm{II}}^2$$

or

$$P_{\mathrm{II}}^2 = \tfrac{1}{2}(P_s^2 + P_d^2 - 2P_{\mathrm{I}}^2),$$

i.e. P_{II} is the line containing the centre of gravity of the triangle whose sides are P_s, P_d and $2P_{\mathrm{I}}$, or, in other words, ACD must be a straight line if it is allowable to add P_{I} and P_{II} geometrically.

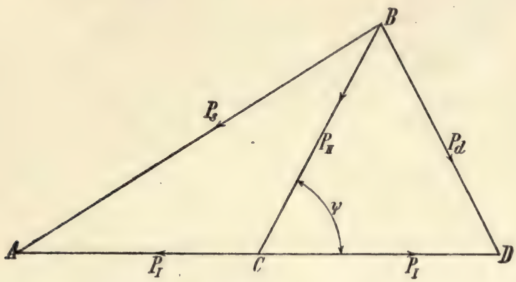


FIG. 270.—Experiment shewing how Pressures can be added.

By means of this proof, we can shew it is allowable to add P_{I} and P_{II} geometrically, if we measure P_{I}, P_{II}, P_d and P_s.

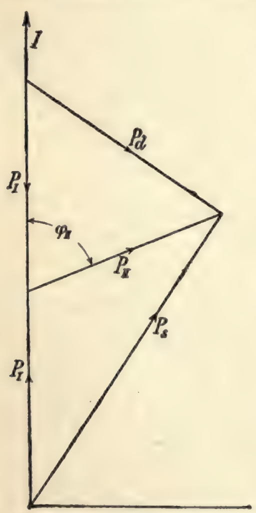


FIG. 271.—Diagram for measuring Power by means of Two Voltmeters.

Instead of the three-voltmeter method for measuring power, the following method can also be used. To measure, for example, the power

$$W = P_{\mathrm{II}} I \cos\phi_{\mathrm{II}}.$$

Since the pressure $P_{\mathrm{I}} = Ir$ (Fig. 271) is in phase with the current I, we have

$$W = \frac{1}{T}\int_0^T p_{\mathrm{II}}\, i\, dt = \frac{1}{T}\int_0^T \frac{p_{\mathrm{I}} p_{\mathrm{II}}}{r}\, dt.$$

By subtracting the above expressions for P_s^2 and P_d^2, we have

$$P_s^2 - P_d^2 = \frac{1}{T}\int_0^T \{(p_{\mathrm{I}} + p_{\mathrm{II}})^2 - (p_{\mathrm{I}} - p_{\mathrm{II}})^2\}\, dt$$

$$= \frac{4}{T}\int_0^T p_{\mathrm{I}} p_{\mathrm{II}}\, dt,$$

thus

$$W = \frac{P_s^2 - P_d^2}{4r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(157)$$

This method has recently been recommended by various writers for cases in which ϕ_{II} is large; but even in such cases it is very inexact. For measuring the power in circuits with large phase displacements, it is advisable to

have special wattmeters with scales only $\frac{1}{3}$ to $\frac{1}{5}$ of those for the ordinary wattmeter. For example, if a wattmeter is made for 60 amperes and 100 volts, the torsion spring governing the movable coil can be set (*i.e.* weakened) so that the instrument has its maximum deflection at 2000 watts instead of at 6000. One may also use the ordinary wattmeter and overload the pressure coil, but this must naturally only be done for short periods.

99. Measurement of the Power in a Polyphase Circuit. In a symmetrical n-phase system which is symmetrically loaded, we found in Chap. XIII. that the power in each phase is $\frac{1}{n}$th of the total power. From this it is obvious that the power in such a system can be measured by a wattmeter inserted in any one of the phases. The same also holds for a balanced two-phase three-wire system, since in this

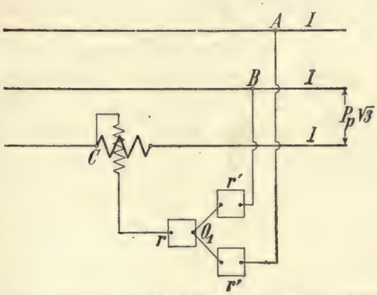


FIG. 272.—Measurement of Power in a Balanced Three-phase System by means of a Wattmeter.

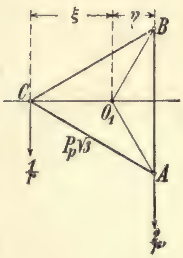


FIG. 273.

case also the two phases produce equal power. This measurement can, however, only be made directly when the system is independent, or in the case of a star system, when the neutral point is available, so that the pressure coil of the wattmeter can be connected up. To carry out the measurement for a ring system, the latter must be opened at some point in one phase and the current coil of the wattmeter inserted, whilst the pressure coil is connected across this phase.

In cases where only the n terminals of the n-phase system are available, we must proceed otherwise. A method suitable for this case was given by Behn-Eschenburg in the *E.T.Z.*, 1896, p. 182. The current coil is connected in series with one of the mains, and the pressure coil between this main and an artificial neutral point O_1 made by means of resistances, as shewn in Fig. 272 for a three-phase system.

If the resistances r' between the two points A and B and the point O_1 are chosen equal, the neutral point O_1 in the equilateral pressure triangle (Fig. 273) will fall on the normal from C on AB; consequently the pressure O_1C is displaced by the phase angle ϕ from the current

in the current coil of the wattmeter. If the pressure between two wires is interlinked, the power in the system is

$$W = 3PI\cos\phi.$$

In the pressure coil, however, we have not the pressure $3P$, but ξ; consequently the reading must be multiplied by the ratio $k_r = \frac{3P}{\xi}$. In the pressure triangle (Fig. 273), the point O_1 is determined by the method for finding the pressure of a load star point, p. 254, and, hence,

$$\frac{\xi}{r} = \frac{2\eta}{r'} = \frac{\eta}{\frac{r'}{2}} = \frac{\xi+\eta}{r+\frac{r'}{2}}.$$

Now $\xi + \eta = \frac{3}{2}P$, since a side of the triangle equals $P\sqrt{3}$.

Hence
$$\frac{\xi}{r} = \frac{3P}{2r+r'}$$

or
$$\frac{3P}{\xi} = \frac{2r+r'}{r} = 2 + \frac{r'}{r} = k_r,$$

and the power in the system equals

$$W = k_r \times \text{measured power}.$$

If we make $r = r'$, we get

$$k_r = \frac{3P}{\xi} = 3$$

and
$$W = 3 \times \text{the measured watts}.$$

If a polyphase system is not quite symmetrical, or is unsymmetrically loaded, the power in the several phases may differ considerably. For this reason, such a system cannot be regarded as balanced, and the total power can only be ascertained in the same way as for any other unbalanced system, as the two following methods shew.

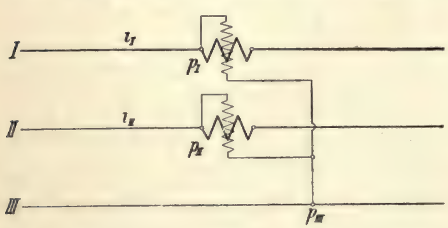


FIG. 274.—Measurement of Power in a Three-wire Three-phase System by means of Two Wattmeters.

(*a*) For the ordinary method of measuring the power in any n-phase system with n wires we need only $n-1$ wattmeters; for any one of the n conductors can be regarded as the return for the $n-1$ currents, since the sum of all the currents in the system equals 0. The current coils of the $n-1$ wattmeters are all connected in the same way in the $n-1$ lines, and the pressure coils between their respective lines and the line where there is no wattmeter. Fig. 274 depicts the connections for a three-phase system.

From the several wattmeters, different powers will be obtained; should any of these be negative, the wattmeter must be reversed and

its power prefixed by the negative sign; the algebraic sum of the powers thus measured gives the total power in the system.

If the power of a symmetrical three-phase system is measured with two wattmeters, the phase displacement of the currents in the system can be found from the wattmeter readings.

Let the three phase-pressures be:

$$p_{\mathrm{I}} = P\sqrt{2}\sin\omega t,$$
$$p_{\mathrm{II}} = P\sqrt{2}\sin(\omega t - 120°),$$
$$p_{\mathrm{III}} = P\sqrt{2}\sin(\omega t - 240°),$$

and the currents,

$$i_{\mathrm{I}} = I\sqrt{2}\sin(\omega t - \phi),$$
$$i_{\mathrm{II}} = I\sqrt{2}\sin(\omega t - \phi - 120°).$$

Then

$$w_{\mathrm{I}} = (p_{\mathrm{I}} - p_{\mathrm{III}})i_{\mathrm{I}}$$

and

$$w_{\mathrm{II}} = (p_{\mathrm{II}} - p_{\mathrm{III}})i_{\mathrm{II}},$$

whence

$$W_{\mathrm{I}} = \sqrt{3}PI\cos(\phi - 30°),$$
$$W_{\mathrm{II}} = \sqrt{3}PI\cos(\phi + 30°).$$

For $\phi \lesseqgtr 60°$, $W_{\mathrm{II}} \gtreqless 0$,

$$\tan\phi = \frac{W_{\mathrm{I}} - W_{\mathrm{II}}}{W_{\mathrm{I}} + W_{\mathrm{II}}}\sqrt{3}. \qquad \text{(158)}$$

The above assumes sine waves for both currents and pressures, and that all phases are equally loaded.

(*b*) The second method for measuring the power in any *n*-phase system consists in using *n* wattmeters, each line containing one; the

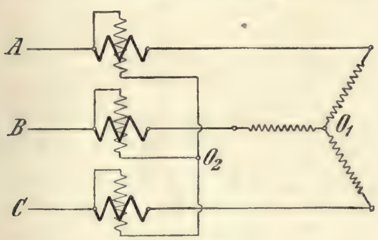


FIG. 275.

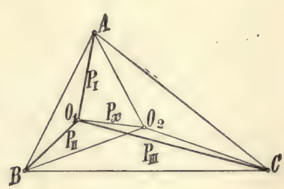


FIG. 276.

Measurement of Power in an Unbalanced Three-phase System by means of Three Wattmeters.

pressure coils can be connected between their respective lines and the neutral line. If no neutral line is present, all the ends may be joined to a neutral point. In the former case, when a neutral wire is present, each wattmeter measures the power in its respective phase. In the latter case, the sum of the readings equals the total power, but the several readings do not, in general, represent the power in the several phases. Consider, for example, a three-phase system without a neutral wire (Fig. 275), and let *ABC*, (Fig. 276), represent its pressure triangle,

with O_1 as the middle point of pressure in the load. The pressures of the three loads are $P_{\rm I}$, $P_{\rm II}$ and $P_{\rm III}$; further, if O_2 is the middle point of pressure for the pressure coils and their resistances and $\overline{O_1O_2}$ is equal to P_x, then the momentary values of the three measured powers are:

$$w_{\rm I} = (p_{\rm I} - p_x) i_{\rm I},$$

$$w_{\rm II} = (p_{\rm II} - p_x) i_{\rm II},$$

$$w_{\rm III} = (p_{\rm III} - p_x) i_{\rm III};$$

hence $$w_{\rm I} + w_{\rm II} + w_{\rm III} = p_{\rm I} i_{\rm I} + p_{\rm II} i_{\rm II} + p_{\rm III} i_{\rm III},$$

which proves the correctness of the measurement.

100. Measurement of the Wattless Component of an Alternating Current. When we wish to determine the wattless component in any circuit, the pressure, current and power factor, $\cos\phi = \frac{W}{PI}$, will serve for finding $\sin\phi$, from which the wattless current can be calculated. At low power-factors, however, the results thus obtained are not accurate.

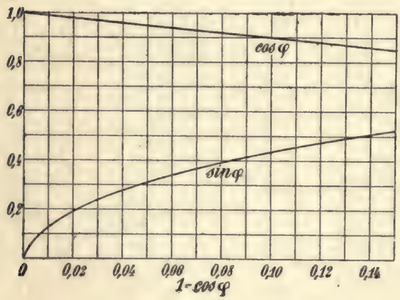


FIG. 277.

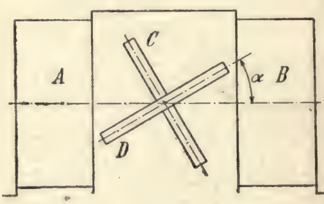


FIG. 278.

As seen from Fig. 277, in which $\sin\phi$ and $\cos\phi$ are plotted as functions of $1 - \cos\phi$, a small error in the reading of the instrument, that is, in $\cos\phi$, causes a large error in $\sin\phi$. If, for instance, the power factor $\cos\phi = 0{\cdot}99$, and unity was read off the instruments (which only amounts to an error of 1 %), the wattless current of 14 % would have escaped notice. Especially in cases where condensers and synchronous motors are used to raise the power factor to unity, it is advisable to have instruments which enable the wattless current to be accurately measured.

Of the various methods which have been published, only a few which have found their way into practice will be described here.

(*a*) The principle of an instrument by Hartmann and Braun is given in Fig. 278. The coil AB is traversed by the current $i = I\sin(\omega t - \phi)$ and produces a field $\Phi = \Phi_0 \sin(\omega t - \phi)$. A current $i' = I' \sin\omega t$ in phase with the pressure is sent through the coil D, whilst a current

$$i' = I' \sin\left(\omega t - \frac{\pi}{2}\right),$$

at right angles to the pressure, is sent through the coil C, which stands at 90° to D.

When D is displaced through the angle α from the field, the torques exerted on the coils are:

$$\delta_1 = \frac{\Phi_0 I'}{2} \sin \alpha \cos \phi,$$

$$\delta_2 = \frac{\Phi_0 I'}{2} \sin \phi \cos \alpha.$$

The movable system will come to rest at such an angle that $\delta_1 = \delta_2$, *i.e.* $\tan \alpha = \tan \phi$. We measure therefore $\tan \phi$, which function, in the neighbourhood of unity, is as sensitive to changes of ϕ as $\sin \phi$ itself.

This instrument is of great use as *synchroniser*, for it serves to denote phase equality and synchronism when paralleling. When a current is sent through AB, proportional to and in phase with the pressure of a generator (or bus bars), and a current in phase with the pressure of the other generator to be paralleled through D, and a current at 90° to this pressure through C, then the position of the movable coils will give the phase difference between the pressures of the two generators.

In large generators, in paralleling which it is highly important to know the exact instant of phase equality, this instrument is of great service. The methods of synchronising by means of lamps or phase voltmeters, since they are not very sensitive, may give rise to heavy rushes of current on switching on. This instrument was first largely used in the power station at Niagara Falls.

It may be further mentioned that neither the angle between C and D nor the phase displacement between the currents in these coils needs to be exactly 90°. So long as there is a space and time angle, the instrument will respond to phase displacement.

(*b*) The principle of displacing the current in the pressure coil 90° from the pressure in an ordinary wattmeter can also be used to measure the so-called imaginary power, or with a constant pressure, the wattless current. Then the instrument will give the value $PI \sin \phi$ instead of $PI \cos \phi$. By connecting a large capacity in series with the pressure coil, a phase displacement of about 90° can easily be obtained.

(*c*) We have still to deal with the application of wattmeters as phase indicators in polyphase systems.

If all the phases of a three-phase system are balanced and the pressure is a sine wave, we can obtain the phase displacement by two readings on one wattmeter from Formula 158, p. 315,

$$\tan \phi = \sqrt{3} \frac{W_{\mathrm{I}} - W_{\mathrm{II}}}{W_{\mathrm{I}} + W_{\mathrm{II}}}.$$

Provided we do not alter the sensitiveness of the instrument we need not know the constants but merely note the readings.

The G.E.C. of America have made instruments with two pressure coils (Fig. 279); the pointer then takes up a position corresponding to the phase displacement. The scale is calibrated on test and reads $\cos \phi$.

Such instruments should only be installed, however, where a knowledge of the wattless current is essential.

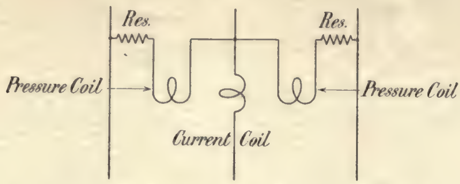


FIG. 279.—Connections of Power-factor Meter (G.E.C.).

The phase meter of the A.E.G. (due to Dolivo von Dobrowolsky) is based on the principle of the induction instrument (see Fig. 258). Here, however, the one-coil system must be traversed by a current in phase with the pressure. The displacement of 90° mentioned on p. 317 is not necessary. The instrument reads $PI \sin \phi$, or at constant pressure, the wattless current $I \sin \phi$.

All instruments, however, which are based on induction effects have the disadvantage that they are largely influenced by frequency and wave shape.

101. Determination of Wave Shape of a Pressure or Current by means of Contact Apparatus and Galvanometer. To determine the instantaneous values of a rapidly varying pressure or current, care must be taken that only one and the same momentary value acts on the instrument—this is attainable with Joubert's disc and contact apparatus. For every revolution of the rotating contact apparatus, the same momentary current is tapped off once. The arrangement of the contact apparatus and the measurement of the current thus tapped off may be accomplished in various ways, only two of which, however, will be given here. The one is a zero method, and is specially suitable for accurate work, whilst the other, due to *Blondel*, is more convenient and needs less time.

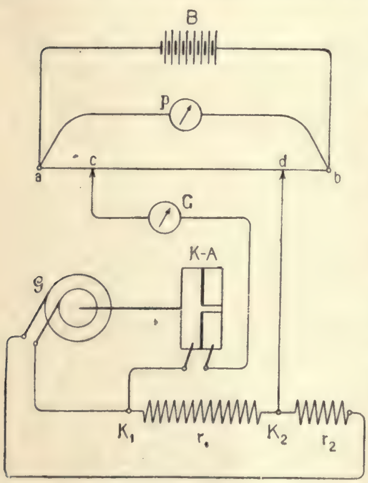


FIG. 280.—Determination of Pressure and Current Curves by the Zero Method.

The *zero* or *compensation method* is given in Fig. 280.

G is the generator from which a current is sent through the resistances r_1 and r_2. Parallel to the resistance r_1, the contact apparatus K-A and a galvanometer are connected,

together with the part *c-d* of the wire *a-b* which is connected to a battery *B*. If the contacts *c-d* are shifted along the wire until the galvanometer shews no deflection, the pressure over the part *c-d* equals the required momentary value. That is, the momentary value being measured $= \frac{cd}{ab} \times p$, where p is the pressure across the whole wire *a-b*. The galvanometer must be a rather heavily damped *Deprez-galvanometer* with a long period of oscillation and high sensitiveness.

Blondel's method as used by Siemens and Halske in conjunction with a synchronous motor is very convenient for practical work, since the apparatus can be used anywhere. Fig. 281 shews the scheme.

The pressure, whose curve is to be found, is applied to the terminals K_1 and K_2. By means of the contact apparatus *K-A*, which is driven by a synchronous motor running from the mains being tested, the condenser C is charged each time contact is made and discharged in the next instant through the galvanometer G, whose throw is thus

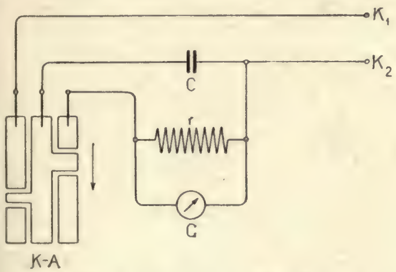


FIG. 281.—Determination of Pressure and Current Curves by Blondel's Method.

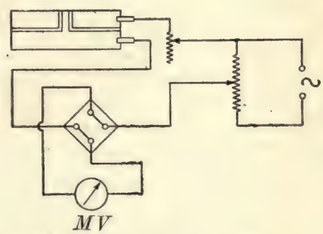


FIG. 282.—Determination of Pressure and Current Curves by means of a Milli-voltmeter.

proportional to the respective momentary pressure. A well-damped milli-voltmeter can also be used to measure this momentary pressure, but it is found that the deflection is not proportional to the momentary value, so that the scale of the milli-voltmeter must be calibrated by means of a direct pressure applied at the terminals K_1 and K_2.

This brings us to the third method, shewn in Fig. 282, in which the capacity is omitted and a sufficiently large deflection of the milli-voltmeter is obtained by introducing a small resistance. Of course, in this case, the instrument will be largely overloaded during the period of contact, which, however, is too short to cause damage.

Care must be taken that the instrument is disconnected before the synchronous motor is switched off, because otherwise the motor might come to rest in such a position that contact was made continuously, which would result in burning out the instrument. To obtain steady deflections, a range of about one-third the scale should not be exceeded. Greater deflections are unsafe owing to the large currents broken at the contact-maker. The good adjustment of the contact spring is of especial value, since the presence of small sparks makes accurate results

impossible. For this reason, only a part of the pressure should be used in taking the pressure curve.

The method of obtaining the current curve is similar to the above, the current being passed through a suitable non-inductive resistance and the pressure curve taken at its terminals. It is desirable to calibrate this also with direct current. Deviations from the proportionality up to 20 % may occur through variations in the contact resistance at the contacts, and it is therefore advisable to clean the contact disc with switch oil.

In the above methods it is chiefly the sparking at the contacts which vitiates the accuracy of the curves. This difficulty is completely overcome by *Owens*' differential galvanometer* (Fig. 283).

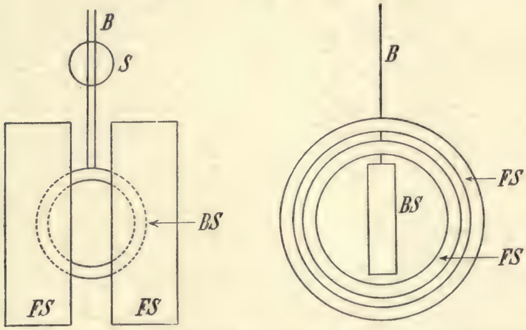


Fig. 283.—Differential Galvanometer for Determination of Alternating-current Curves (*R. B. Owens*).

The alternating current to be measured is sent through the two outer fixed coils and a direct current through the two inner ones or vice versâ. A synchronous motor, driven by the current under consideration, carries a contact-maker, which periodically closes a direct-current circuit containing the movable coils. The four fixed coils act on the movable turns only when the circuit is closed. Hence, if the direct-current flowing in the inner fixed coils is varied, the momentary value of the alternating current, which only flows in the moving coils at the moment the circuit is closed, can be compensated, so that the coils do not deflect. A calibration is here necessary, in order to know what relation the direct current bears to the momentary values of the alternating current.

For exact measurements the arrangement of the contact apparatus shewn in Figs. 284*a* and *b* and devised by *G. Schade* is very useful. In this arrangement, there are no rubbing surfaces, but contact is made by pressure. The time of contact is very small, so that rapid changes in the curve can be accurately measured.

The instrument consists of the contact-disc S coupled to the shaft of

* *Amer. Inst. of Elect. Eng.* 1902, p. 753.

the machine, a segment T and the sliding piece of ebonite G which can

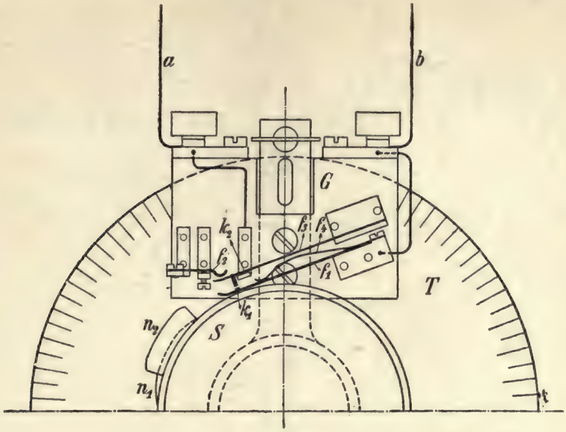


FIG. 284*a*.

be shifted over this segment. G carries the contact springs f_1 and f_3 insulated from it and the control springs f_2 and f_4. S carries a contact cam n_1. The wires a and b are connected once in every revolution, when n_1 comes underneath f_1. f_1 is thus lifted until k_1 makes contact with f_3. In the next instant the circuit between a and b is broken, due to f_1 and f_3 being raised together, thereby shifting k_2 from its position. To prevent n_1 from making further contact, a second cam n_2 behind n_1 is provided. This keeps f_3 raised until f_1 returns to its original position.

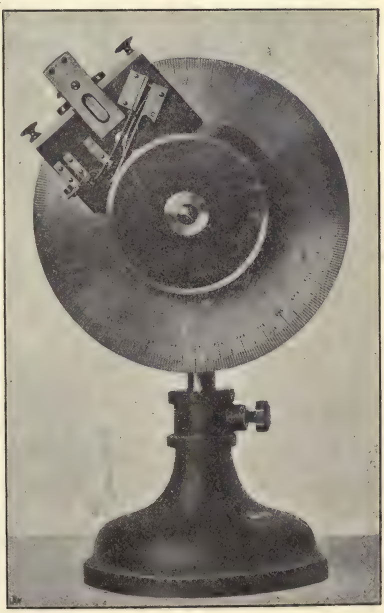


FIG. 284*b*.—Contact-maker for determining Alternating-current Curves (*G. Schade*).

102. The Oscillograph. The point-by-point methods just described for delineating alternating-current curves have many great disadvantages. In the first place they require much time, and secondly they are often inexact. Point-by-point methods are, of course, out of the question altogether when the successive waves are not identical. In this case, instruments known as *oscillographs*

can be used for taking such curves, especially as they have been much improved of late years. In Vol. XXVIII. (1899) of the *Journal of the Inst. of Elec. Engineers*, *Duddell* and *Marchant* described an oscillograph constructed according to a suggestion by Blondel. The following is a summary of this description.

In Fig. 285 the instrument is shewn diagrammatically. In the narrow gap between the poles NS of a powerful electromagnet are

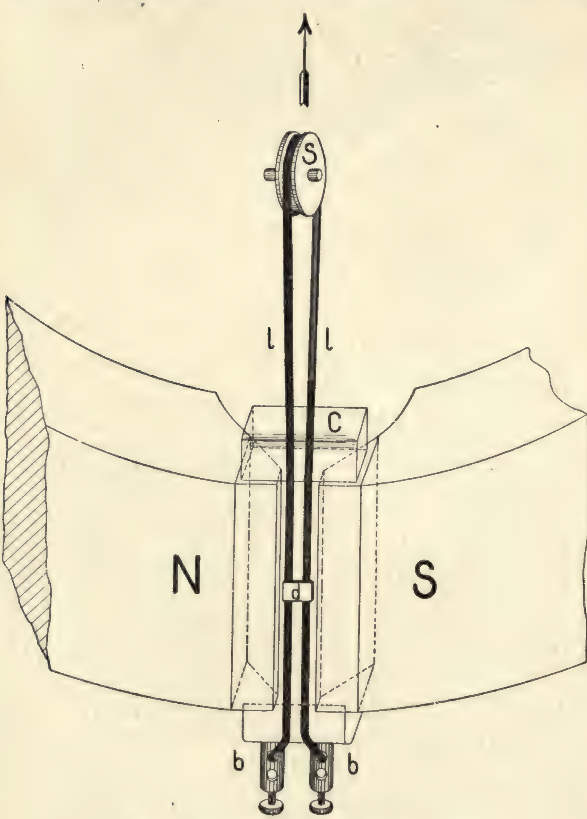


Fig. 285.—Diagrammatic View of Oscillograph.

stretched the two parallel sides ll of a metal strip which passes over a small disc S. At the bottom the strip is fixed at bb, and above it presses against the bridge C. The current flows up one side of the strip and down the other. Owing to the electromagnetic action brought into play, the one conductor will be displaced forwards and the other backwards, whereby the small mirror d, fixed to the two conductors, will be deflected through an angle, which, for small

deflections, will be proportional to the current flowing through the strip. An oscillograph should fulfil the following conditions :

1. The time of natural oscillation of the conductors ll must be very small compared with the period of the alternating-current being measured.

2. The instrument must be damped so as just to prevent the movement becoming oscillatory.

3. The apparatus must have a negligible self-induction.

4. The sensitiveness must be sufficiently large.

The requisite damping is obtained by surrounding the conductors and mirror with oil, the case for the oil being formed by the pole-faces for the sides, a brass plate for the back and a lens for the front.

In order to observe the movements of the mirror, a ray of light is reflected from it by means of another rotating mirror; or by suitable arrangements, the moving ray of light can be photographed.

Actually the instrument is provided with two strips, each strip occupying a separate space in the magnetic circuit, so that the pressure and current curves can be taken simultaneously. In addition to this, between the two movable mirrors there is a small fixed mirror. The ray of light reflected from this fixed mirror then gives the zero line.

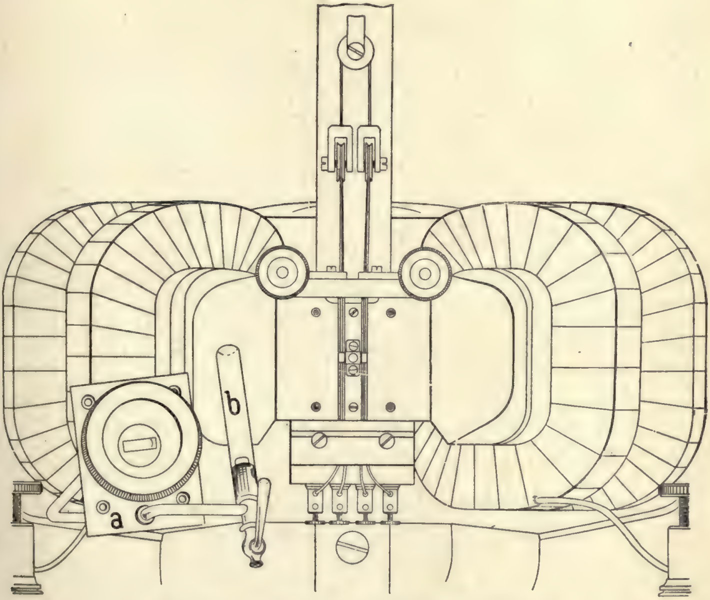


Fig. 286*a*.—*Duddell* and *Marchant's* Oscillograph.

Fig. 286*a* shows a front view of the instrument. The front part, together with the lens, is removed and placed on the left at *a*.

The glass tube b fixed to this part is for inserting the damping oil. The optical system of the apparatus is shewn in Fig. 286b. O is the oscillograph with the two vibratory mirrors s_1 and s_2, whilst s_3 is the fixed mirror and l is the lens. The beam of light is supplied by the direct-current arc lamp lantern L.

The light passes through a system of lenses and a vertical slit d (about 1·5 mm wide). The slit d is about 270 cm away from the lens l.

The photographic plate is dropped through an arrangement at S. During its motion (vertical) the plate passes a horizontal slit some 6 mm wide, through which the light from the mirrors falls on to the plate. The vertical distance of the case, which holds the plate, above the slit must be chosen so that the mean velocity of the plate in moving over the slit is 640 cm per sec. For bringing the plate to rest after

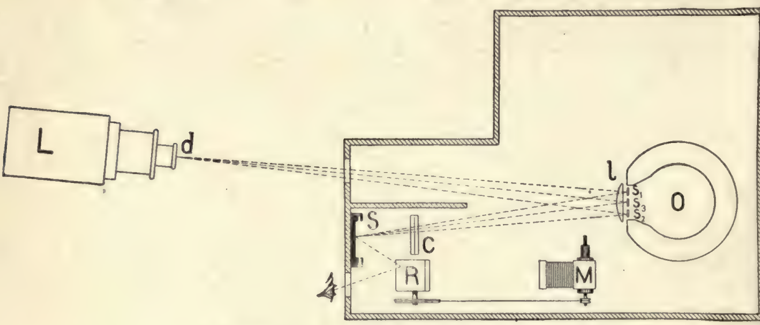


Fig. 286b.—Arrangement of Oscillograph O.

passing the slit there is a braking arrangement which acts by pressing a spring against the back of the plate. The plates are brought to and taken from the apparatus by means of light-tight bags and wooden cases.

In front of the slit there is a cylindrical lens C whose axis is horizontal. This serves to concentrate the light coming from the vertical slit d and to produce a sharp point of light at S. R is the rotating mirror driven by the small direct-current motor M. A strip of film can be used instead of the plate to obtain a continuous photograph of the curve.

In order to observe the curve continuously, a white plate is fixed at S exactly behind the falling plate. The rays reflected from the small mirrors s_1, s_2 and s_3 then fall on the white screen, and the wave-shape can be observed in the rotating mirror at the same time as the exposure is taken.

The Cambridge Scientific Instrument Co. constructs such an oscillograph in which the time of natural vibration of the strip is less than $\frac{1}{10000}$ second.

The maximum permissible current for these oscillographs is 0·1 amp. Usually, however, the desired amplitude of the wave can be obtained with a considerably smaller current.

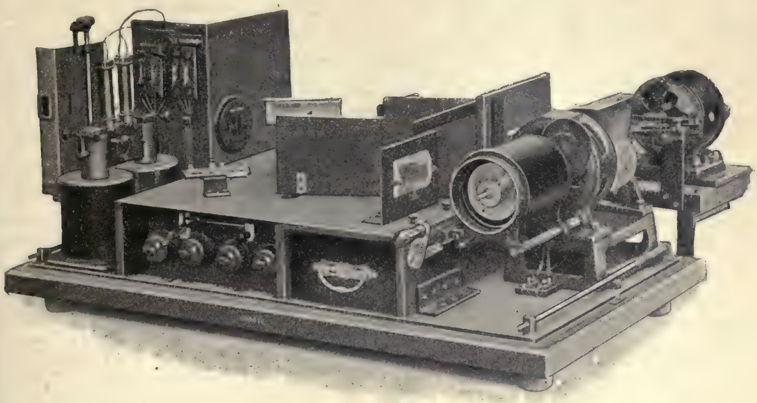


Fig. 287.

A further great advantage of the oscillograph is that the shape of the curve can be inspected before it is photographed.

Fig. 287 shews an oscillograph on the same principle, made by Siemens & Halske.

103. Braun's Tube. The cathode rays, emitted in an exhausted tube from the surface of the cathode where the current lines from the anode strike, are diverted by a magnetic field into a plane perpendicular to the direction of the lines of force. In a rotary field of constant strength, therefore, the cathode rays will describe a conical surface. Since a cathode ray causes chalk, Balmain's luminous paint and many other bodies which it meets to glow brilliantly, the magnetic field can be represented by means of a luminous curve, which can be photographed. If the vector representing the rotary field fluctuates periodically, the luminous curve will be the polar diagram of this vector. This method is very sensitive, and can be so arranged that even fields of $\frac{1}{10}$ C.G.S. unit can be detected.

If the field is merely alternating, the ray will only be diverted in one plane, and will swing with the frequency of the current. The luminous line thus formed will represent a curve on a uniformly revolving mirror. The curve can, however, be also seen directly on the screen, when the cathode ray is given a uniform velocity perpendicular to the plane in which it swings, by means of a variable auxiliary current. This auxiliary current can be obtained, for example, by means of a contact C (Fig. 288) moved uniformly along the wire AB.

The current traversing the coil S will then be nearly proportional to the time. This is most easily obtained by placing the wire AB on the periphery of a disc revolving synchronously with the alternating-current, whilst C remains stationary. In this way corresponding points of the current curve always fall on the same part of the luminous screen, so that the curve on the latter appears stationary and can be photographed.

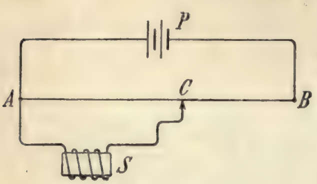


FIG. 288.

104. Measurement of the Frequency of an Alternating Current. (*a*) To measure the frequency of an alternating-current the effects of *resonance* may be used, because these phenomena always depend on the frequency, no matter whether we are dealing with the resonance between a current and a tuning-fork or with electric resonance.

Fig. 289 shews a steel fork vibrating under the influence of an alternating-current magnet. In such an apparatus resonance occurs between the alternating magnetic field and the fork, when the natural time of vibration of the latter is an exact multiple of the frequency of the current. If either is altered, the vibrations disappear, together with the note given out by the fork.

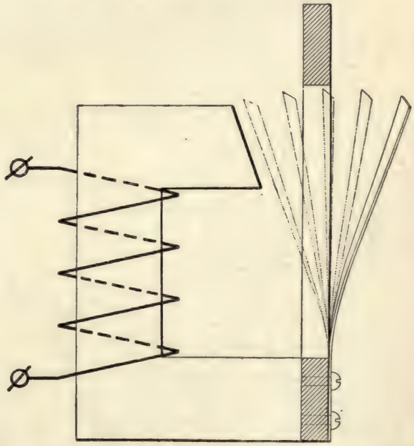


FIG. 289.—Diagrammatic Representation of Flat Spring vibrated by an Alternating-current Magnet.

In the *E.T.Z.*, 1899, p. 873, an instrument of this nature for finding the frequency is described by *E. Stöckhart.* The chief part of the instrument consists of a soft-iron tuning-fork carrying weights which can be moved along its limbs to vary the time of vibration. Between the ends of the fork there is a soft-iron core wound with a coil through which the alternating-current is sent. Each of the movable weights carries a pointer which moves along a fixed scale, from which the frequency of the current is read off directly. To take the measurement the weights are displaced until the note given out becomes loudest.

In the *E.T.Z.*, 1901, p. 9, *Kempf-Hartmann* described a different method for directly measuring the frequency. The instrument has 32 steel tongues (similar to that in Fig. 289) having different natural periods of vibration, all of them being fixed in a ring with their free

ends pointing upwards. By turning a screw the tongues can be passed across the poles of an electromagnet. As soon as the tongue corresponding to the frequency of the current enters the field, it commences to give out its note. The frequency is then read off directly on the scale. The loudness is immaterial; the vibrations of the tongue can even be observed through a glass plate—and the adjustment is made so as to obtain the maximum amplitude of vibration.

With these acoustic instruments it is possible to determine the frequency to within about one-fifth of a whole period.

Figs. 290*a* and *b* shew *Frahm's* frequency measurer.* A series of springs *f*, made from spring steel as used for clocks, are adjusted for different periods of vibration and fastened to a common bar *s*. This bar is connected to the plate *p* by means of two steel springs *bb* (called bridges), so that it can move somewhat about its longitudinal axis.

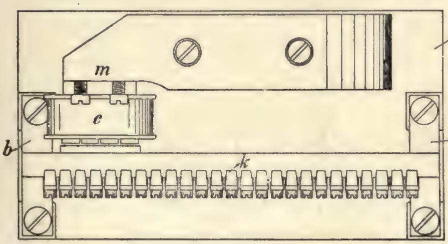


Fig. 290*a*.

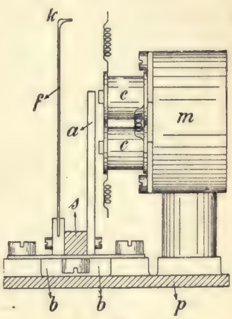


Fig. 290*b*.

Frahm's Frequency Measurer.

On this bar also a flat piece of iron *a* is fixed, which forms the armature for the magnet *m*. The magnetism of this latter is alternately strengthened and weakened by the current whose frequency is being measured—the current being sent through the coils *cc*. The bar, together with the springs attached to it, are thus set vibrating synchronously with the alternating-current, and the particular spring whose natural period of vibration harmonises with this motion is set swinging to a sufficient extent to enable the motion at the head *k* to be distinctly observed.

(*b*) A black disc, having a white line drawn on it radially, is used for the *stroboscopic* method of measuring the frequency. The disc is mounted on the shaft of a motor and is lit up by an arc lamp working on the alternating-current being investigated.

The light of the arc lamp varies periodically with the frequency of the current, and when the speed of the stroboscopic disc is equal to this frequency, the white line will always be illuminated in the same place and appear to be at rest. If the speed of the disc is less than the

* See *E.T.Z.*, 1905, p. 264.

frequency, the line will appear to rotate in the opposite direction to the disc, and if greater, in the same direction.

This method of measuring the frequency is similar to that for determining the slip of an induction motor, which is treated fully in *Wechselstromtechnik*, Bd. V., Part I., Sect. 74.

105. Instrument Transformers. In the measurement of very heavy currents or high pressures, it is often not possible to connect the instruments directly in the respective circuits, for instruments suitable for these extreme values of current and pressure would become both expensive and impracticable, whilst such instruments in connection with high pressures could not be used without danger. In such cases, therefore, instrument transformers are used.

In Fig. 291, T_V shews the connections of a pressure transformer for measuring the pressure across the bars S. T_A is a current transformer for measuring the current flowing in the line L.

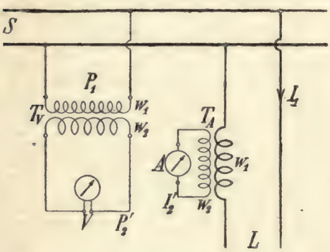


FIG. 291.—Connections of Pressure and Current Transformers.

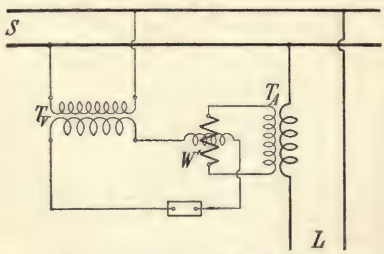


FIG. 292.—Connections of Wattmeter with Instrument Transformers.

As a first approximation, where the various losses in the transformers are neglected, the pressures will be directly proportional to the numbers of turns and the currents inversely proportional, thus

$$P_1 \simeq \frac{w_1}{w_2} P_2' = u_e P_2' ; \qquad I_1 \simeq \frac{w_2}{w_1} I_2' = \frac{1}{u_i} I_2'.$$

Usually the instruments are provided with scales to read the primary values directly.

If the instrument transformers are connected to a wattmeter, as shewn in Fig. 292, and again neglecting the losses, the power supplied to the line L will be

$$W = \frac{u_e}{u_i} W',$$

where W' is the reading of the wattmeter.

On load, the pressure transformer works as on no-load, for the voltmeter current must be very small. The current transformer, on the other hand, is practically on short-circuit, for the terminal pressure of the ammeter is very small.

When the range of a voltmeter is increased by placing resistance in

series, that of an ammeter by placing resistance in parallel, the losses increase as the range is enlarged. On the other hand, instrument transformers allow of extreme values of current and pressure being measured without causing larger losses than exist in the instruments on their normal range, when the losses in the transformers themselves are neglected.

(*a*) *Pressure Transformer.* To investigate instrument transformers, we start from the secondary values of current, pressure and impedance, reduced to the primary, and write

$$I_2' = \frac{w_2}{w_1} I_2; \qquad P_2' = \frac{w_1}{w_2} P_2; \qquad z_2' = \left(\frac{w_1}{w_2}\right)^2 z_2.$$

From eq. 88, p. 157, we have for the pressure transformer,

$$\frac{P_1}{P_2} = C_1 + C_2 \frac{I_2}{P_2} z_{K1} = C_1 + C_2 z_{K1} y_v,$$

where y_v is the admittance of the voltmeter reduced to primary. Further, as shewn before,

$$C_1 = 1 + z_1 y_a,$$

$$C_2 = 1 + z_2 y_a,$$

$$z_{K1} = z_1 + \frac{z_2}{C_2}.$$

z_{K1} in the equivalent circuit (Fig. 293) is the short-circuit impedance measured between the terminals 1–1 when the terminals 2–2 are short-circuited. Let z_{K2} denote the short-circuit impedance between 2–2 when the terminals 1–1 are short-circuited.

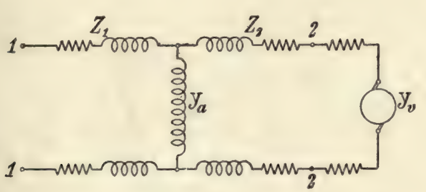


FIG. 293.—Equivalent Circuit of Pressure Transformer.

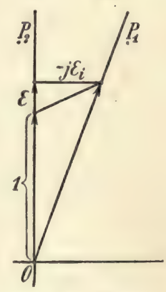


FIG. 294.—Pressure Diagram.

Then

$$C_2 z_{K1} = C_1 z_{K2},$$

$$\frac{P_1}{P_2} = C_1(1 + z_{K2} y_v) = (1 + z_1 y_a)(1 + z_{K2} y_v)$$

$$= 1 + \epsilon - j\epsilon_i, \qquad (159)$$

where ϵ is the percentage pressure rise in phase with P_2, and ϵ_i the percentage pressure rise leading P_2 by 90° (Fig. 294).

Since y_v must be kept very small, it is sufficiently accurate to put

$$\frac{P_1}{P_2} \simeq 1 + z_1 y_a + z_{K2} y_v$$
$$= 1 + z_1 y_a + (z_1 + z_2) y_v,$$

and

$$\left.\begin{aligned}\epsilon &\simeq r_1(g_a + g_v) + x_1(b_a + b_v) + r_2 g_v + x_2 b_v,\\ \epsilon_i &\simeq r_1(b_a + b_v) - r_2 b_v + x_1(g_a + g_v) + x_2 g_v.\end{aligned}\right\} \quad \text{.............}(159a)$$

Since the imaginary part of this expression is very small compared with the real, the ratio between the effective values of the pressures can be written

$$\frac{P_1}{P_2} \simeq 1 + r_1(g_a + g_v) + x_1(b_a + b_v) + r_2 g_v + x_2 b_v, \quad \text{...........}(160)$$

or, if the current taken by the voltmeter is very small, *i.e.* y_v is very small, then

$$\frac{P_1}{P_2} \simeq 1 + r_1 g_a + x_1 b_a. \quad \text{.......................}(160a)$$

The pressure transformer should be constructed, therefore, so that $1 + z_1 y_a$ is as near unity as possible, that is, $z_1 y_a$ is as small as possible, for in this case the pressures are as nearly as possible in proportion to the numbers of turns. Further, this is also advantageous when the transformer is graduated, for then the changes of g_a and $x_1 b_a$ are least affected by variations in the saturation and frequency. On the contrary, the secondary resistance r_2 and reactance x_2 have no effect when the voltmeter current is small.

The conductance g_a is due to the hysteresis and eddy losses in the iron. Whilst the latter part is independent of the pressure, the part due to hysteresis varies inversely as the 0·4th power of the pressure. Owing to this decrease in the hysteresis conductance with increasing pressure, the deviation in the secondary pressure is greater at low pressures than at high. To make this error as small as possible, the primary resistance r_1 must be as small as possible.

The susceptance b_a varies inversely as the permeability with varying pressure. It is therefore large at low pressures, attains a minimum at an induction $B = 7000$ to 9000, and then rises again. With low pressures when the induction is below 7000 to 9000, b_a changes in the same way as g_a, and with increasing pressure causes an increase in the secondary pressure compared with the primary. At higher pressures, the increase of b_a acts against the decrease of g_a, and the ratio of the pressures will be more constant.

With changing frequency c, the hysteresis conductance varies inversely as $c^{0·6}$. Hence, qualitatively, the same changes occur as with varying pressures.

(*b*) *Current Transformer.* From eq. 89, we have for the current transformer

$$\frac{I_1}{I_2} = C_2 + C_1 \frac{P_2}{I_2} y_{01} = C_2 + C_1 y_{01} z_A.$$

Here z_A denotes the impedance of the ammeter, reduced to the primary. Also

$$y_{01}=\frac{y_a}{C_1} \quad \text{and} \quad y_{02}=\frac{y_a}{C_2},$$

where y_{02} is the no-load admittance between the secondary terminals; hence

$$\frac{I_1}{I_2}=C_2(1+y_{02}z_A)=C_2+y_a z_A=1+(z_2+z_A)y_a$$
$$=1+\iota+j\,.\,\iota_i. \qquad (161)$$

Here $$\iota=(r_2+r_A)g_a+(x_2+x_A)b_a \qquad (162a)$$

is the percentage increase of current in phase with I_2, and

$$\iota_i=(r_2+r_A)b_a-(x_2+x_A)g_a \qquad (162b)$$

is the percentage current increase lagging 90° behind I_2 (see Fig. 295).

Since the imaginary part is here very small compared with the real, we can write

$$\frac{I_1}{I_2}\simeq 1+(r_2+r_A)g_a+(x_2+x_A)b_a. \qquad (163)$$

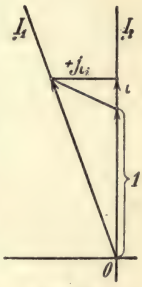


FIG. 295.

From this it is at once seen that the primary impedance of the current transformer has no effect on the measurements. On the other hand, care must be taken to keep the secondary resistance and reactance as small as possible—just the reverse of a pressure transformer. It is therefore immaterial where the primary coil is arranged; often the bus bar is merely led through an iron ring, thus making one turn in the primary winding. To make the effect of changes in g_a and b_a as small as possible, the impedance of the ammeter z_A, reduced to primary, must be kept as low as possible. Thus the apparent volt-ampere consumption of the ammeter should be kept very small, so that the current transformer is practically on short-circuit.

To make g_a and b_a as small as possible, the induction must not be made too low. Since the induced E.M.F. is very small, only a small iron section is required.

Since the E.M.F. increases as the current rises in the same way as when the pressure increases in a pressure transformer, the secondary current increases in proportion to the primary current, owing to the decrease of g_a and b_a. Fig. 296a shews the increase of this ratio very clearly for a current transformer made by Siemens & Halske. The abscissa axis represents the current in per cent. of the range of the instrument, whilst the ordinates shew the percentage deviations of the current ratio from its mean value. The curves A, B and C are for different impedances z_A. As eq. 161 shews, the secondary current decreases for larger z_A. At the same time the effect of changes in g_a and b_a is increased, so that the lower curves B and C rise more rapidly than the upper curve A.

As mentioned in connection with pressure transformers, a decrease in the frequency c must act in the same direction as an increase in z_A.

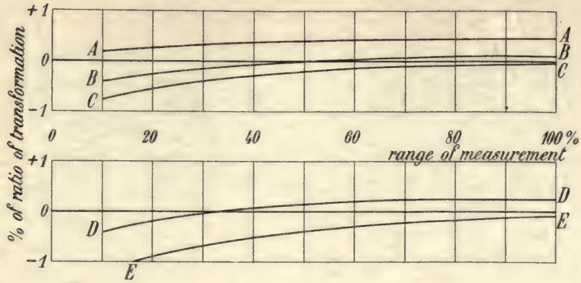


FIG. 296*a* and *b*.

This is clearly seen in the curves D and E (Fig. 296*b*), which are taken for frequencies of 50 and 25.

(*c*) *Wattmeter Transformers.* For the measurement of power, current and pressure transformers are used. As before, let y_v denote the secondary admittance of the pressure transformer and z_A the secondary impedance of the current transformer. Further, we will let the suffix V denote the constants of the pressure transformer, and the suffix A those of the current transformer. The primary and secondary powers are then represented by the vectors,

$$\dot{W}_1 = \dot{P}_1 \dot{I}_1'; \qquad \dot{W}_2 = \dot{P}_2 \dot{I}_2',$$

where $\dot{I}_1'$ and $\dot{I}_2'$ denote the conjugate vectors of $\dot{I}_1$ and $\dot{I}_2$. We have then

$$\frac{\dot{W}_1}{\dot{W}_2} = \frac{\dot{P}_1}{\dot{P}_2}\frac{\dot{I}_1'}{\dot{I}_2'} = (\dot{C}_1 + \dot{C}_2 \dot{z}_{K1} \dot{y}_V)_V (\dot{C}_2 + \dot{C}_1 \dot{y}_{01} \dot{z}_A)_A'$$
$$= \dot{C}_1 \dot{C}_2' (1 + \dot{z}_{K2} \dot{y}_V)_V (1 + \dot{y}_{02}' \dot{z}_A')_A.$$

The symbols marked $'$ denote the conjugate vectors. Introducing equations 158 and 161, we get further

$$\left.\begin{aligned} \frac{\dot{W}_1}{\dot{W}_2} &= (1 + \epsilon - j\epsilon_i)(1 + \iota - j.\,\iota_i) \\ &\simeq 1 + \epsilon + \iota - j(\epsilon_i + \iota_i). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (164)$$

If an ammeter is placed in series with the current coil and a voltmeter in parallel (or in series) with the pressure coils of the wattmeter, we can at the same time measure the real part W_2 of the secondary power and also the secondary current and the secondary pressure P_2.

We then get $\qquad \dot{W}_2 = W_2 + jW_{2i},$

where $\qquad W_{2i} = \sqrt{(I_2P_2)^2 - W_2^2}.$

Similarly, if we write $\quad \dot{W}_1 = W_1 + jW_{1i},$

we have

$$\left.\begin{aligned} W_1 &= (1 + \epsilon + \iota) W_2 + (\epsilon_i + \iota_i) W_{2i}, \\ W_{1i} &= (1 + \epsilon + \iota) W_{2i} - (\epsilon_i + \iota_i) W_2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (165)$$

If the secondary phase displacement is small (*i.e.* W_{2i} small), the primary power W_1 to be measured is found by increasing the reading W_2 by the percentage pressure drop and percentage decrease in current. The measurement of the primary imaginary power W_{1i} or primary phase displacement is then inaccurate, because the term $-(\epsilon_i + \iota_i) W_2$ may be large.

With very large phase displacements, the imaginary primary power W_{1i} is obtained by increasing the imaginary power W_{2i}, measured in the secondary, by the percentage pressure drop and current decrease. The measurement of the real primary power W_1 is then inaccurate, since the term $(\epsilon_i + \iota_i) W_{2i}$ can be comparatively large.

106. Electricity Meters. The energy consumed in a circuit is

$$A = \int pi\,dt = \int PI \cos\phi\,dt.$$

If the pressure remains constant,

$$A = P \int I \cos\phi\,dt.$$

If I and ϕ are constant,

$$A = I \cos\phi \int P\,dt.$$

Finally, if the momentary power is constant, then

$$A = PI \cos\phi \int dt.$$

Corresponding to the above equations we can distinguish between watt-hour meters, ampere-hour meters, volt-hour meters and electricity meters. Since it is difficult to construct instruments to respond only to the watt component of the current, ampere-hour meters are not largely used with alternating-currents. We shall therefore deal chiefly with *watt-hour meters.* These work partly on the dynamometer principle and partly on the laws of induction. We can distinguish between *motor meters* where the current to be measured itself causes a movement, the speed of which is directly proportional to the current, and *pendulum meters* where the alternating action of two coils carrying current is made to influence an already existing motion. The latter possess the disadvantages of being complicated, on account of the many axes and moving parts, and that of being continually in motion and therefore always subjected to wear. Moreover, the permanent control possible with the motor meter is an advantage which must not be under-estimated. Thus, whilst the motor meter is more reliable in working than the pendulum meter, yet the induction meter, in which there are no current leads and rubbing contacts, has a still more certain action. As an example of the *pendulum meter* we shall consider the *Aron watt-hour meter.*

This instrument is provided with two pendulums, each possessing a pressure coil. Under each pendulum a coil carrying the line current is placed, and connections are made so that the one pendulum is

accelerated, the other retarded. If the pendulums swing synchronously when no current is flowing, and operate a counting device which only records the difference of their swings, then the readings will be approximately proportional to the current flowing.

The time of oscillation t of a pendulum of length l is

$$t=\pi\sqrt{\frac{l}{g}},$$

where g is the acceleration due to gravity. When current flows through the coils, we can write

$$t_1=\pi\sqrt{\frac{l}{g+g_1}} \quad \text{and} \quad t_2=\pi\sqrt{\frac{l}{g-g_1}}.$$

If the pointer on the indicator moves one division when one pendulum has completed N swings more than the other, then one division will correspond to $N\frac{t_1t_2}{t_2-t_1}$ seconds, and the consumption per division, or the so-called constant of the instrument, is

$$K=\frac{t_1t_2}{t_2-t_1}N\frac{PI}{3600\times 1000} \text{ kilowatt-hours}$$

or

$$=\frac{NPIt}{1000\times 3600}\frac{g}{g_1},$$

where the higher powers of $\frac{g_1}{g}$ are neglected.

That these instruments read correctly for alternating-currents is seen directly, when we remember that the dynamometer action depends only on the watt component of the current. Against the disadvantages of the several axes and moving parts, these instruments have many advantages, since they are independent of the frequency and wave-shape, and further are very sensitive and possess no permanent magnets whose magnetism can vary with age.

Motor meters have been constructed in many forms and placed on the market. They consist, in principle, of one or more fixed current coils, an armature to which a current proportional to the pressure is supplied and a damping device, usually consisting of a disc of aluminium or copper which revolves between the poles of a permanent magnet. If the instrument is to read correctly for alternating-currents, no iron must be present. Since a large resistance is placed in series with the armature, the induced E.M.F. is small compared with the network pressure, and the current in the armature is practically proportional to the pressure. The torque will therefore be proportional to PI_w and the power to PI_wn, where n is the speed of rotation.

In the damping device E.M.F.'s are induced directly proportional to the speed, so that the power consumed in the disc is proportional to the square of the speed.

Since now—neglecting losses—the power taken must equal that supplied, then $PI_W n = Cn^2$ or the power taken from the line is proportional to the speed of the motor. Hence a counting device coupled to the axis of the motor can be made to read the power directly.

The principle of the motor meter is only free from objection when the pressure coil is entirely non-inductive. A phase displacement ψ between pressure and current in the pressure coil changes the formula $W = PI \cos \phi$ into

$$W = PI \cos \phi \cos \psi \frac{\cos(\phi - \psi)}{\cos \phi},$$

where $$\psi = \tan^{-1} \frac{\omega L}{r};$$

$$\left.\begin{array}{l} L = \text{coefficient of self-induction} \\ r = \text{resistance} \end{array}\right\} \text{of the pressure coil.}$$

Since, however, the arrangements necessary to eliminate this error make the instrument too costly, they are only provided in standard meters. In general, when the phase angle ϕ is not too large and the resistance in series with the pressure coil is sufficiently high, the accuracy is not materially affected, and a correction becomes unnecessary for practical purposes.

The error due to friction loss can be eliminated by placing sufficient turns on the current coil and sending through them the current in the pressure coil until their mutual action can just compensate for this loss. The friction losses, however, do not remain constant—after a time they may decrease with wear, and then the meter may come to possess the worst possible fault in the eyes of the consumer, viz. the instrument rotates when no current is being supplied.

Consequently, artificial friction resistance is often added, the magnitude of which is large compared with the original, and remains constant. Moreover, these artificial resistances have the advantage of being adjustable. They can be provided in various ways, but a complete description would take us too far here. One practical device consists in allowing a pin on the revolving axis to strike against one or more springs at every revolution.

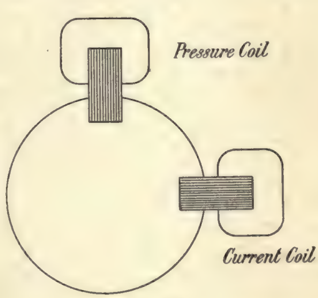


FIG. 297.

Induction meters, from their principle, are only applicable for alternating-currents. Like the induction instruments for measuring current and pressure described above, these also depend on the alternating action of two magnetic fields—displaced from one another in phase—on a closed revolving conductor (Fig. 297). If the line current I flows through one coil and a current i proportional

to the pressure through the other, then the torque exerted on the revolving conductor will be proportional to $Ii \sin \phi$, where ϕ is the phase displacement between the two currents.

Since now the power of an alternating-current is proportional to the cosine of the angle of phase displacement between the pressure and current, it follows that we must displace the current in the pressure coil by 90° in order to use this instrument directly for measuring energy. There are several methods whereby an instrument capable of reading sine functions can be transformed into one for reading cosine functions. The simplest arrangement would consist in making the pressure circuit as inductive as possible, so that the ohmic resistance becomes negligible in comparison with the reactance (Fig. 298). Since this is not sufficiently accurate in practice, other arrangements are used, which, however, we cannot enter into here.

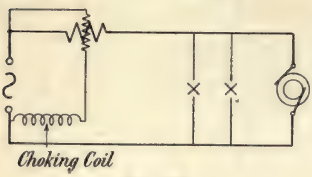


Fig. 298.

107. Calibration of Alternating-Current Instruments. In the description of the instruments it has been pointed out which of them can be calibrated by means of continuous current. With these, either a comparison can be made with standard instruments, or potentiometer methods can be used for determining pressures, currents being then calculated from the terminal pressure and standard resistances. Instruments which cannot be calibrated with continuous currents, however, must be compared with instruments that can be used on either alternating or continuous-current circuits. In calibrating ammeters, the pressure is usually transformed down, so that it is not necessary to dissipate such large amounts of power in resistances. This principle is also useful in calibrating the current coils of wattmeters. To obtain any desired phase displacement between current and pressure *phase changers* are used.

For these, a three-phase induction motor, whose rotor can be fixed in any position, can be used. By means of this arrangement we can test if a wattmeter reads correctly on all power factors.

For calibrating electricity meters, a comparison with calibrated instruments can be made; or better, a constant current at a constant pressure may be passed through the instrument for a definite time and the reading compared with the calculated amount of energy.

Since, however, the wheels in the instrument always have some play, the accuracy of the measurement may be largely affected. It is preferable to observe only the movement of a part of the meter which is directly actuated by the current (period of oscillation of pendulum, number of revolutions in a motor meter, etc.), and from this determine the constants of the instrument by means of the ratio of conversion, which is either known or can be easily found.

CHAPTER XVIII.

MAGNETIC PROPERTIES OF IRON.

108. Magnetisation by Continuous Current. 109. Magnetisation by Alternating Current. 110. Magnetising Current with Sinusoidal Pressure. 111. Eddy-Current Losses in Iron. 112. Effect of Eddy-Currents on the Flux Density and Distribution in Iron. 113. Effect of Frequency and other Influences on the Iron Losses. 114. Flux Distribution in Armature Cores. 115. Iron Losses due to Rotary Magnetisation. 116. Testing and Predetermination of Losses in Iron Stampings. 117. Calculation of the Magnetising Ampere-turns with Continuous and Alternating Current. 118. The Magnetic Field in a Polyphase Motor.

108. Magnetisation by Continuous Current. If an iron ring made of laminations or wire (Fig. 299), which has been completely demagnetised, is magnetised quite gradually by a continuous current by uniformly increasing the magnetising ampere-turns from zero, the magnetic induction in the ring will increase with the magnetising force. *Ewing* and *Lord Rayleigh* found that with a very feeble magnetising force H, the induction continues to rise for some time—even for some minutes—after H has reached its maximum value. A large number of experiments has demonstrated that this so-called "magnetic creeping" is smaller, the thinner the laminations or wire, the harder the iron or the greater H is. Recently *Klemencic* has proved that this phenomenon, which he calls "magnetic after-effect," is also present with very fine wire (0·3 mm). It appears, however, that this magnetic after-effect only commences after a certain time (some hundredths of a second), so that with rapid changes of H it has not to be considered.

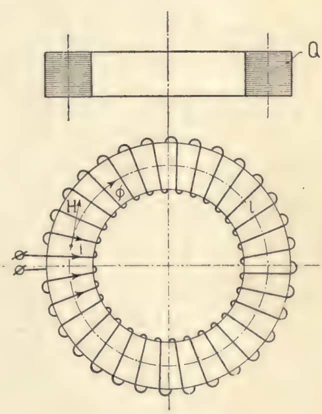


Fig. 299.—Toroid.

Ewing explains this phenomenon as being due to the retentive powers of the magnetic molecules, when they are arranged so as to form groups. The splitting up of these groups takes time; it begins with the molecules on the surface of the wire, which are less closely held together and therefore more movable, and moves gradually inwards. With fine wires there are relatively more movable surface molecules; consequently, in this case the combinations of molecules are disturbed much more quickly.

By plotting the magnetic induction B as a function of the magnetic force H, we get the static magnetisation curve of the material—which is found most accurately by means of a ballistic galvanometer.

Since
$$\int H\,dl = 0{\cdot}4\pi i w,$$

the ampere-turns per cm-length will be

$$aw = \frac{iw}{l} = 0{\cdot}8H.$$

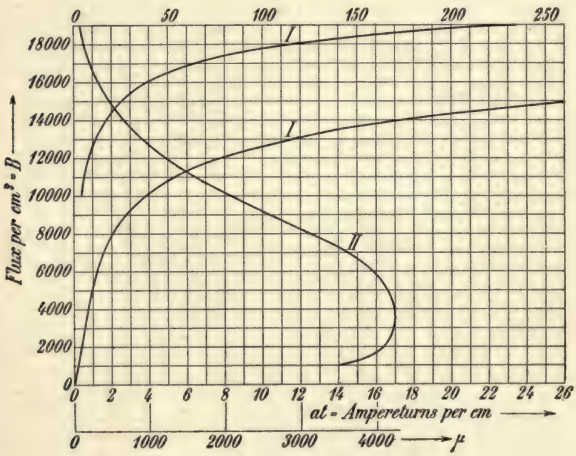


Fig. 300.—Magnetisation Curve for Iron Stampings.

For practical purposes, it is more convenient to plot B as a function of aw instead of H. Such a magnetisation curve for iron stampings is shewn in Fig. 300 by curve I; curve II shews the permeability

$$\mu = \frac{B}{H} = \frac{B}{1{\cdot}25aw}$$

as a function of B.

If the magnetisation of the iron is taken through a cycle by uniformly varying the magnetising force between the two values $-H_{max}$ and $+H_{max}$, B can again be determined ballistically and plotted as a function of H or aw.

Instead of an iron ring (or toroid), the Hopkinson's yoke (Fig. 301) can be used. The test-bar S is here clamped at both ends to a soft iron yoke J having low magnetic reluctance, thus forming a closed magnetic circuit.

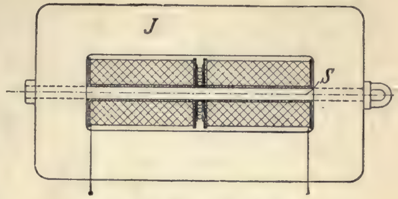


Fig. 301.—Hopkinson's Yoke.

Since the induction does not depend alone on the effective magnetising force H at the moment considered, but is also dependent on the magnetic induction at the previous moment—the latter property being due to the retentivity of the iron—the cyclic magnetisation curve for iron is a closed curve, the so-called *hysteresis loop* H_y (Fig. 302). The curve in this case is obtained by static magnetisation.

Since the induction in iron is—as shewn—a many-valued function of the magnetising force, a magnetisation curve—such as is represented

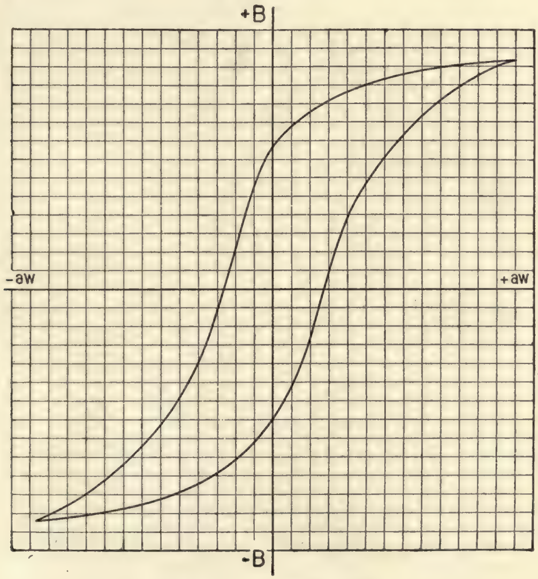


Fig. 302.—Hysteresis Loop.

in Fig. 300—can only give one value of induction for one magnetising force, which depends on the means by which it is measured.

Usually such curves are taken on the ballistic galvanometer by measuring the throw on the galvanometer when the current is reversed.

After a few reversals, this throw remains constant whilst the induction changes from a positive value to the same negative value. The measurements are taken for various field strengths by starting with the lowest magnetising current and increasing the latter step by step, and determining the throw on the galvanometer for each step after a certain number of reversals. Previous to taking the measurements, the iron should be demagnetised as completely as possible. It is important to start with the small inductions and gradually increase to the higher, for a higher magnetisation wipes out the after-effect of a smaller magnetisation more easily than vice versa.

The magnetisation curve taken in this way (the so-called rising magnetisation curve) represents—as is seen—*the locus for the peaks of the static hysteresis loops of the iron.*

The area of the hysteresis loop represents a loss of energy, for, according to the definition of the potential energy for electric current (see p. 15), the work done in a unit of time is

$$\frac{iw}{10}\,d\Phi \text{ ergs,}$$

where iw denotes the ampere-turns interlinked with the flux Φ. If the ring (Fig. 299) has a constant section Q and a mean length l, then

$$\frac{iw}{10}\,d\Phi = \frac{aw}{10}\,lQ\,dB = \frac{aw}{10}\,dB\,.\,V,$$

where $V = Q\,.\,l =$ volume of the iron ring in cm³.

The work done during one period is accordingly

$$V\int_{H_y}\frac{aw}{10}\,dB = V\,.\,W_h,$$

and the *hysteresis loss in ergs per second for one cm³*

$$W_h = \int_{H_y}\frac{aw}{10}\,dB = \frac{1}{4\pi}\int_{H_y}H\,dB, \quad\ldots\ldots\ldots\ldots\ldots\ldots(166)$$

and *is thus proportional to the area of the hysteresis loop H_y.*

Formula 166 is deduced on the assumption that the magnetisation of the iron sample is uniform, and that the magnetic force is due solely to the electric current. It is easy to shew that this formula holds quite generally,—for instance, in the case when various inductions are present in the several parts of the iron and magnetising forces other than those due to electric currents act on the iron. In this case, however, the loss in each part of the iron must be determined by itself. Further, it must be noted that the energy loss due to hysteresis may not only be supplied by electric currents, but also by external mechanical forces, as in generators.

If a test piece is magnetised cyclically between equal positive and negative values of the maximum induction, it is found that the shape

and area of the hysteresis loop varies with the maximum induction. The nature of this change is shewn in Fig. 303, which represents a hysteresis curve (due to Ewing) for annealed piano wire. Here the induction is always varied from one value to a somewhat greater value of the opposite sign.

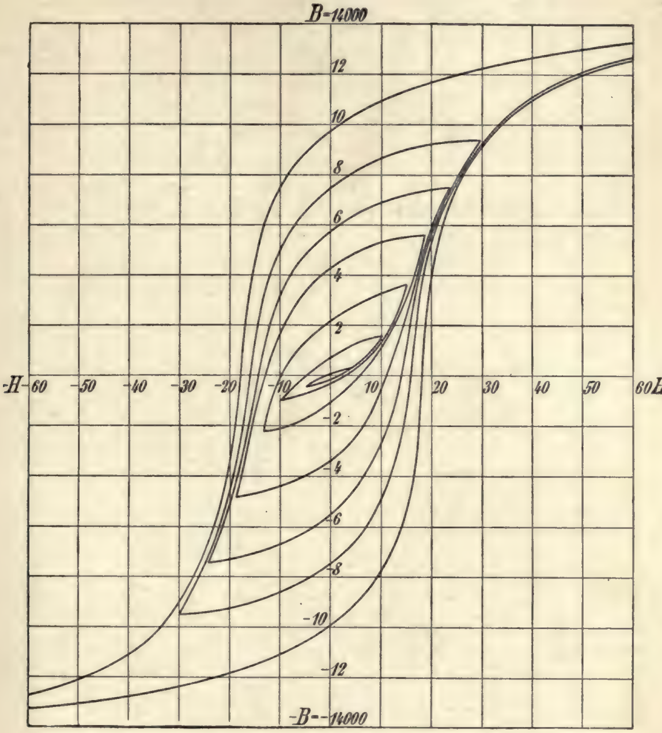


FIG. 303.—Hysteresis Loops, Piano Wire (Ewing).

If we plot the areas of the hysteresis loops divided by 4π, taken at different maximum inductions, as a function of these latter, we get a curve which represents the work in ergs per cycle and per cm^3 due to the hysteresis of the iron in terms of the maximum induction. Fig. 304 shews such a curve as given by Ewing for soft iron plates. It is seen that the loss increases more rapidly than the induction. *Steinmetz* has given the following empirical equation for the curve:

$$A_h = \eta B^{1 \cdot 6} \text{ ergs.} \qquad \text{(167)}$$

η is called the hysteresis constant. For soft, annealed dynamo plates η varies from 0·001 to 0·003.

If c is the frequency at which the iron is magnetised, *i.e.* the number

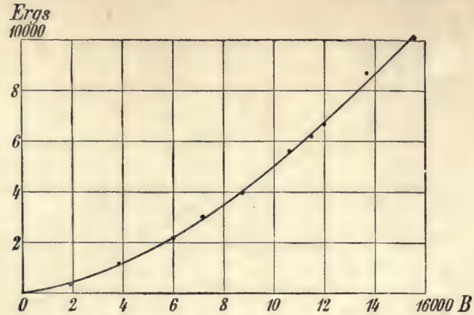


FIG. 304.—Hysteretic Energy per Cycle as Function of the Induction.

of complete cycles the magnetisation passes through per second, the effective loss due to hysteresis will be

$$W_h = \eta c B^{1\cdot 6} \text{ ergs per sec.}$$

$$= \eta c B^{1\cdot 6} 10^{-7} \text{ watts.}$$

The loss per dm³ is $W_h = \eta c B^{1\cdot 6} 10^{-4}$ watts. ....(167a)

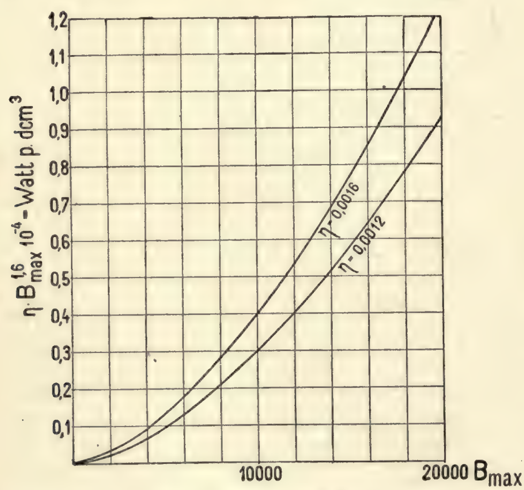


FIG. 305.—Hysteretic Loss for One Cycle per Second for Different Kinds of Iron.

In Fig. 305 curves are given which represent the hysteresis loss per dm³ for one cycle per second in watts, *i.e.*

$$W_h = \eta B^{1\cdot 6} 10^{-4},$$

as function of B. The curves are calculated for $\eta = 0{\cdot}0012$ and $\eta = 0{\cdot}0016$.

When we multiply and divide by $1000^{1\cdot6}=63100$ in formula (167*a*), we get the following expression for the hysteresis loss per dm³, which is more convenient for calculation;

$$W_h=(631\eta)\frac{c}{100}\left(\frac{B}{1000}\right)^{1\cdot6}\text{ watts}$$

$$=\sigma_h\left(\frac{c}{100}\right)\left(\frac{B}{1000}\right)^{1\cdot6}\text{ watts,} \quad\quad (168)$$

where $\sigma_h=631\eta$.

The above expression has been developed on the assumption that the hysteresis loss per cycle is independent of the rate at which the latter is completed. More recent experiments, however, shew that this is not quite true.

In comparing the magnetic conditions accompanying static magnetisation with that due to alternating-current, the first difference we may mention is the *eddy currents* set up in the iron in the latter case. When the magnitude of the induction is rapidly varied, E.M.F.'s are induced in the iron which give rise to currents whose directions are such as to tend to hinder the pulsations of the flux. This has the effect of reducing the flux for a given magnetising current, or for a given flux a larger alternating-current is required than when the same flux is produced by continuous current. In addition to this, the eddy currents produce a loss in the iron which is proportional to the square of these currents.

109. Magnetisation by Alternating Current. Let the pressure

$$p=\sqrt{2}P\sin\omega t$$

be applied at the terminals of the winding on the iron ring shewn in Fig. 299,—then a current will flow through the winding. This current is called the magnetising current, and excites a magnetic flux Φ in the iron which induces an E.M.F. e in the winding,

where $$e=-\frac{d(w\Phi)}{dt}.$$

If r denotes the ohmic resistance of the winding, then we have

$$p+e=ir.$$

If we choose the relations so that i and r are both small, we can write with close approximation,

$$p=-e=\frac{d(w\Phi)}{dt}=\sqrt{2}P\sin\omega t,$$

whence $$w\Phi=-\sqrt{2}\frac{P}{\omega}\cos\omega t=\sqrt{2}\frac{P}{\omega}\sin\left(\omega t-\frac{\pi}{2}\right).$$

$$\therefore\ \Phi=\sqrt{2}\frac{P}{\omega w}\sin\left(\omega t-\frac{\pi}{2}\right).$$

From this we see that when the applied pressure p varies in a sine wave, the flux Φ also obeys a sine law. The flux, moreover, is seen to lag 90° in phase behind the applied pressure. The maximum value of the flux is

$$\Phi_{\max}=\sqrt{2}\,\frac{P}{\omega w}=\frac{\sqrt{2}}{2\pi}\,\frac{P}{cw}=\frac{P}{4{\cdot}44cw},$$

where the pressure P is measured in absolute units. When the effective terminal pressure P is measured in volts,

$$\Phi_{\max}=\frac{P\,.\,10^8}{4{\cdot}44cw}\text{ maxwells, } \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(169)$$

$$P=4{\cdot}44cw\,\Phi_{\max}\,10^{-8}\text{ volts. }\quad\ldots\ldots\ldots\ldots\ldots(170)$$

The induced E.M.F. E is numerically equal to the terminal pressure P and directly opposed to it in direction; thus E lags 90° behind the flux Φ.

We will now consider the case when the applied pressure is not sinusoidal, but is merely some periodic function of the time—the only assumption we now make is that momentary values taken 180° apart are numerically equal and of opposite sign. The pressure curve will then only possess odd harmonics. In this case the flux curve also will have no even harmonics, that is to say, instantaneous values taken half a period apart are likewise equal and opposite. Since, in general,

$$p=-e=w\frac{d\Phi}{dt}$$

or

$$p\,dt=w\,d\Phi,$$

or, again,

$$\Phi=\frac{1}{w}\int p\,dt,$$

the curve for the flux Φ as a function of the time is the integral curve of the pressure curve with regard to time. If we integrate $p\,dt$ over a semi-period and choose the limits so that the integral becomes a *maximum*, then we denote

$$\frac{2}{T}\int_t^{t+\frac{T}{2}}p\,dt=P_{\text{mean}}$$

as the *mean value* of the periodic pressure—and this passes through a *positive half-wave* in the time from t to $t+\frac{T}{2}$, where T denotes the time of a complete period. Denoting the magnitude of the flux at time t by $\Phi_{\min}$ and at time $t+\frac{T}{2}$ by $\Phi_{\max}$, then

$$\Phi_{\max}-\Phi_{\min}=\frac{T}{2}\,\frac{1}{w}\,P_{\text{mean}}$$

is the largest increase the flux can pass through in a semi-period.

Further, since from the above

$$\Phi_{min} = -\Phi_{max},$$

then Φ_{min} is an absolute minimum and Φ_{max} an absolute maximum of the flux. Then, we get

$$\Phi_{max} = \frac{T}{4}\frac{1}{w}P_{mean},$$

where the pressure is measured in absolute units. Since $T = \frac{1}{c}$ we get

$$P_{mean} = 4cw\Phi_{max}10^{-8} \text{ volts}, \quad \text{.....................(171)}$$

which *is quite independent of the wave-shape.* On page 217 the form factor of an alternating-current curve was defined as the ratio

$$f_\epsilon = \frac{\text{effective value}}{\text{mean value}} = \frac{P}{P_{mean}}.$$

For any wave-shape, therefore, we have the following expression:

$$P = 4f_\epsilon cw\Phi_{max}10^{-8} \text{ volts.} \quad \text{.....................(172)}$$

For a sine-wave $f_\epsilon = 1{\cdot}11$, and by substituting this we get formula (170).

110. Magnetising Current with Sinusoidal E.M.F. We again consider the magnetisation of the iron ring shewn in Fig. 299, and assume a sinusoidal pressure is applied at the terminals of its winding. We shall take the pressure drop in the winding to be so small that it is allowable to assume the induced E.M.F. at any instant is equal and opposite to the impressed voltage. Then, as already shewn, the flux must also follow a sine law. Now, to produce this flux Φ we need a magnetising current which alternates periodically with the induction in the core.

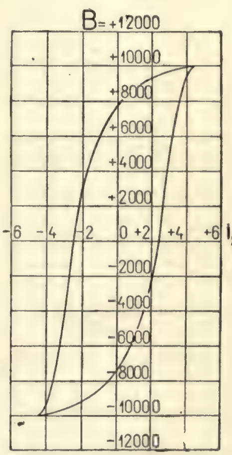


FIG. 306.

At any point of the sinusoidal flux curve or induction curve we can find the respective momentary value of the magnetising current from the *hysteresis loop.* We have shewn above that the area of this loop gives a measure for the energy which is necessary to magnetise the iron through one cycle. This energy, which has to be supplied from outside by the primary no-load current, is converted into heat.

The curve of magnetising current, which we get from the hysteresis loop by calculating for a sinusoidal flux, is not sinusoidal and is unsymmetrical with respect to its maximum ordinate. In Fig. 306 a hysteresis loop is represented, whilst Fig. 307 shews e the curve of induced E.M.F., Φ the corresponding flux curve and i_0 the curve of the magnetising current, which latter is obtained from Fig. 306.

The curve of the magnetising current i_0 can be split up into a first harmonic i_1 and a curve i_d which contains the higher harmonics. Let the effective values of these two curves be I_1 and I_d respectively.

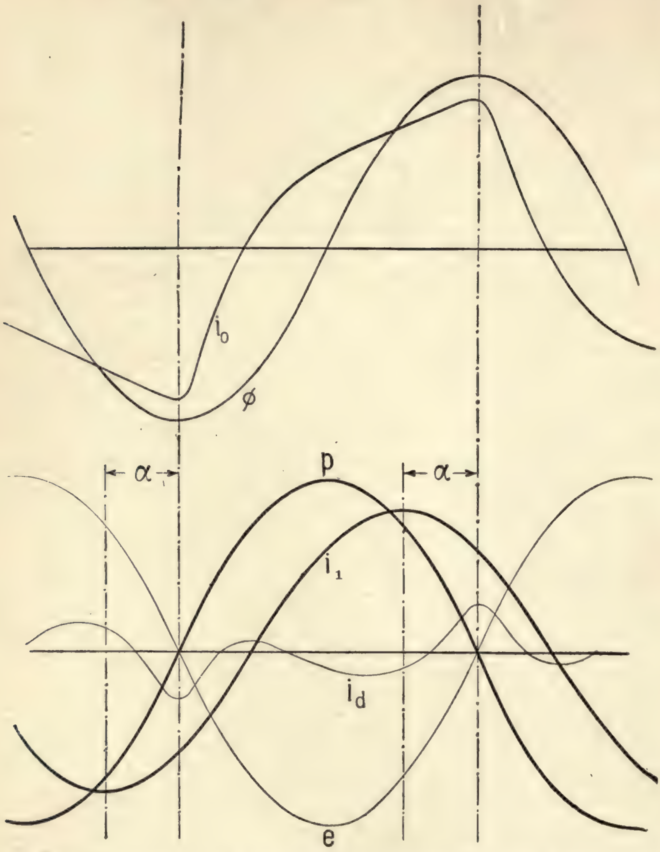


FIG. 307.—Determination of Magnetising Current with Sine-wave Pressure by means of Hysteresis Loop.

We draw the curve of applied pressure $p = -e$ and analyse the sinusoidal curve i_1 into a component i_{1W} in phase with p, and a component i_{1WL} which lags 90° behind the applied pressure. Since the current curve i_d is wattless with respect to the sinusoidal applied pressure, the component i_{1W} will represent the total watt component of the magnetising current, and the hysteresis loss is

$$W = P . I_{1W},$$

where I_{1W} is the effective value of the current i_{1W}.

The wattless component of the magnetising current is made up of the wattless component of the first harmonic $I_{1\,WL}$, and of the effective value of the higher harmonics I_d. These components can therefore be substituted by an equivalent sinusoidal current whose effective value is

$$I_{WL} = \sqrt{I^2_{1\,WL} + I^2_d}.$$

The total magnetising current can now be replaced by an equivalent sinusoidal current whose effective value is I. Written symbolically,

$$\underset{\cdot}{I} = I_W + jI_{WL},$$

where

$$I_W = I_{1\,W}.$$

Thus

$$I = \sqrt{I^2_W + I^2_{WL}} = \sqrt{I^2_{1\,W} + I^2_{1\,WL} + I^2_d}. \quad \text{......(173)}$$

Graphically, the magnetising current can be represented as shewn in Fig. 308. Here $\underset{\cdot}{P}$ the applied pressure is set off along the ordinate axis, while the flux Φ is set off to the left along the abscissa axis. The component

$$\overline{OA} = \underset{\cdot}{I}_W = \underset{\cdot}{I}_{1\,W}$$

is set off in the direction of the pressure and the wattless component

$$\overline{OB} = \underset{\cdot}{I}_{WL} = \sqrt{I^2_{1\,WL} + I^2_d}$$

in the direction of the flux. The sinusoidal current $\underset{\cdot}{I}$, which is equivalent to the magnetising current, is given in magnitude and direction by the vector $\overline{OC}$.

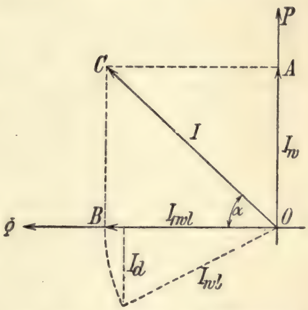


Fig. 308.—Diagram of Magnetising Current.

If we measure the consumed power W, the effective pressure P and the effective magnetising current I, the vector of the equivalent current can be at once determined, for

$$W = PI\cos(90 - \alpha) = PI\sin\alpha = PI_W,$$

$$\sin\alpha = \frac{W}{PI},$$

$$I_W = \frac{W}{P},$$

$$I_{WL} = \sqrt{I^2 - \left(\frac{W}{P}\right)^2}.$$

The angle α, by which the equivalent sinusoidal current of the magnetising current leads the flux, is called the *hysteretic angle of advance.*

The ratio

$$\frac{I}{P} = y$$

is the *admittance* of the magnetising winding. Similarly, the wattless component of the magnetising current is

$$I_{WL} = I \cos \alpha = bP,$$

and the watt component of the same

$$I_W = I \sin \alpha = gP,$$

where b denotes the *effective susceptance* and g the *effective conductance.*

The hysteresis loss is then

$$W = gP^2.$$

If we calculate the effective resistance and reactance corresponding to g, b and y from formulae 37 and 38, p. 55,

$$r = \frac{g}{y^2} = \frac{g}{g^2 + b^2},$$

$$x = \frac{b}{y^2} = \frac{b}{g^2 + b^2};$$

then r represents an effective resistance which is independent of the ohmic resistance of the winding. This effective resistance equals the ohmic resistance which the magnetising winding would have if the hysteresis loss W were consumed in the winding by the magnetising current I,

i.e. $$W = I^2 r$$

or $$r = \frac{W}{I^2},$$

and the effective reactance is

$$x = \sqrt{\left(\frac{P}{I}\right)^2 - r^2}.$$

If we assume—as above—that the ohmic resistance of the winding is negligible, then P, I and W represent the measured values.

In the above we have neglected the effect of the eddy currents. These can easily be taken into account experimentally; for with a sine-wave pressure the flux and along with it the eddy currents vary after a sine-wave. The eddy currents increase both the magnetising currents and the losses, and cause an increase both in the wattless component and in the watt component of the sinusoidal part of the magnetising current. Consequently, nothing is altered in the calculations and considerations as given above, when these eddies are taken into account, and the same diagrams can be used for the experimental values obtained with alternating-currents. The analytical treatment of eddy currents will be found in sections 111 and 112.

In Fig. 309 the curve B represents the induction in dynamo plates of average quality in terms of the momentary values of the ampere-

turns per cm length of the magnetic path. Curve AT gives the *maximum* induction in terms of the *effective* value of the ampere-turns per cm for sinusoidal magnetisation, curve AT_1 shews the effective value of the first harmonics, AT_d the effective value of all the higher harmonics.

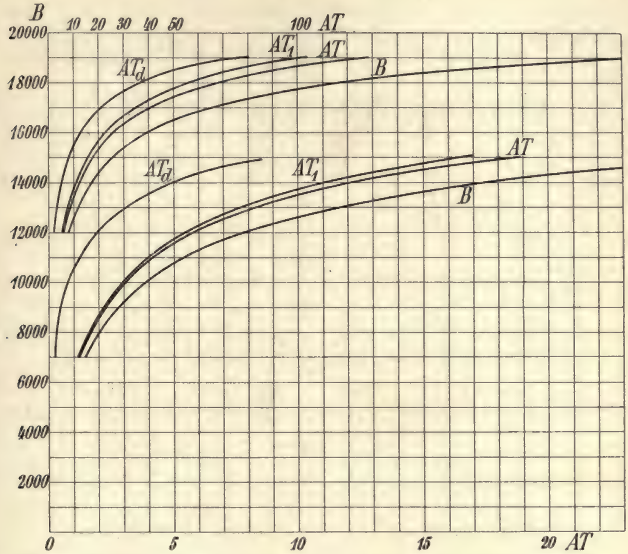


FIG. 309.—Magnetisation Curves for Armature Stampings as Function of AT per cm.

111. The Eddy-Current Losses in Iron. When iron is magnetised by means of alternating currents, eddy currents are always set up in the iron. Suppose a surface to be taken through the iron perpendicular to the direction of the induction, and let a closed curve be drawn in this surface, then along this curve an E.M.F. is induced equal to the rate of change of the enclosed flux. The currents thereby set up flow in a direction such that they oppose any change in the main flux, and dissipate themselves in heat corresponding to the energy they take from the magnetising current. If the reversals in the magnetisation in the iron are caused by the movement of the latter in the field, the loss will be supplied by the mechanical force causing the movement. In some cases, the losses are partly supplied from electrical and partly from mechanical sources of power.

The most effective means of reducing eddy currents consists in laminating the iron. The laminations must run parallel to the lines of induction.

In what follows, the eddy-current losses will be calculated in each case, on the assumption that the induction is uniformly distributed over the whole section.

Let the iron be made up of wires, and the induction, whose maximum value is B, be uniformly distributed over the section of the wire. In a ring of radius x, the induced E.M.F. will be then (see Fig. 310):

$$E_x = 4f_\epsilon cB\pi x^2\, 10^{-8} \text{ volts},$$

where f_ϵ denotes the form factor of the E.M.F. wave.

For a length of wire 1 cm, the resistance of the ring of thickness dx is

$$\rho\frac{2\pi x}{1 \times dx},$$

where ρ denotes the resistance per cm³ of the iron expressed in ohms.

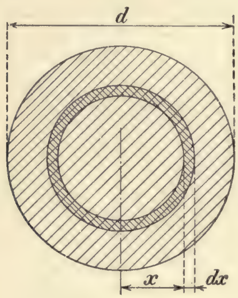


FIG. 310.—Path of Eddy Currents in Round Wire.

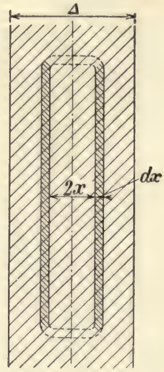


FIG. 311.—Path of Eddy Currents in Iron Stamping.

The heating loss in the ring is then

$$E_x^2\frac{dx}{2\pi x}\frac{1}{\rho} = 8\pi f_\epsilon^2 c^2 \frac{1}{\rho} B^2 x^3 dx\, 10^{-16} \text{ watts}.$$

From this we get the heating loss per cm length,

$$\frac{\pi}{8}\frac{1}{\rho}c^2 f_\epsilon^2 B^2 d^4\, 10^{-16} \text{ watts};$$

thus per cm³, $$w_w = \frac{1}{2}\frac{1}{\rho}c^2 f_\epsilon^2 B^2 d^2\, 10^{-16} \text{ watts}.$$

For the volume V_e measured in dm³ and d in mm, we have

$$W_w = \frac{1}{2}\frac{10^{-5}}{\rho}\left(d\frac{c}{100}\frac{f_\epsilon B}{1000}\right)^2 V_e \text{ watts}$$

$$= \sigma_w\left(d\frac{c}{100}\frac{f_\epsilon B}{1000}\right)^2 V_e \text{ watts. } \qquad \text{.................(174)}$$

For soft iron, $\rho = 5\,.\,10^{-5}$ to 10^{-5} ohms,

whence $\sigma_w = 0{\cdot}1$ to $0{\cdot}5$.

Next, let us assume the iron to be made up of thin plates. Fig. 311 shews a section through a plate perpendicular to the lines of induction.

In a sheet of current 1 cm long, at distance x from the centre line of the plate, an E.M.F. is induced:

$$E_x = 4f_e \,.\, c \,.\, B \,.\, x \,.\, 10^{-8} \text{ volts.}$$

The resistance for 1 cm depth of plates (measured perpendicularly to the plane of the paper) is $\frac{\rho}{dx}$ ohms. The loss in a sheet of current 1 cm long, 1 cm deep and of thickness dx cm is

$$E_x^2 \frac{dx}{\rho} = \frac{16}{\rho} c^2 f_e^2 B^2 x^2 dx \, 10^{-16} \text{ watts.}$$

For the whole thickness of the plate the loss is

$$2\int_0^{\frac{\Delta}{2}} E_x^2 \frac{dx}{\rho} = \frac{4}{3} \frac{c^2}{\rho} f_e^2 B^2 \Delta^3 \,.\, 10^{-16} \text{ watts.}$$

The loss per cm³ is therefore, when Δ is measured in cm,

$$w_w = \frac{4}{3} \frac{c^2}{\rho} f_e^2 B^2 \Delta^2 \,.\, 10^{-16} \text{ watts. } \qquad \text{.....................(175)}$$

For a volume V_e in dm³ and for Δ in mm, we have

$$W_w = \frac{4}{3} \frac{10^{-5}}{\rho} \left(\Delta \frac{c}{100} \cdot \frac{f_e B}{1000} \right)^2 V_e \text{ watts}$$

$$= \sigma_w \left(\Delta \frac{c}{100} \cdot \frac{f_e B}{1000} \right)^2 V_e \text{ watts, } \qquad \text{.....................(176)}$$

where $\sigma_w = \frac{4}{3} \frac{10^{-5}}{\rho}$ is the eddy-current coefficient of the plate.

If we substitute $\rho = 5 \,.\, 10^{-5}$ to 10^{-5} ohms in the above, we get

$$\sigma_w = 0{\cdot}267 \text{ to } 1{\cdot}33.$$

112. Effect of Eddy Currents on the Flux Density and Distribution in Iron. In a piece of iron of circular or rectangular section (Figs. 310 and 311) let Φ_m denote the pulsating flux which the magnetising current i_m would produce alone when no eddy currents were present. This induces an E.M.F. e_w in the shaded circuit, which—for a sinusoidal flux variation—can be represented by a vector lagging 90° behind the flux vector, as in Fig. 312.

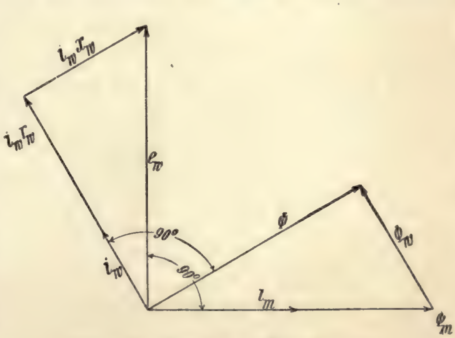


FIG. 312.—Reaction of Eddy Currents.

The E.M.F. e_w produces an eddy current i_w, which in its turn produces a flux represented by Φ_w in Fig. 312. The eddy-current circuit thus possesses inductance corresponding to the flux Φ_w, and i_w lags behind e_w.

The resultant flux of Φ_m and Φ_w is Φ, and we see that the effect of all the eddy currents is first to cause the resultant flux Φ to lag behind the magnetising current i_m in phase, and secondly the flux is reduced from Φ_m to Φ.

Both the weakening and the lag of the induction is greatest in the middle of the iron, and decreases towards the surface, where it is zero. *Oberbeck and J. J. Thomson* have made calculations to determine the weakening of the induction in iron cores due to eddy currents (not as above, due to a single eddy). These calculations shewed that the weakening in very thin wires and plates can be entirely neglected, whilst with thicker plates the weakening rapidly increases with the thickness. This is best seen from a short calculation. In a circuit of radius x (Fig. 310) a maximum E.M.F. $2\pi c\Phi_x 10^{-8}$ volts is induced, and in a circuit of radius $x+dx$ the maximum E.M.F. induced will be

$$2\pi c(\Phi_x + 2\pi x B_x\,dx)\,10^{-8} \text{ volts.}$$

Hence in the outer circuit the E.M.F. per cm length is larger by the amount

$$dE_x = 2\pi c B_x\,dx\,10^{-8} \text{ volts.}$$

In order to get the induction B_x from this, we must find a further relation between E_x and B_x. This is obtained from the fundamental principle of electromagnetism, which states that the induction B_x increases from the radius x to the radius $x+dx$ by the amount corresponding to the M.M.F. of the current in the circular ring. The maximum value of this current is

$$I_x = \frac{E_x}{\rho}\,dx,$$

where ρ is the specific resistance of the iron. The increase in induction corresponding to this current is

$$dB_x = 0{\cdot}4\,I_x\mu = 0{\cdot}4\pi\frac{\mu}{\rho}\,E_x dx,$$

where μ denotes the permeability of the iron. Passing to symbolic values and taking the phases of the different quantities into account, we get the two equations:

$$d\dot{E}_x = j2\pi c\dot{B}_x\,dx\,10^{-8}$$

and

$$d\dot{B}_x = -0{\cdot}4\pi\frac{\mu}{\rho}\,\dot{E}_x\,dx.$$

Substituting $\dot{E}_x$ from the second equation into the first, we get

$$\frac{d^2\dot{B}_x}{dx^2} = -j0{\cdot}8\pi^2 c\frac{\mu}{\rho}\,10^{-8}\dot{B}_x.$$

This is a homogeneous linear differential equation of the second degree, whose solution is

$$B_x = A\epsilon^{\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}x} + B\epsilon^{-\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}x}.$$

A and B are two constants which have the same value in this case, since B_x has the same value, but of opposite sign, at diametrically opposite points at the same distance x. Putting further $B_x = B_{max}$ for $x = r$, *i.e.* at the surface of the cylinder, we get

$$B_x = A\left(\epsilon^{\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}x} + \epsilon^{-\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}x}\right),$$

$$B_{max} = A\left(\epsilon^{\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}r} + \epsilon^{-\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}r}\right),$$

whence by division

$$B_x = B_{max}\frac{\epsilon^{\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}x} + \epsilon^{-\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}x}}{\epsilon^{\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}r} + \epsilon^{-\sqrt{-j0\cdot 8\pi^2 c\frac{\mu}{\rho}10^{-8}}r}}.$$

Since $$\epsilon^{\sqrt{-2j}x} = \epsilon^{(1-j)x} = \epsilon^x(\cos x - j\sin x)$$

and we write for brevity $$\frac{2\pi}{10^4}\sqrt{\frac{c\mu}{10\rho}} = \lambda, \qquad \text{.............................(177)}$$

the induction B_x can be written

$$B_x = B_{max}\frac{\epsilon^{(1-j)\lambda x} + \epsilon^{-(1-j)\lambda x}}{\epsilon^{(1-j)\lambda r} + \epsilon^{-(1-j)\lambda r}}$$

or $$B_x = B_{max}\frac{(\epsilon^{\lambda x} + \epsilon^{-\lambda x})\cos\lambda x - j(\epsilon^{\lambda x} - \epsilon^{-\lambda x})\sin\lambda x}{(\epsilon^{\lambda r} + \epsilon^{-\lambda r})\cos\lambda r - j(\epsilon^{\lambda r} - \epsilon^{-\lambda r})\sin\lambda r}.$$

By comparing this expression with the formula on p. 133 for the distribution of the pressure along a long line, we see at once that the induction from the surface to the interior of the cylinder follows a sine law.

The length of a complete wave is found from

$$\lambda x = 2\pi \quad \text{or} \quad \frac{2\pi}{\lambda} = \frac{10^4}{\sqrt{\frac{c\mu}{10\rho}}}.$$

Over such a wave-length the phase of the induction passes through 360°.

Since $$I_x = \frac{dB_x}{0\cdot 4\pi\mu} = \frac{E_x}{\rho}dx,$$

the eddy currents are propagated in the iron according to the same exponential law as the induction.

For an iron plate (Fig. 311) the same differential equation for B_x is obtained, and consequently the same flux distribution over the section as in a cylinder. In this case x and $r = \frac{\Delta}{2}$ do not represent radii, but the distances of the respective points or surface from the centre of the plate.

Fig. 313 shews the magnitudes of the flux distribution over a plate for different thicknesses of plates at $c = 100$. An idea of the alteration

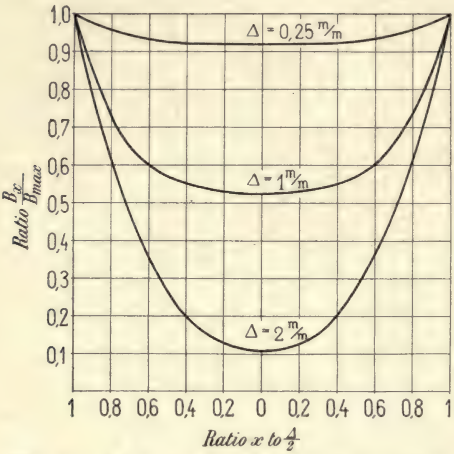


FIG. 313.—Distribution of Induction across a Stamping for 100 Cycles per sec.

in the phase of the induction throughout the plate is obtained by remembering that the wave-length for $c = 100$, $\mu = 2000$ and $\rho = 10^{-5}$ is

$$\frac{10^4}{\sqrt{\frac{c\mu}{10\rho}}} = \frac{10^4}{\sqrt{0{\cdot}1 \times 100 \times 2000 \times 10^5}} = 0{\cdot}224 \text{ cm} = 2{\cdot}24 \text{ mm}.$$

Thus at the centre of a 2 mm plate the induction is displaced $\frac{360^\circ}{2{\cdot}24} = 160^\circ$ in phase from that at the surface. The induction $B_{\max}$ at the surface only corresponds to the effect of the external magnetising forces, which we suppose in this case to act uniformly over the whole length of the cylinder or width of the plate. If we ascertain the greatest mean value B_{mean} of the flux density which can exist at any instant, this must be less than the mean value of the amplitudes of the induction at the different sections, as found from Fig. 313. In Fig. 314, the ratio of the maximum mean value B_{mean} to the maximum induction $B_{\max}$ is plotted as function of the plate thickness for $c = 100$. From the figure it is clearly seen that with a plate 1 mm thick only about 55 % is utilised, and with $\frac{1}{2}$ mm plate about 95 %. The 1 mm plate

therefore would only increase the flux in the ratio of 55 to 47·5 with the same maximum induction. This agrees with J. J. Thomson's statement that a thick plate does not conduct an alternating flux of 100 cycles any better than two thin plates each of $\frac{1}{4}$ mm thickness,

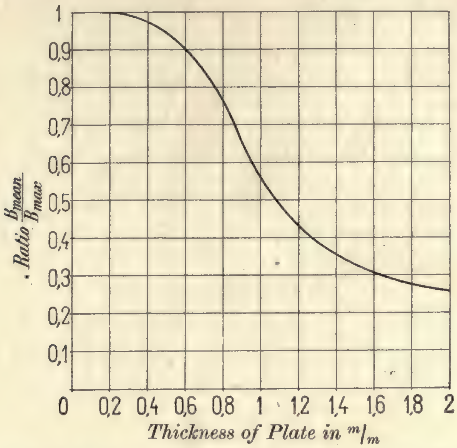


FIG. 314.—Ratio of Maximum to Mean Induction for Different Thicknesses of Stampings at 100 Cycles per sec.

i.e. the total permeance of a thick plate at this frequency only equals that of the two outside layers of $\frac{1}{4}$ mm each. For this layer a simple formula can be obtained, which gives fairly accurate results for plates of high permeability. When λ is very large, $\epsilon^{-\lambda x}$ can be neglected compared with $\epsilon^{\lambda x}$. Then we get

$$B_{\text{mean}} = \frac{2}{\Delta}\int_{x=0}^{x=\frac{\Delta}{2}} B_x\,dx$$

$$= \frac{2}{\Delta}\int_{x=0}^{x=\frac{\Delta}{2}} B_{\text{max}} \frac{\epsilon^{(1-j)\lambda x}}{\epsilon^{(1-j)\lambda\frac{\Delta}{2}}}\,dx$$

$$= \frac{2B_{\text{max}}}{(1-j)\lambda\Delta}\left(1-\epsilon^{-(1-j)\lambda\frac{\Delta}{2}}\right) \simeq \frac{B_{\text{max}}}{(1-j)\lambda\frac{\Delta}{2}}.$$

Coming back to the absolute values:

$$B_{\text{mean}} = \frac{B_{\text{max}}}{\sqrt{2}\lambda\frac{\Delta}{2}}$$

or

$$\frac{\Delta}{2}B_{\text{mean}} = \frac{B_{\text{max}}}{\sqrt{2}\lambda},$$

whence it follows that the thickness of the equivalent plate, instead of $\frac{\Delta}{2}$, is only $\delta = \frac{1}{\lambda\sqrt{2}} = \frac{10^4}{2\pi}\sqrt{\frac{5\rho}{c\mu}}$ cm.

For $c = 100$, $\mu = 2000$ and $\rho = 10^{-5}$ we get $\delta = 0{\cdot}253$ mm, which agrees with Thomson's investigations. It is also clear that the induction rapidly decreases towards the interior since $\epsilon^{-\lambda x} = \epsilon^{-2\pi} = 0{\cdot}0019$, where x is a wave-length, *i.e.* the amplitude of a magnetic wave is reduced to a two-thousandth of its value for every wave-length completed towards the interior of the iron.

For the eddy-current loss it follows that at a given mean value B_{mean}, the induction is increased on account of the unsymmetrical distribution of the flux. In electromagnetic machines, however, such thin plates are used that the induction is almost uniformly distributed over the whole core, whence it is admissible to calculate the eddy losses by means of formulae 174 and 176.

113. Effect of the Frequency and Other Influences on the Iron Losses. If the induced effective E.M.F. E in an electromagnetic apparatus is constant, then

$$cB = \frac{E10^8}{4f_\epsilon wQ_e} = \text{constant.}$$

Now, from equation (179), the *eddy-current loss* is proportional to

$$(cf_\epsilon B)^2 = \left(\frac{E10^8}{4wQ_e}\right)^2. \quad \text{.................................(178)}$$

From this it follows *that the eddy-current loss is proportional to the square of the effective induced E.M.F. independently of the frequency and wave-shape of the latter.*

This only holds, however, up to a certain value of the frequency, when the induction becomes unsymmetrically distributed over the section.

The *hysteresis loss* is, from equation (167), proportional to

$$cB^{1{\cdot}6} = \frac{(cB)^{1{\cdot}6}}{c^{0{\cdot}6}} = \left(\frac{10^8}{4wQ_e}\right)^{1{\cdot}6}\frac{E^{1{\cdot}6}}{f_\epsilon^{1{\cdot}6}c^{0{\cdot}6}}. \quad \text{..................(179)}$$

From this we see: *The hysteresis loss is inversely proportional to the* $0{\cdot}6^{\text{th}}$ *power of the frequency.*

The greater the frequency the smaller the hysteresis loss (for the same pressure), and up to a certain limit this holds for the total iron losses. As the frequency increases a point is reached beyond which, on account of the unsymmetrical distribution, the eddy losses increase faster than the loss due to hysteresis decreases.

Further, it is held that, in addition to eddy currents, there are yet other differences between static and alternating magnetisation. *Max Wien* has attempted to shew experimentally, in *Wiedemanns Annalen*, Bd. 66, that the so-called *magnetic inertia* or *viscosity* at rapid reversals

causes a decrease in the permeability and an increase in the hysteresis loss per cycle at a constant maximum induction. Thus, a similar effect is ascribed to magnetic inertia as to eddy currents. To prove this, Max Wien took care to make the eddy losses quite negligible in every respect, whilst the experiments were undertaken throughout with sinusoidal E.M.F.'s and very different frequencies. From Figs. 315 and 316, based on Max Wien's experiments, it is easily seen that the flux

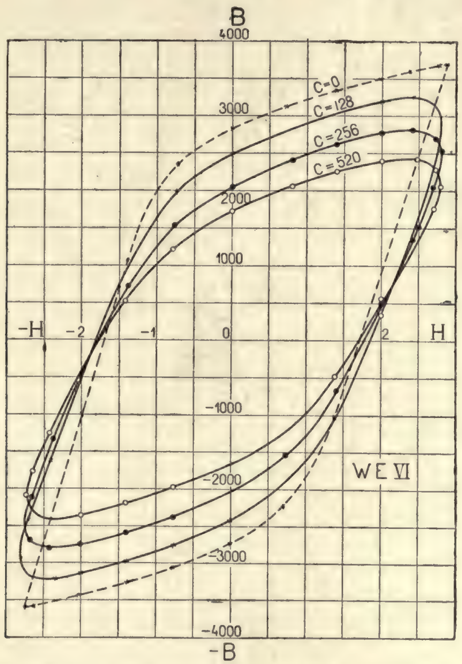


FIG. 315.—Shortening of the Hysteresis Loop due to Increasing Frequency.

at rapid reversals cannot quite follow the magnetising force, consequently the hysteresis loops under these conditions appear different from those taken with slow changes.

At the close of his paper, Max Wien writes as follows on the relation between magnetic after-effect and inertia: "Whilst inertia becomes noticeable with flux variations completed within $\frac{1}{1000}$th of a second, the magnetic after-effect does not begin before a lapse of several tenths of a second (Klemencic-Martens). This after-effect is greatest for weak fields, where the differences of the permeability and hysteresis loss at the various frequencies are scarcely noticeable. These differences attain their greatest value at maximum permeability, at which point the magnetic after-effect vanishes. On the other hand, there are several analogies between the two phenomena—chiefly the

dependence on the diameter of the wire magnetised and the decrease with the hardness of the iron."

Like magnetic inertia, other magnetic phenomena can also be explained by Ewing's molecular theory.

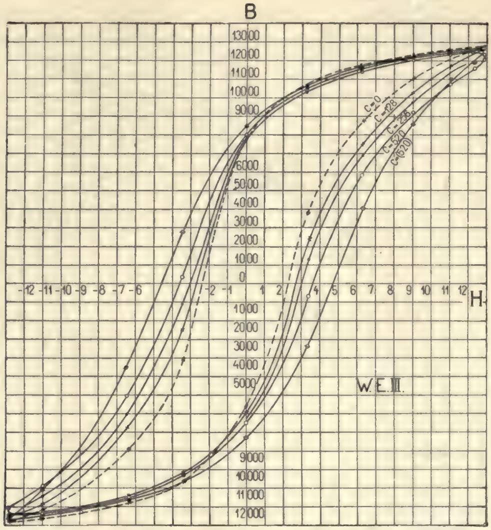


FIG. 316.—Widening of the Hysteresis Loop due to Increasing Frequency.

Vibration decreases hysteresis loss. This is especially so with soft iron and weak fields.

The conviction is now fairly general that the hysteresis loss depends much more on the *physical* nature of the iron than on the *chemical*. *Pressure* increases the hysteresis loss and decreases the permeability, even when the force is removed.

Mordey found that a pressure of 270 kg per cm^2 caused an increase of 20 % in the hysteresis loss; on removing the pressure, the loss sank to its original value.

In one and the same plate, the hysteresis loss varies from point to point, and this variation may amount to 28 %. Near the edge and perpendicular to the direction in which the plate has been rolled, the loss is greatest, and in the inside portion parallel to this direction the loss is least.

Layers of oxidation on the plate, which have a low permeability, lead to an increase in the hysteresis losses. Iron plates are *annealed* to reduce hysteresis loss. The latter, plotted as a function of the annealing temperature, gives a curve shewing that minimum loss occurs at 950° C. When we come above this annealing temperature the loss curve rises rapidly. At higher temperatures the plates may stick together and be destroyed.

Up to about 200° C., the hysteresis loss is almost independent of the temperature, whilst between 200° and 700° C. the loss decreases from 10 to 20 %.

With continuous heating, however, the hysteresis loss increases—this process is known as *ageing*. The higher the annealing temperature, the more pronounced does this property shew itself.

The curves in Fig. 317 were taken by *A. H. Ford* on four different transformers of 1 to 2 K.W. The transformers were fully loaded during the whole of the experiments. Ford maintains that ageing can be reduced by rapidly cooling the red-hot plates.

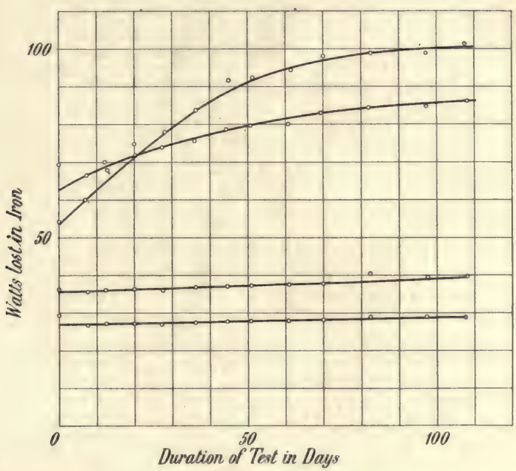


FIG. 317.—Ageing of Iron.

Mauermann * investigated a number of plates with respect to ageing, some of which were annealed at 700-750° C. and the remainder at 950-1000° C. Those plates which were annealed at 950-1000° C. shewed a noticeable increase in hysteresis loss after one week's heating at 56° C., whilst the plates annealed at the lower temperature shewed little change. After being heated at 77° C. for a fortnight, the latter plates still shewed little change, whilst the increase for the plates annealed at the higher temperature remained about the same.

Consequently, on account of ageing, it would seem that the annealing temperature should not be too high.

The investigations of a committee on Hysteresis appointed by the *Verband deutscher Elektrotechniker* gave the following results (*E.T.Z.* 1904, p. 501):

1. After lying in the temperature of the laboratory for some months, some transformers shewed a higher loss coefficient † than on entering;

* *E.T.Z.* 1901, p. 861. † Total iron loss in 1 kg at $c=50$ and $B=10000$.

on the contrary, the loss coefficient of the testing-transformer, kept at the temperature of the room, shewed no change during the $2\frac{1}{2}$ months' continuous experiments, so that it appears the loss coefficient got worse at the beginning when the iron was brought into the laboratory temperature and then remained constant.

2. Only one plate shewed no signs of ageing—all the others shewed a tendency to this, which was more marked with the 0·35 mm plates than with the 0·5 mm; in general, the ageing is very small (3 to 8 %), with the exception of one plate, which was found to be non-homogeneous on delivery. In this case the loss increased 25 %.

3. Marked ageing, on the other hand, was observed in the alloyed plates, and was found to be larger in those with 2 % Al. (33 %) than in those containing 1 % Al. (15 %).

4. An increase in the loss due to hysteresis was always the cause of the loss coefficient becoming worse (η getting worse by 47 %), whilst the eddy-current loss in general remained constant and in the alloyed plates rather decreased (12 to 17 %). The figures obtained from static methods—in so far as could be expected from the uncertainty of the separation—agreed in general with those obtained by wattmeters.

From recent experiments by *Dr. E. Kolben*, on the influence of silicon on the ageing of iron, it appears that this phenomenon of ageing disappears rapidly as the amount of silicon increases, until with iron containing 3·5 % of silicon it vanishes almost entirely.

The *wave-shape of the pressure*, like the frequency, has no effect on the eddy-current loss at low and moderate frequencies. At high frequencies, however, the eddy losses are larger when the pressure curve deviates from a sine wave, because the higher harmonics cause larger eddy losses than the fundamental. From formula (182) it is seen that the hysteresis loss varies inversely as the 1·6th power of the form factor. Since peaked pressure curves have the largest form factors, the hysteresis loss is smaller for such than for flat-shaped curves. This follows also from the fact that the maximum induction B is proportional to the area of the pressure curve, whilst this area is inversely proportional to the form factor for the same effective value. Consequently, the maximum induction is inversely proportional to the form factor and the hysteresis loss to the 1·6th power of the form factor.

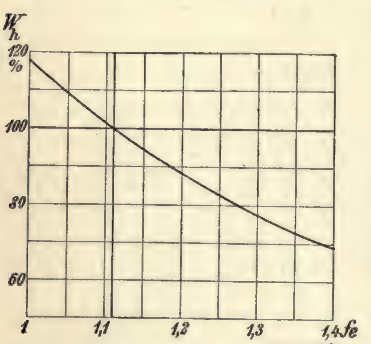


FIG. 318.—Influence of the Form Factor on Hysteresis Losses.

To give an idea of the influence of the wave-shape on the hysteresis losses, the latter have been calculated for various form factors as a percentage of the hysteresis losses

for a sinusoidal pressure, assuming the applied E.M.F. constant in every case. The results are plotted as a curve in Fig. 318.

It seldom occurs that a pressure curve has a form factor greater than 1·3 to 1·35; with such a wave-shape the hysteresis loss will be reduced some 25 %. Such highly peaked curves, however, are a disadvantage in other ways—especially on account of the heavy strain placed on the insulation. In addition to this the eddy losses are increased with peaked curves, so that they are not so efficient with regard to the iron losses as indicated by Fig. 318.

114. Flux Distribution in Armature Cores. In most electrical machines the iron is not continually magnetised and demagnetised in diametrically opposite directions, but the induction often remains more or less constant, whilst its direction rotates. Such a magnetisation occurs in the armature of the four-pole dynamo in Fig. 319. A rotating induction of this kind can always be split up into two components perpendicular to one another.

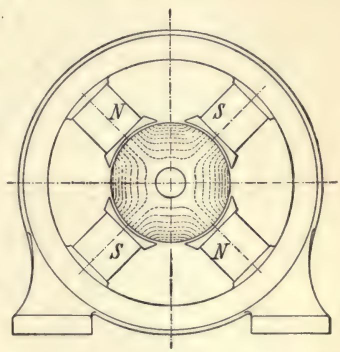


FIG. 319.—Flux Distribution in Four-pole Armature.

To determine these components, we start with the assumption that the induction at the surface of the armature is sinusoidally distributed,—a field thus distributed is called a sine-wave field. To calculate the flux distribution inside the armature, we can suppose that magnetic charges exist on the surface of the armature, the density of which $I \simeq \frac{B}{4\pi}$ varies after a sine wave. These magnetic masses exert magnetising forces H in the interior of the core, in accordance with the law of magnetic potential—these forces cause the magnetic induction B.

R. Rüdenberg* has calculated the components of this induction from differential equations of the magnetic potential, on the assumptions that the permeability μ of the plate is constant at all points and in all directions, and that the distribution of induction is not affected by eddy currents.

In polar co-ordinates, the radial component is

$$b_r = \frac{1}{r}(Ar^p - Br^{-p})\cos p\phi,$$

and the tangential component

$$b_\phi = -\frac{1}{r}(Ar^p + Br^{-p})\sin p\phi,$$

* *E.T.Z.* 1905 and R. Rüdenberg, *Energie der Wirbelströme. Sammlung electr. Vorträge* (Stuttgart), 1906.

where p is the number of pole-pairs in the machine and A and B two constants. These are obtained from the limiting conditions for the inside and outside radius

$$(1)\ r=r_i, \qquad b_r \simeq 0;$$

$$(2)\ r=r_a, \qquad b_r = B_l \cos p\phi,$$

assuming a sine-wave flux distribution B_l in the gap.

Hence follows
$$A = B_l \frac{r_a^{1-p}}{1-\left(\frac{r_i}{r_a}\right)^{2p}}$$

and
$$B = B_l \frac{r_a^{1+p}}{\left(\frac{r_a}{r_i}\right)^{2p}-1}.$$

If we change r_i and r_a these formulae hold for machines with rotating poles. In Fig. 320 the flux distribution in the machine in Fig. 319 is shewn, as calculated by Rüdenberg from the above formulae.

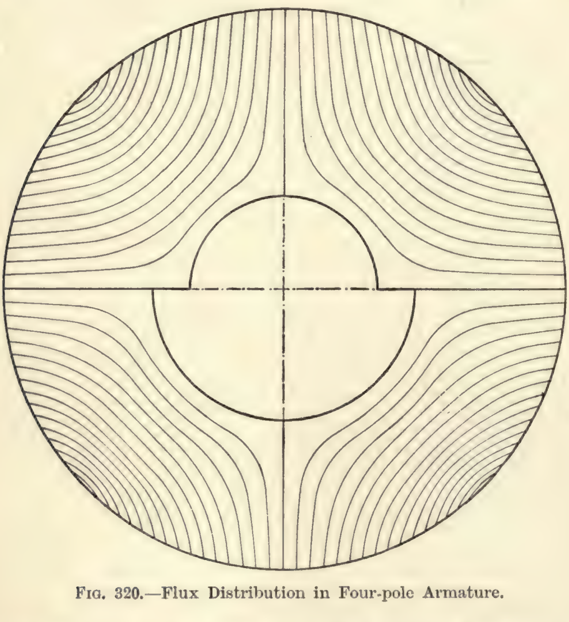


FIG. 320.—Flux Distribution in Four-pole Armature.

From the formulae it is seen that the induction at every point of a revolving armature is made up of two components, one of which varies with $\cos p\phi$ and the other with $\sin p\phi$. If the $2p$-polar armature revolves at n revolutions per minute, E.M.F.'s will be induced in the

armature conductors at a frequency np per minute or $c = \frac{np}{60}$ cycles per second. Further

$$\phi = \frac{2\pi n}{60} t,$$

where t is the time in seconds taken by the armature to rotate through the angle ϕ; hence

$$p\phi = \frac{2\pi pn}{60} t = 2\pi ct = \omega t.$$

The two components can therefore be expressed thus:

$$b_r = B_r \cos \omega t,$$

$$b_\phi = B_\phi \sin \omega t,$$

and the resultant induction can be represented by a vector $\overline{OB}$ revolving about O, as in Fig. 321.

The angular velocity of this rotation is variable, and its average is ω. The extremity B of the rotating vector moves over an ellipse (elliptic induction, elliptic rotary field). Near the external surface of the armature, $B_r = B_l$ and

$$B_\phi = \frac{r_a^{2p} + r_i^{2p}}{r_a^{2p} - r_i^{2p}} B_l;$$

hence, for $r_i = 0$, or when p is very large, at the surface where $r = r_a$, we have

$$B_r = B_\phi = B_l.$$

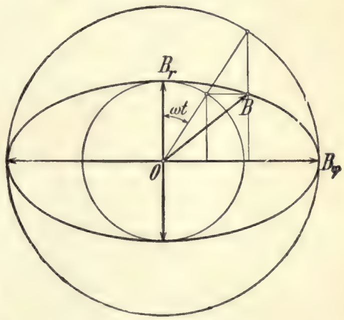


FIG. 321.—Representation of Radial and Tangential Components of Induction.

At the internal surface of the armature, where $r = r_i$, then $B_r = 0$ and

$$B_\phi = B_l \frac{2 r_a^{1+p} r_i^{p-1}}{r_a^{2p} - r_i^{2p}}.$$

Whilst the radial component always decreases from the outside to the inside surface of the core, this is only the case for the tangential component when the number of poles is greater than two. The ellipses, after which the induction varies, become flatter the deeper we go into the core. At the interior surface it becomes a line, because the induction here varies in diametrically opposite directions, as in a transformer core. The ellipse only becomes a circle in the theoretical case when the inside diameter is zero, and only the induction at the outside layer of such an armature follows a uniform rotation like a circular vector (perfect rotary field). Assuming that the molecular theory of magnetism corresponds to the physical phenomena in iron, we see that the molecules have the tendency to rotate when the

armature rotates, the mean velocity corresponding to the frequency of the E.M.F.'s induced in the armature winding.

If the field in the gap is not a sine wave, the flux curve can be analysed by Fourier's Series into its fundamental and higher harmonics, and the calculations repeated for each field. By superposing the inductions due to the several fields, we get the resultant flux distribution in the armature. Naturally, the fields with the largest numbers of poles penetrate the least distance into the core.

If p is very large or equal to ∞, the equations assume the following forms when rectangular co-ordinates are introduced. The tangential component becomes

$$b_x = \left(A\epsilon^{\frac{\pi}{\tau}y} + B\epsilon^{-\frac{\pi}{\tau}y}\right)\sin\frac{\pi}{\tau}x,$$

and the radial
$$b_y = -\left(A\epsilon^{\frac{\pi}{\tau}y} - B\epsilon^{-\frac{\pi}{\tau}y}\right)\cos\frac{\pi}{\tau}x,$$

when τ is the pole-pitch and A and B two constants which are found from the two limiting conditions

$$(1)\ y=h, \qquad b_y=0;$$

$$(2)\ y=0, \qquad b_y=B_l\cos\frac{\pi}{\tau}x.$$

We then get
$$A=\frac{B_l}{\epsilon^{2\frac{\pi}{\tau}h}-1},$$

$$B=\frac{B_l}{1-\epsilon^{-2\frac{\pi}{\tau}h}};$$

h is the core-depth. Thus, in the first formulae, $h=r_a=r_i$ and $\tau=\dfrac{\pi r_a}{p}$. The last formulae give an insight into the flux distribution in the laminated pole-shoes of a continuous or alternating-current machine with open or semi-enclosed slots in the armature. On the mean induction B_l, a magnetic wave, with its maximum value B_n opposite the teeth and its minimum value $-B_n$ opposite the slots, is superposed (Fig. 322).

At a depth $y=\dfrac{t_1}{2}=\tau$, the magnetic waves have practically vanished, since they are here reduced to

$$\epsilon^{-\frac{\pi}{\tau}y}=\epsilon^{-\pi}=0{\cdot}0435,$$

i.e. $4\frac{1}{2}\%$ of their original value.

The two assumptions on which we have based all our calculations, viz. that the permeability is constant throughout, and that the eddy

currents do not affect the flux distribution, are not always quite true. Since, however, the permeability increases towards the interior, the

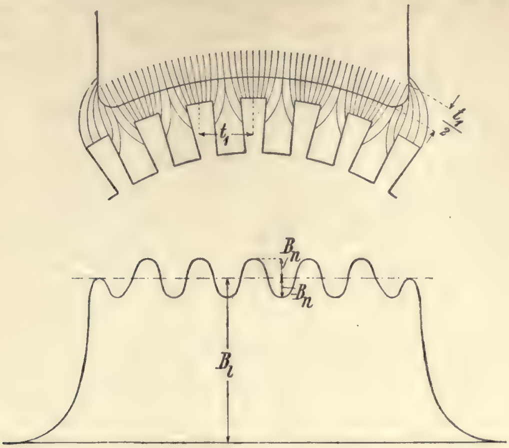


FIG. 322.—Flux Pulsations in the Gap, due to Slot-openings.

induction inside will be somewhat larger than that given by the formulae. The eddy currents have just the opposite tendency, and strive to keep the flux towards the exterior. Figs. 323 and 324 shew the distribution of the flux in a smooth-cored and in a toothed armature. These pictures of the lines of force are reproduced from photographs taken by *W. M. Thornton* * carried out by the method due to *Hele-Shaw, Hay and Powell.* The method is based on the fact that the fundamental equations for the magnetic lines of force agree with the fundamental equations for the flow in two dimensions of an ideal—*i.e.* frictionless and incompressible—fluid. A perfectly frictionless fluid does not exist, but it is sufficient to take an

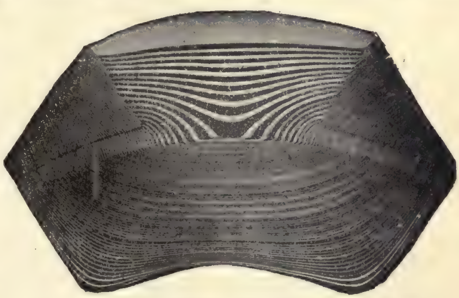


FIG. 323.—Flux Distribution in a Smooth-cored Armature.

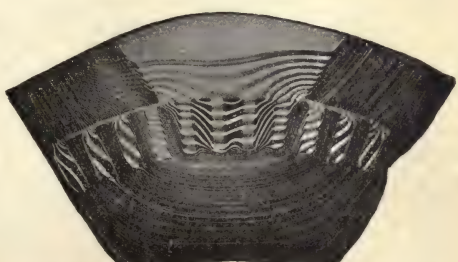


FIG. 324.—Flux Distribution in a Toothed Armature.

* *Electrician* 1905/06, p. 959.

ordinary liquid flowing in a very thin layer between two parallel surfaces. By forcing a coloured liquid in streaks between two parallel glass plates, Hele-Shaw and others succeeded in producing stream lines which agreed with the lines of force in a magnetic field. The coloration of the liquid was obtained by forcing an aniline dye into the liquid from a tube containing a large number of fine holes at small equal distances from one another—thus forming sharply-defined stream lines of extraordinary regularity.

Further, it can be proved that the velocity of the fluid under like conditions varies with the cube of the thickness of the layer. This fact gives a suitable means for producing a mechanical analogy for the various permeances of the several parts of the current path. The parts of the one plate which is to represent the air-gap are covered with a layer of wax, and the other plate is brought so near to this that only

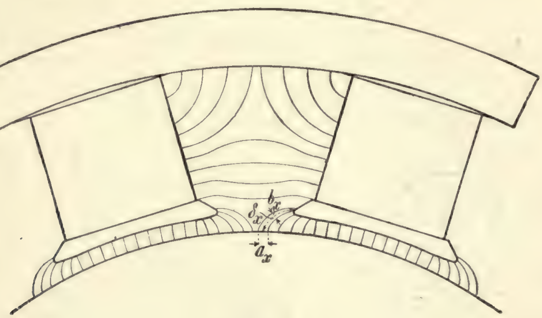


FIG. 325.

a minimum gap is left between them; if, for instance, this gap was a tenth of that at the part not covered with wax, the "permeability" would be reduced to a thousandth. The liquid used was glycerine, which was led in at one pole and out at the other. As shewn by the photographs, the paths of the "lines of force" correspond exactly with those obtained from complicated calculations.

In the calculation and construction of diagrams of the lines of force it is best to make several pictures of the lines of force by estimation, split these up into tubes of force and calculate the permeance of the tubes. Since the path of the lines of force is always such that the total flux is a maximum, the diagram giving the greatest permeance can be taken as the best. It is often well to draw in the equi-potential lines of the flux, and from these obtain the position of the lines of force. This is only advisable, however, in cases where the equi-potential lines can at once be drawn more easily and accurately than the lines of force. If we have now the figure of the lines of force—as, for instance, between the pole surface and armature surface in Fig. 325—and have found that this possesses the largest permeance, we then pass on to calculate

exactly the flux between the pole and armature surface. The permeance λ_x of a tube of force is

$$\lambda_x \simeq \frac{b_x}{0{\cdot}8\,\delta_x},$$

where b_x is the mean width and δ_x the mean length of the tube of force. The breadth of the tube perpendicular to the plane of the paper is assumed to be 1 cm. If the magnetic potential difference between the pole and armature surface is AW_δ, the flux in the tube in question will be

$$\Phi_x = \frac{b_x}{0{\cdot}8\,\delta_x} AW_\delta,$$

and the flux density at the armature surface

$$B_x = \frac{\Phi_x}{a_x} = \frac{b_x AW_\delta}{0{\cdot}8 a_x \delta_x}, \quad \text{.........................(180)}$$

since the tubes always enter the iron at right angles. If the flux density has to be found at a point in the gap, then Φ_x must be divided by the part of the equi-potential surface at the place in question, which is cut by the tube of force. In this way, the flux in all the tubes and the flux density at any point can be found with fair accuracy.

115. Iron Losses due to Rotary Magnetisation. (**a**) The *eddy current losses* in the iron with rotary magnetisation are obtained by simply adding the losses produced by the two components of the induction.

If the iron is magnetised by a pure rotary field, then $B_r = B_\phi = B$, and we get just double the eddy losses obtained with a linear alternating magnetisation to the same value B.

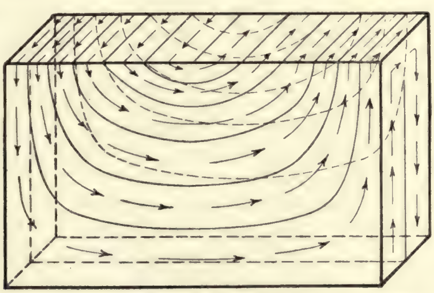


FIG. 326.—Distribution of Eddy-currents due to Rotating Magnetisation.

Starting from the formula in Section 114 for the flux distribution, *R. Rüdenberg* has analytically investigated the eddy currents in revolving armatures and obtained the interesting result that the stream lines of the eddy currents are identical with the lines of force of the magnetic field except at the boundary surfaces where the currents are reversed. The current distribution is illustrated by Fig. 326.

For the eddy-current losses, Rüdenberg obtained the same formula as above:

$$W_w = \sigma_w \left(\Delta \frac{c}{100} \frac{B_{\text{mean}}}{1000}\right)^2 V_e \text{ watts.}$$

Here $B_{\text{mean}} = \frac{\tau}{\pi h} B_t$ is the mean tangential induction in the neutral zone where $b_r = 0$. Only the eddy-current coefficient for rotary magnetisation is larger than for linear magnetisation, and, as seen from the following formulae, depends largely on the armature dimensions. For a rotating armature, we have

$$\sigma_w = \frac{\pi^2}{6} \frac{\pi h}{\tau} \frac{1 + \left(1 - \frac{\pi}{p} \frac{h}{\tau}\right)^{2p}}{1 - \left(1 - \frac{\pi}{p} \frac{h}{\tau}\right)^{2p}}. \quad \text{(181)}$$

For $p = \infty$, *i.e.* for a flat armature surface,

$$\sigma_w = \frac{\pi^2}{6} \frac{\pi \frac{h}{\tau}}{\tanh \frac{\pi h}{\tau}}, \quad \text{(182)}$$

and for hollow armature cores such as stators,

$$\sigma_w = \frac{\pi^2}{6} \pi \frac{h}{\tau} \frac{\left(1 + \frac{\pi}{p} \frac{h}{\tau}\right)^{2p} + 1}{\left(1 + \frac{\pi}{p} \frac{h}{\tau}\right)^{2p} - 1}. \quad \text{(183)}$$

In Fig. 327 the values of σ_w for different numbers of poles are plotted as functions of $\frac{h}{\tau}$. All these curves start from $\frac{\pi^2}{6}$ for $\frac{h}{\tau} = 0$, corresponding to alternating-current magnetisation. Bi-polar rotating armatures have the lowest eddy-current coefficient and bi-polar stator cores the largest. These formulae are deduced under the assumption of uniformly distributed induction over the width of each plate and for constant permeability μ. These assumptions are only partly correct, so that the eddy losses are always somewhat larger than those given by the formulae. These losses are further increased by the filing, etc., done in building the core, so that the experimental values of the eddy-current coefficient usually lie between 5 and 10, and in continuous current machines may be still higher. This is largely due to the fact, that in addition to the eddy losses in the armature plates there are also the further losses in the pole shoes, due to the teeth passing over. A similar effect is produced in an induction motor. These losses must of course be separated, as will be shewn in the latter part of this section.

(b) With respect to the hysteresis loss due to rotary magnetisation (so-called rotary hysteresis), not many investigations have been made. As shewn, the iron molecules in a revolving armature strive to rotate at a frequency c corresponding to the mean angular velocity ω, but are prevented from following the magnetising force by the friction between them and the neighbouring molecules rotating in the opposite direction.

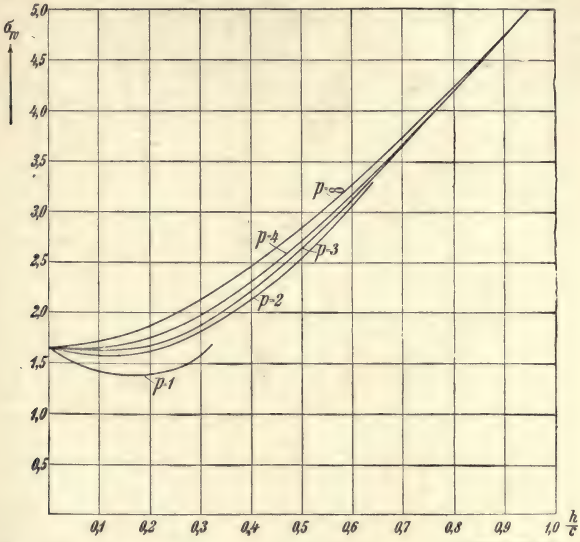


FIG. 327.—Relation of Eddy-current Coefficient σ_w to Core Depth with Different Numbers of Poles.

Consequently, losses occur here which *a priori* are not necessarily equal to the hysteresis loss due to alternating magnetisation, for in this case the magnetising force does not alter in direction but only in strength. The most recent researches, however, shew that the hysteresis loss with rotary magnetisation has about the same value as alternating magnetisation for low induction up to about 10,000. At higher inductions, on the other hand, the hysteresis loss is somewhat smaller than with alternating magnetisation. Various writers have even asserted that the rotary hysteresis loss reaches a maximum at flux densities of 16,000 to 20,000, and then at higher values falls off very rapidly to a very low value. It has been attempted to explain this phenomenon by means of Ewing's molecular theory, but neither the explanation nor the experiments seem to be free from objection. The hysteresis losses obtained with alternating magnetisation in formula (168) are therefore generally used directly for rotary magnetisation also, and calculated for the mean tangential induction $B_{\text{mean}} = \frac{\tau}{\pi h} B_t$.

(c) *Losses in Pole Shoes.* With a slotted armature the induction over the surface of the pole shoe is not constant, but varies along a wave corresponding to the teeth and slots. When the armature revolves, the maxima and minima of this wave move over the pole shoe, so that at any point in the latter the induction pulsates at a frequency corresponding to the number of teeth Z moving across the pole per second. As a consequence of this, eddy currents are induced in the shoes having the frequency $c_n = \frac{Zn}{60}$ and penetrate to a depth h, where the induction is constant. The direction of these currents is such as to damp the oscillations of the flux, that is, they exert a screening effect and are therefore chiefly confined to the surface of the shoe; below the surface, they are rapidly damped out.

If the pole shoes are laminated, the eddy-current loss due to the teeth can be calculated from formula (176) for $p = \infty$. It must be remembered, however, that the gap density B_l must be replaced by the amplitude B_n of the flux pulsations at the surface of the shoe and the pole pitch τ by the half slot-pitch. The depth of the laminations is taken as $\frac{t_1}{2}$, for if they were deeper, this would have but little effect on the calculation, since the magnetic waves—as shewn—are practically damped out at this depth. Thus, in a pole shoe of length l cm, width b cm and depth $\frac{t_1}{2}$ cm, the eddy-current loss will be

$$W_w = \sigma_w \left(\Delta \frac{c_n}{100} \frac{B_n}{\pi 1000}\right)^2 \frac{blt_1}{2000} \text{ watts,}$$

since

$$B_{\text{mean}} = \frac{B_n}{\pi}.$$

Here

$$\sigma_w = \frac{\pi^2}{6} \frac{\frac{\pi h}{\tau}}{\tanh \frac{\pi h}{\tau}} = \frac{\pi^2}{6} \frac{\pi}{\tanh \pi} = \frac{\pi^3}{6}$$

and the frequency

$$c_n = \frac{100v}{t_1},$$

where v is the peripheral speed of the armature in metres per second. Inserting these values:

$$W_w = \frac{\pi^3}{6}\left(\Delta v \frac{B_n}{1000}\right)^2 \frac{lbt_1}{2000\pi^2 t_1^2} \text{ watts}$$

$$= \frac{\pi}{120 t_1}\left(\Delta \frac{v}{10} \frac{B_n}{1000}\right)^2 lb \text{ watts, } \qquad \text{(184)}$$

where l, b and t_1 are in cm and Δ in mm.

The hysteresis losses are approximately

$$W_h = \sigma_h \frac{c_n}{100}\left(\frac{B_n}{\pi 1000}\right)^{1\cdot 6} \frac{lbt_1}{2000} \text{ watts}$$

$$= \frac{\sigma_h}{400\pi} \frac{v}{10}\left(\frac{B_n}{1000}\right)^{1\cdot 6} lb \text{ watts.}$$

In this formula, as in the earlier, the flux distribution is taken as constant over the whole plate. For most pole shoes, however, this does not hold, partly because the plates are often 1 mm or more thick and partly because the frequency c_n lies between 500 and 1500. The thickness δ of the equivalent layer of a plate in a pole shoe, where $c = 1000$, $\rho = 10^{-5}$ and $\mu = 2000$, is

$$\delta = \frac{10^4}{\pi}\sqrt{\frac{\rho}{0\cdot 8 c_n \mu}} = \frac{1}{40\pi} \text{ cm} = 0\cdot 08 \text{ mm},$$

thus being much less than half the thickness of the plate. In such cases the values obtained from the formulae are too low. It is seen, however, that it is extremely important not to use too thick plates for pole shoes. It is therefore of interest to calculate the eddy losses in a solid pole shoe and compare these with the losses in laminated shoes. For this calculation we shall use the method given by Rüdenberg in the *E.T.Z.* 1905, p. 182.

The magnetic wave entering the shoe will again be represented by

$$b_n = B_n \cos \frac{2\pi}{t_1} x.$$

In each element at the surface of the pole shoe and parallel to the axis, the E.M.F. induced per cm length is

$$e_w = v b_n 10^{-6} \text{ volts,}$$

where v is the peripheral speed of the armature in m/sec. This E.M.F. produces an eddy current near the surface

$$i_w = \frac{e_w}{\rho} = v \frac{b_n}{\rho} 10^{-6} \text{ amp.}$$

In section 112 it was shewn that the eddy currents are propagated in solid iron in accordance with the exponential function $\epsilon^{-\lambda y}$, where

$$\lambda = \frac{2\pi}{10^4}\sqrt{\frac{c_n \mu}{10\rho}},$$

a constant depending on the iron, and y the distance of the point in question from the surface.

Hence the general expression for the eddy currents can be written

$$i_w = \frac{v}{\rho} B_n \epsilon^{-\lambda y} \cos \frac{2\pi}{t_1} x 10^{-6}.$$

We take now the expression $i_w^2 \rho \, dv$, which represents the loss due to eddy currents in the element of volume dv, and integrate over the

surface of the pole shoe $t_1 l$. It is convenient to extend the integration with respect to y to ∞, but the magnetic waves do not extend even a wave length into the iron; we then get the total eddy-current loss in a slot pitch:

$$w_w = \int_0^{t_1} dx \int_0^{\infty} dy \int_0^{l} dz\, i_w^2 \rho$$

$$= \int_0^{t_1} dx \int_0^{\infty} dy \int_0^{l} dz \frac{v^2 B_n^2}{\rho 10^{12}} \epsilon^{-2\lambda y} \cos^2 \frac{2\pi}{t_1} x,$$

hence
$$w_w = \frac{v^2 B_n^2}{\rho 10^{12}} \frac{t_1}{2} \frac{l}{2\lambda}.$$

Integrating over the whole polar arc b, instead of over a slot pitch t_1, we get the total eddy-current loss

$$W_w = \frac{v^2 B_n^2}{\rho 10^{12}} \frac{bl}{4\lambda} \text{ watts}$$

$$= \frac{v^2 B_n^2}{10^8 8\pi\rho \sqrt{\dfrac{c_n \mu}{10\rho}}} bl \text{ watts},$$

and with $c_n = \dfrac{100v}{t_1}$,

$$W_w = \frac{1}{80\pi} \left(\frac{B_n}{1000}\right)^2 \left(\frac{v}{10}\right)^{1\cdot 5} \sqrt{\frac{t_1}{\rho\mu}}\, bl \text{ watts}, \qquad (185)$$

where b, l and t_1 are in cm and v in m/sec.

As seen, this expression differs considerably from that for laminated shoes. They are in the ratio

$$\frac{\pi^2}{1\cdot 5 t_1^{1\cdot 5}} \left(\frac{v}{10}\right)^{0\cdot 5} \Delta^2 \sqrt{\mu\rho} = \frac{6\cdot 6\Delta^2}{t_1} \sqrt{\frac{v\mu\rho}{10 t_1}}$$

to one another.

For $\Delta = 0\cdot 5$ mm, $t_1 = 2$ cm, $v = 20$ m/sec., $\mu = 2000$ and $\rho = 10^{-5}$, this ratio becomes

$$\frac{6\cdot 6 \times 0\cdot 5^2}{2} \sqrt{\frac{20 \times 2000}{2 \times 10^5 \times 10}} = 0\cdot 116.$$

In this case, therefore, the losses in the laminated pole shoes are little more than one tenth of those in the solid pole shoes. To obtain this result, however, the plates of the laminated shoes must not be more than $2\delta = 0\cdot 16$ mm, for $c_n = \dfrac{100v}{t_1} = 1000$ cycles per second.

Since these thin plates are not practicable, the eddy losses in the actual laminations will have a value between the above.

116. Testing and Pre-determination of Losses in Iron Stampings. For investigating iron, the apparatus should be arranged so that the magnetic circuit is entirely composed of the sample to be tested.

In the standards of the *Verband Deutscher Elektrotechniker* the arrangement shewn in Fig. 328 is proposed for the testing of iron plates.

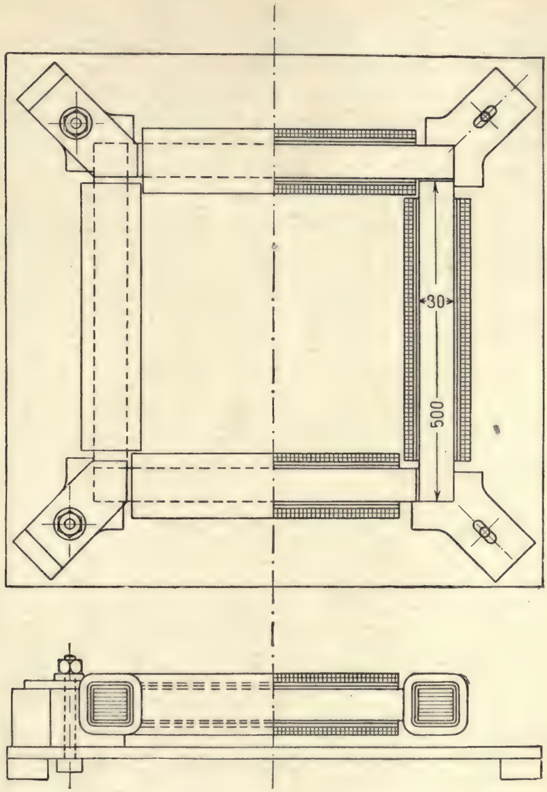


FIG. 328.—Apparatus for Testing Iron Stampings.

The magnetic circuit is made up of four cores each 500 mm long, 30 mm wide and at least $2\frac{1}{2}$ kg in weight. The several plates are insulated from one another by tissue paper. The cores are held in position by wooden clamps and at the junctions separated by a 0·15 mm strip of presspahn. Special care is to be taken that the cores are strictly in line, correct position being detected by minimum noise and minimum magnetising current. The exciting coils are wound on presspahn spools, on each of which there are 150 turns of wire of 14 mm^2 section.

The stampings—according to these instructions—shall be taken from a sample of four lots weighing at least 10 kg. From the total losses

measured by the wattmeter, the loss in the winding is to be deducted in order to obtain the iron loss W_e. From formulae (168) and (176), the total iron losses are:

$$W_e = W_h + W_w = \left[\sigma_h \frac{c}{100}\left(\frac{B}{1000}\right)^{1\cdot 6} + \sigma_w\left(\Delta \frac{c}{100}\frac{f_\epsilon B}{1000}\right)^2\right] V_e. \quad \text{...(186)}$$

The coefficients σ_h and σ_w can be found by experiment, by testing the sample at a constant induction B with alternating-currents and variable frequency c. For this purpose we have only to maintain the excitation of the generator constant and vary its speed; for then the E.M.F. varies in proportion to the frequency and the flux remains constant. The losses measured by the wattmeter are then divided by the volume of iron to obtain the loss per dm³. These values divided by their respective frequencies c are plotted as functions of the induction B, and must—according to the above equation—give a straight line.

The intercept of this straight line on the ordinate axis equals $\frac{\sigma_h}{100}\left(\frac{B}{1000}\right)^{1\cdot 6}$, whilst the height of a point on the straight line above this point of intersection with the ordinate axis is

$$c\sigma_w\left(\frac{\Delta}{100}\frac{f_\epsilon B}{1000}\right)^2.$$

In Fig. 329 the above-mentioned lines have been determined for 0·5 mm dynamo plates at the inductions $B = 6000$, 10,000 and 15,000, and the values of σ_h and σ_w calculated from the same are given.

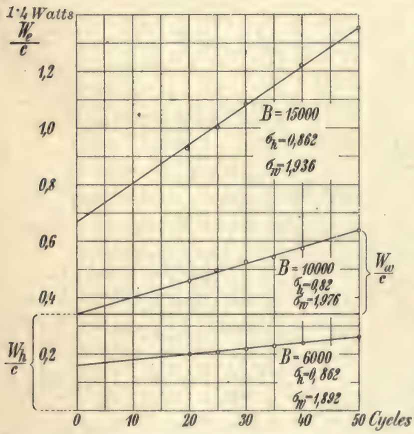


FIG. 329.—Separation of Iron Losses by Frequency Method.

This method of separating the hysteresis and eddy losses is based on the assumption that the hysteresis loss per cycle is independent of the frequency. This is not, as we have seen, strictly correct, for the same increases somewhat as the frequency increases. Consequently, by this method of separation the eddy-current loss will appear somewhat greater, and the hysteresis loss somewhat smaller than is actually the case. But in any case the method enables us to see what part of the losses is proportional to the frequency and what part to the square, which is of importance for pre-determining the losses and obtaining the coefficients σ_h and σ_w experimentally. Further, we have seen that the eddy currents—especially at high frequencies—cause a non-uniform distribution of the induction over the section of the plates.

In consequence of this, the hysteresis loss will be further increased with increasing frequency, which appears as an increase of the eddy-current coefficient σ_w in the above separation. This coefficient, therefore, will generally be found considerably greater when determined by this means, than when it is deduced from the thickness and permeance of the plates. If the paper between the plates does not insulate properly, or if a direct path for currents from plate to plate is made during erection or construction, as is often unavoidable in practice, the eddy-current coefficient may be still further considerably increased.

The total loss in watts in a kilogram of iron at an induction of 10,000 and frequency of 50 is called the *specific loss* of the iron. Assuming a specific gravity of 7·77, the iron tested in Fig. 329 has a specific loss of 4·1.

According to Ewing, the best result obtained by him was from iron having the following composition:

Carbon 0·02 %.	Phosphorus 0·02 %.
Silicon 0·032 %.	Sulphur 0·003 %.
Traces of manganese.	Iron 99·925 %.

This iron ages considerably, however. By adding 3 % of silicon or aluminium it has recently be found possible to produce an iron, in which the hysteresis loss is less than that of the best Swedish iron. This iron is also considerably less affected by ageing. The permeability of such an alloyed iron is, however, lower than that of ordinary iron, and likewise its mechanical strength.

Since such *alloy* plates have 4 to 5 times the electrical resistance of ordinary plates and therefore smaller eddy losses, they are particularly suitable for transformers and other electromagnetic apparatus with large iron losses and poor cooling.

For the specific loss the Bismarck hütte—whose plates are largely used in Germany at the present day—guarantees:

Ordinary plates - - -	3·6 watts per kg.
0·6 to 0·7 % Silicon Alloy -	3·2 „
3·0 to 3·5 % „ -	1·8 „

The composition of alloy plates is usually as follows:

Carbon 0·03 %,	Phosphorus 0·01 %,
Silicon 3·4 %,	Sulphur 0·04 %,
Manganese 0·3 %,	Iron 96·2 %,

and they have a specific resistance of 0·5 ohm.

117. Calculation of the Magnetising Ampere-turns with Continuous and Alternating-Current. To calculate the ampere-turns in a magnetic circuit excited by direct current, we divide the magnetic circuit into parts made of the same material and having approximately a constant induction. Starting, for example, with the value Φ_1 of the flux in the first part, we find the induction $B_1 = \frac{\Phi_1}{Q_1}$, where Q_1 is the mean section

of this part. Similarly, the induction at another part $B_x = \frac{\Phi_x}{Q_x} = \sigma_x \frac{\Phi_1}{Q_x}$, where σ_x denotes the leakage coefficient of the part x with respect to part 1. We now need the *magnetisation curves* of the respective materials. These curves give the inductions B for the different materials as functions of the ampere-turns aw per cm length of the magnetic path. Such curves are determined by the above-mentioned ballistic measurements, or by means of some form of permeameter, and take no account, therefore, of the effect of hysteresis. The error hereby introduced is usually not considerable. In Fig. 330 the magnetisation curves for the commonly-used magnetic materials of average quality are given.

The permeability of good cast steel is independent of the amount of carbon present up to 0·25 % of the latter. Above this value the steel becomes harder both mechanically and magnetically and its permeability rapidly decreases.

Let at_1, at_2, etc., denote the values of the ampere-turns per cm length, as given by these curves, for the inductions B_1, B_2, etc., in the several parts; then for the whole magnetic circuit we have the total ampere-turns

$$AT_k = at_1 L_1 + at_2 L_2 + \dots,$$

where L_1, L_2, etc., denote the lengths of the several parts.

If we carry out this process for a number of values of the flux Φ_1, we get a curve shewing Φ_1 as a function of AT_k (cp. magnetisation curve or no-load characteristic of machines).

The calculation of the magnetic circuit with an alternating flux, as in the case of transformers or induction motors, is quite similar.

Here we have usually the *maximum* value of the sinusoidal alternating flux either given or assumed, whilst the *effective* value of the magnetising ampere-turns or current is to be calculated. Further, this effective value has to be split up into an energy or watt component and a wattless component. If the magnetic circuit is made up of several parts, the problem cannot be solved accurately, unless we have the hysteresis loops for the several inductions in the various parts. From these the hysteresis loop for the whole magnetic path could be calculated point by point and the curve of magnetising current found, similarly to that shewn in Fig. 307.

Since this method is much too roundabout for practical purposes, it is better to use the following approximate method.

On a test-ring of the particular material, as shewn in Fig. 299, with various applied pressures P, the effective current I and consumed watts W are measured. If the pressure is sinusoidal,

$$\Phi_{\max} = \frac{P \cdot 10^8}{4{\cdot}44\, cw},$$

and the maximum induction

$$B = \frac{P \cdot 10^8}{4{\cdot}44\, cwQ},$$

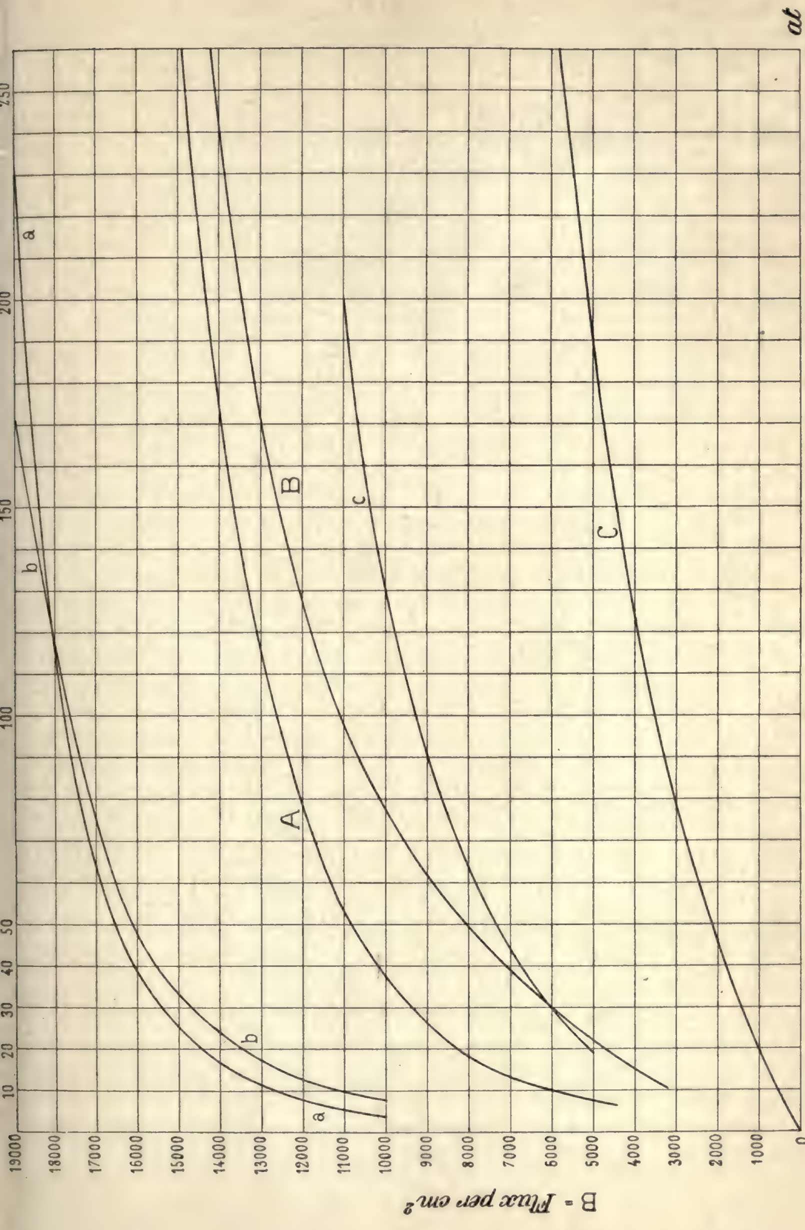


FIG. 330.—Magnetisation Curves for *A*, *a* Armature Stampings; *B*, *b* Cast Steel; *C*, *c*, Cast Iron.

where Q equals the section of the material. The effective value of the magnetising ampere-turns per cm length of the ring is

$$at = \frac{Iw}{L_m},$$

where L_m is the mean length of the ring.

Further, the watt component of the magnetising current is

$$I_W = \frac{W}{P},$$

and the watt-component of the corresponding ampere-turns per cm length is

$$at_W = \frac{I_W w}{L_m} = \frac{Ww}{PL_m}.$$

The wattless component of the magnetising current and of the corresponding ampere-turns per cm length of the magnetic path are

$$I_{WL} = \sqrt{I^2 - I_W^2},$$

$$at_{WL} = \frac{I_{WL} w}{L_m} = \frac{w}{L_m}\sqrt{I^2 - I_W^2} = \sqrt{at^2 - at_W^2}.$$

In Fig. 331 the values of at_W and at_{WL} are plotted for different values of B at 50 cycles per sec. The curves are taken for iron plates of various qualities and thicknesses, curves I and II being for dynamo plates 0·5 mm and 0·35 mm thick, and curve III for alloy plates 0·33 mm thick.

To calculate a magnetic circuit for alternating-current, the procedure is similar to that for a circuit excited by continuous current. After the circuit has been divided into parts of the same material and with approximately constant inductions B_1, B_2, etc., then, by means of the curves, we can get the watt ampere-turns AT_{kW} for the whole circuit

$$AT_{kW} = at_{W1}L_1 + at_{W2}L_2 + \dots, \quad \text{.................. (187)}$$

and likewise the wattless ampere-turns

$$AT_{kWL} = at_{WL1}L_1 + at_{WL2}L_2 + \dots. \quad \text{................ (188)}$$

The resultant ampere-turns are then

$$AT_k = \sqrt{(AT_{kW})^2 + (AT_{kWL})^2}. \quad \text{.................. (189)}$$

By this method, we not only take into account the effect of magnetic hysteresis, but also the influence of the eddy-current losses on the magnetising current.

The calculation of the watt ampere-turns is quite accurate, since these are *sinusoidal* and give the total watts lost in the circuit

$$W = I_W P = I_W \, 4{\cdot}44 \, cw \, \Phi_{\max} 10^{-8}$$

$$= AT_{kW} \, 4{\cdot}44 \, c \, \Phi_{\max} 10^{-8} \text{ watts.}$$

The calculation of the wattless ampere-turns in the whole circuit by summing up the wattless ampere-turns in the several parts is not quite exact, since these components contain higher harmonics which have different relations to the fundamental in the several parts. This method, therefore, gives a somewhat too high value for the wattless ampere-turns, especially when strongly saturated iron is in series with feebly saturated or with air.

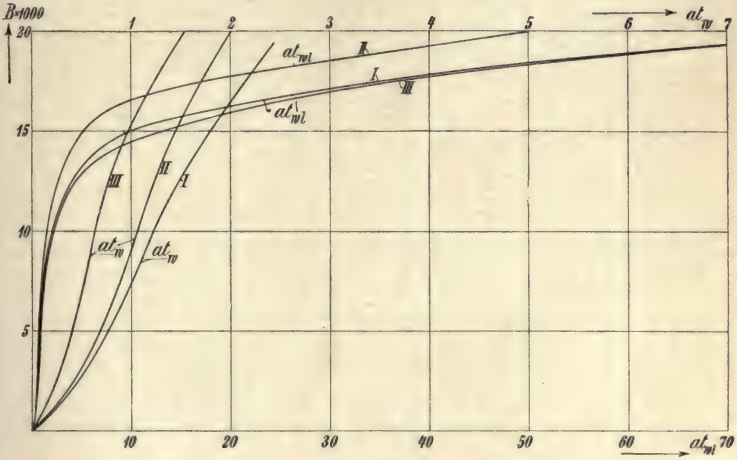


FIG. 331.

The error can be reduced somewhat by splitting up the ampere-turns at_{WL} into a fundamental $at_{1\,WL}$ and a component at_d comprising the higher harmonics. The latter is found from the equation

$$at_d = \sqrt{(at_{WL})^2 - (at_{1\,WL})^2}.$$

In Fig. 332 the curves for $at_{1\,WL}$ and at_d are calculated for laminations of the material used for curve I, Fig. 331.

Similarly, as in the above, we can now calculate from the curves for the whole magnetic circuit

$$AT_{k\,W} = at_{W1}L_1 + at_{W2}L_2 + \dots,$$

$$AT_{1\,k\,WL} = at_{1\,WL_1}L_1 + at_{1\,WL_2}L_2 + \dots,$$

$$AT_{dk} = at_{d\,1}L_1 + at_{d\,2}L_2 + \dots,$$

whence $$AT_{k\,WL} = \sqrt{at^2_{1\,k\,WL} + at^2_{dk}}$$

and $$AT_k = \sqrt{AT^2_{kW} + AT^2_{k\,WL}} = \sqrt{AT^2_{kW} + AT^2_{1\,k\,WL} + AT^2_{dk}}. \quad \dots\dots(190)$$

At the present time, plates with low losses are usually used for static transformers, which make it possible to work at high densities.

In these special plates, however, saturation is usually reached comparatively early, so that the magnetising current quickly becomes distorted. On this account, in the diagram of such transformers, the magnetising current cannot be considered sinusoidal, and therefore cannot be added geometrically to the sinusoidal load current in the ordinary way; but, as shewn above, the sinusoidal part of the wattless

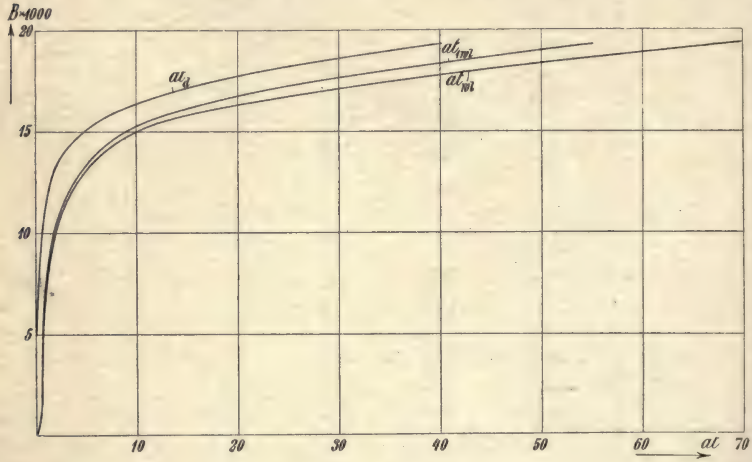


FIG. 332.

component of the magnetising current must first be added directly to the wattless component of the load current and then the components of the higher harmonics at 90° to these geometrically added, in order to obtain the total wattless component of the primary current supplied to the transformer. By means of this accurate procedure the wattless component of the primary current will appear smaller than the sum of the wattless components of the magnetising current and the secondary load current, which is usually the one calculated. The error introduced, however, by the latter simple method is generally negligible.

118. The Magnetic Field in a Polyphase Motor. For the sake of simplicity we will consider the actual case of a symmetrical two-pole three-phase induction motor. The stator coils of the three phases are displaced from one another by 120° in space. To the three phases the following symmetrical pressures are applied:

$$p_{\mathrm{I}} = P_{\max} \sin(\omega t + \psi),$$

$$p_{\mathrm{II}} = P_{\max} \sin(\omega t + \psi - 120°),$$

$$p_{\mathrm{III}} = P_{\max} \sin(\omega t + \psi - 240°).$$

These pressures produce the following fluxes, which are interlinked with the windings of the three phases:

$$\Phi_{\mathrm{I}} = -\Phi_{max} \cos(\omega t + \psi),$$

$$\Phi_{\mathrm{II}} = -\Phi_{max} \cos(\omega t + \psi - 120^\circ),$$

and

$$\Phi_{\mathrm{III}} = -\Phi_{max} \cos(\omega t + \psi - 240^\circ).$$

These fluxes are displaced by 120° in space, whilst in time they succeed one another after one-third of a complete period.

The resultant flux in a direction x, which encloses the angle x with the perpendicular to the coils of the first phase, can therefore be written:

$$\Phi_x = -\Phi_{max} \cos(\omega t + \psi - x).$$

Suppose the direction x rotates with the angular velocity ω, then we can write

$$x = x_0 + \omega t,$$

and we get

$$\Phi_x = -\Phi_{max} \cos(\psi - x_0), \quad \text{.........................(191)}$$

i.e. *the flux along an axis revolving with the angular velocity of the current is constant.* Such a field is called a rotary field.

If we take the initial position $x_0 = \psi$, i.e. so that the flux at the instant $t = 0$ is a maximum in the direction x_0, then this direction x corresponds with the maximum flux at every instant.

Hence, in a polyphase motor we have a constant flux rotating with a constant angular velocity ω, the direction of flux coinciding with the perpendicular to the coils of each phase at the instant when the pressure of the respective phase is zero. The flux distributes itself in the gap in practically a sine wave over the armature periphery.

To calculate the magnetising current in each phase, the effect of all three phases in producing the common rotary field must now be taken into account. Consider, for example, the instant when the flux is a maximum in the first phase, then the resultant magnetising ampere-turns along the perpendicular to the coils in this phase are also a maximum and equal

$$AT_{max} = i_{\mathrm{I}} w \cos 0^\circ + i_{\mathrm{II}} w \cos 120^\circ + i_{\mathrm{III}} w \cos 240^\circ,$$

and this AT_{max} has to produce the maximum flux density B_l in the gap along the perpendicular to the first phase. w equals the number of turns per pole and phase. Since the magnetising currents are practically wattless, i_{I} is a maximum, since the phase pressure is zero at this moment. Hence, we have

$$\begin{aligned} AT_{max} &= w\left[I_{max} \cos 0^\circ \sin \frac{\pi}{2} + I_{max} \cos 120^\circ \sin\left(\frac{\pi}{2} + \frac{2}{3}\pi\right) \right. \\ &\qquad \left. + I_{max} \cos 240^\circ \sin\left(\frac{\pi}{2} + \frac{4}{3}\pi\right)\right] \\ &= I_{max} w (\cos^2 0^\circ + \cos^2 120^\circ + \cos^2 240^\circ) \\ &= \tfrac{3}{2} I_{max} w, \end{aligned}$$

that is to say, the magnetising current per phase required to produce the rotary field in a three-phase motor is only $\frac{2}{3}$ of the current required to produce an equal alternating field by means of a single phase.

For an n-phase motor we should have

$$AT_{\max} = I_{\max} w \left(\cos^2 0^\circ + \cos^2 \frac{2\pi}{n} + \cos^2 \frac{4\pi}{n} + \ldots + \cos^2 \frac{2(n-1)\pi}{n} \right)$$

$$= \frac{n}{2} I_{\max} w. \quad \text{.....(192)}$$

Hence in an n-phase motor, the magnetising current in each phase required to produce the rotary field is only $\frac{2}{n}$ of the magnetising current required to produce a corresponding alternating field.

In a two-phase motor, where $n = 2$,

$$AT_{\max} = I_{\max} w.$$

In this motor the total flux is produced by one phase when the flux is a maximum along the perpendicular to this phase. Suppose the two phases of the two-phase motor produce alternating fields b_{I} and b_{II} of the same maximum density B_l, which are displaced by 90° both in space and time, then, as shewn in Fig. 333, these combine to produce a rotary field of constant intensity B_l. From the above it is clear that to produce a rotary field, twice as many ampere-turns are needed as to produce an alternating flux. Whence it follows further, that a single-phase induction motor at no-load (i.e. running light) takes twice the magnetising current that it takes at rest, since at rest an alternating field is produced, and when running a rotary field.

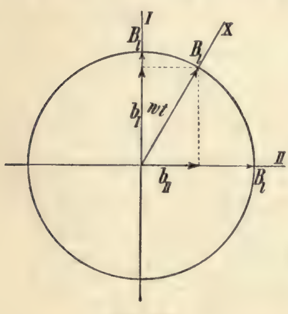


FIG. 333.

If the three-phase motor is wound for $2p$ poles, the rotary field will again move over a double pole-pitch in a period,—thus through $\frac{1}{p}$th of a revolution. Hence the rotary field in a $2p$-pole motor moves p times more slowly than in a bi-polar, i.e. at the speed $\frac{\omega}{p}$. With the same magnetic reluctance per unit-tube of flux, the $2p$-pole motor requires p times the magnetising current that the bi-polar takes, since there are p times as many fields to produce.

CHAPTER XIX.

THE FUNDAMENTAL PRINCIPLES OF ELECTROSTATICS.

119. The Electric Field. (a) By the term "electric field" is understood a space where electric forces can be observed. The electric field has several properties in common with the magnetic field, though in several points, on the other hand, there is a marked difference. For example, the total quantity of magnetism in a magnet is always zero. With bodies in electric fields this is not always so; a body, for example, may contain only positive electricity, in which case it is said to be positively electrified or charged. Electrically-charged bodies produce in their neighbourhood an electric field, which becomes weaker the further we go from the charged body. The repelling force exerted on one another by two small bodies carrying the charges q_1 and q_2 in air or in vacuo can be calculated from Coulomb's Law:

$$K=\frac{q_1q_2}{r^2}, \qquad \text{.......................................(193)}$$

where r is the distance in cm between the bodies. If the charges are expressed in electrostatic units, the force K will be given in dynes. In the electrostatic system of units, therefore, the electric quantity or charge has the same dimensions $(L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1})$ as the magnetic quantity in the electromagnetic system of units. If we have an electric charge $+1$ in an electric field, it will be acted on by the mechanical force f. This force f is termed the *electric field-strength*, and has the same dimension $(L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1})$ as the magnetic field-strength in the electromagnetic system of units.

As in a magnetic field there are magnetic lines and tubes of force, similarly in an electric field there are electric lines and tubes of force. An electric line of force is defined as a line such that its tangent at any point coincides in direction with the field-strength. The number of unit tubes of force passing through a surface of 1 cm²

perpendicular to the direction of the force is taken as numerically equal to the field-strength at the respective point.

(**b**) Every point in a constant electric field possesses a *potential.* At any point in the field the potential is

$$P=\Sigma\frac{q}{r}, \qquad (194)$$

where q denotes the electric charge of a point at the distance r from the point considered. The summation has to be extended over all the electric charges in the field.

If we calculate the work A done when the electric charge $+1$ at distance r_1 from the charge q_1 is removed to infinity, we have

$$A=\int_{r=r_1}^{r=\infty} K\,dr=\int_{r_1}^{\infty}\frac{q_1}{r^2}dr=\left(-\frac{q_1}{r}\right)_{r_1}^{\infty}=\frac{q_1}{r_1}=P_1.$$

The work A is thus equal to the potential of the charge q_1 at a distance r_1. Since this work is independent of the path s over which the unit charge is conveyed, the potential will be

$$P=\int_{r_1}^{\infty} f_s\,ds=\int_{\infty}^{r_1} -f_s\,ds.$$

By differentiating, we get the field-strength in the direction s

$$f_s=-\frac{dP}{ds} \qquad (195)$$

equal to the fall of potential in this direction. From this, the potential difference between two points A and B is

$$P_A-P_B=\int_A^B f_s\,ds.$$

A surface perpendicular at all points to the direction of the field-strength, and hence the locus of all points having the same potential, is called an *equi-potential surface.* The earth's potential is usually taken as zero, and in this case the potential of a point can be calculated as the work done in moving positive unit charge from earth to the point considered.

(**c**) *Gauss and Green's Theorem.* The total flux ϕ leaving a closed surface F is equal to 4π times the sum of the electric charges q inside the sphere. This theorem can be directly deduced from Coulomb's Law. Symbolically

$$\phi=\int_F f_n\,dF=4\pi\Sigma q, \qquad (196)$$

where f_n is the normal component of the electric field-strength, directed outwards, on the elemental surface dF, and the integral is taken over the whole closed surface F.

Inside a solid conductor, maintaining equilibrium, the electric field-strength f is everywhere zero. Thus if the closed surface is placed inside a conductor where $f=0$ everywhere, then $\Sigma q=0$; i.e. *no electricity can exist inside a charged conductor.* The electricity inside the conductor mutually

repels itself to the surface, where the total electrical charge of the conductor is therefore located. *The quantity of electricity per unit of surface is called the surface density σ of the electric charge.*

On the element of surface dF the charge is

$$dq = \sigma dF.$$

If a closed surface—as shewn in Fig. 334—is placed very near to the elemental surface dF, then as the electric field-strength inside the conductor is zero, and from Gauss's Theorem we have

$$\int f_n dF = f\,dF = 4\pi\Sigma q = 4\pi\sigma\,dF$$

or
$$f = 4\pi\sigma. \quad \text{.................................(197)}$$

Hence the electric field-strength at a point near the surface of a charged conductor is 4π times the surface density. From this it follows that the surface of a conductor forms an equi-potential surface, and that the electric lines of force leave the surface perpendicularly when it is positively charged, and enter perpendicularly when it is negatively charged. The positive and negative charges form the termini of the tubes of electric force.

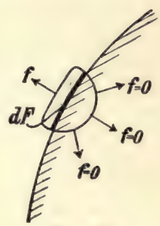


FIG. 334.

(d) The electric field-strength at a point in the surface of a conductor is not equal to the field-strength at a point just outside.

Just outside the surface, both the electric charge $\sigma\,dF$ and all the other electric charges on the conductor exert their effect; hence we can put $f_a = f_1 + f_2$, where the field-strength f_2 is due to the charge $\sigma\,dF$. At a point on the surface, the charge $\sigma\,dF$ exerts no force f_2, so that the resultant field-strength here is $f_0 = f_1$. At a point just inside the conductor the charge $\sigma\,dF$ exerts the force $-f_2$, directed inwards, since the point is on the opposite side of the surface element. Since the electric field-strength inside a conductor is zero, then $f_i = f_1 - f_2 = 0$, i.e. $f_1 = f_2 = \frac{1}{2}f_a$. Consequently, the electric field-strength at a point on the surface is

$$f_0 = \tfrac{1}{2}f_a = 2\pi\sigma.$$

In a field of this intensity there acts on every unit of surface having the surface density σ, the mechanical force

$$K = f_0\sigma = 2\pi\sigma^2 = \frac{f_0^2}{2\pi} = \frac{f_a^2}{8\pi}, \quad \text{.......................(198)}$$

which is always directed outwards, and is known as the *electrostatic tension.* Its presence can be observed by electrifying a soap bubble, which grows larger and finally bursts.

If the conductor is a solid body and the electrostatic tension becomes too high, the conductor will discharge itself into the air. At ordinary atmospheric pressure and temperature, such a discharge occurs when $K = 400$ to 500 dynes. This tension corresponds to a mercury column of 0·3 mm.

The distribution of the surface density σ over the surface is usually non-uniform. On a conductor removed from all other conductors, it only depends on the shape of the surface; the density at any point is inversely proportional to the radius of curvature at this point. The greatest density, therefore, is at points and edges of the conductor, so that the discharge occurs first in these places.

(e) Electric conductors are not only charged with electricity by direct contact, but also by electrostatic induction. If a conductor is brought into an electric field, then negative charges will collect on the part of its surface where the lines of force enter the conductor and positive charges where the lines of force leave. *The algebraic sum of the charges of electricity thus produced is always zero.*

To protect a body against static induction it can be enclosed in a conducting cover. No lines of force enter the hollow space, thus the conducting cover acts as an electric screen against all external electric forces. This property is employed in electrostatic measuring instruments. In the interior of a hollow conductor, no electricity can exist.

120. Capacity. By the capacity C of a conductor is understood the ratio of its charge Q to its potential P; hence

$$Q = CP. \qquad (199)$$

Since the potential $P = \Sigma \frac{q}{r}$, capacity has the dimension of a length in the electrostatic system of units.

(a) If the electric charge Q is concentrated at a point, then the electric field-strength at a distance ρ is

$$f = \frac{Q}{\rho^2},$$

and the potential P at the point in question is found from

$$\frac{dP}{d\rho} = -f,$$

and is
$$P = -\int f\,d\rho = -\int \frac{Q}{\rho^2}\,d\rho = \frac{Q}{\rho} + \text{const.}$$

Since $P = 0$ when $\rho = \infty$, the constant disappears, and the potential is

$$P = \frac{Q}{\rho}.$$

Since $P =$ constant for surfaces at the same potential, ρ is constant for such surfaces. Hence the equi-potential surfaces are spheres about the charged point as centre. Considering the space enclosed by one of these spheres when the enclosing cover is metal, then the whole charge Q passes to the surface without the electric field being affected in any way. For, from Gauss's Theorem, the total flux ϕ through the several equi-potential surfaces is not altered; this is

$$\phi = 4\pi Q = 4\pi\rho^2 f,$$

and the surface density on a spherical surface is therefore

$$\sigma = \frac{f}{4\pi} = \frac{Q}{4\pi\rho^2} = \frac{\phi}{16\pi^2\rho^2}.$$

The potential at the surface of a sphere of radius r and charge Q is thus

$$P = \frac{Q}{r}. \quad \text{............(200)}$$

Hence, it follows that in air the capacity of a sphere equals its radius. Inside the sphere the potential is everywhere zero, irrespective of whether the sphere is hollow or solid.

Consider a straight line of infinite length (Fig. 335) with the charge Q per unit length. The field-strength due to it at a point distant ρ from the straight line is

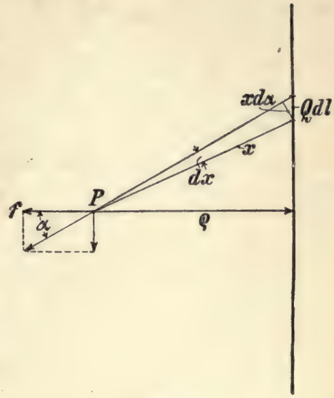


FIG. 335.

$$f = \int_{\alpha=-\frac{\pi}{2}}^{\alpha=+\frac{\pi}{2}} \frac{Q\,dl}{x^2}\cos\alpha = \int_{\alpha=-\frac{\pi}{2}}^{\alpha=+\frac{\pi}{2}} \frac{Qx\,d\alpha}{x^2} = \int_{\alpha=-\frac{\pi}{2}}^{\alpha=+\frac{\pi}{2}} \frac{Q\cos\alpha}{\rho}\,d\alpha = \frac{2Q}{\rho}. \quad (201)$$

The potential at this point is

$$P = -\int f\,d\rho = -\int \frac{2Q}{\rho}\,d\rho = \text{const.} - 2Q\log_\epsilon \rho.$$

The equi-potential surfaces also satisfy the equation $\rho = \text{const.}$ here, i.e. they are cylinders about the straight line as axis. Suppose again an equi-potential surface to be metallic, then the charge Q will pass to this metal cylinder, without affecting the electric field. The electric flux for the length l of the cylinder is in this case

$$\phi = 4\pi Ql = 4\pi\rho\frac{f}{2}l = 2\pi\rho l f,$$

and the surface density is

$$\sigma = \frac{f}{4\pi} = \frac{2Q}{4\pi\rho} = \frac{Q}{2\pi\rho} = \frac{\phi}{8\pi^2\rho l}.$$

The potential and capacity of an infinitely long cylinder cannot be expressed in finite terms, since there are no limiting conditions for the constants. Later, however, we shall return to special cases.

Lastly, we can consider an infinitely large plane with the surface density σ; the field-strength at a point near the plane is $f = 2\pi\sigma$, since half of the $4\pi\sigma$ lines per unit surface go out perpendicularly on the one side, and the other half on the other side. On the surface itself $f_0 = 0$.

(**b**) To calculate the capacity of a line, it is best to proceed as follows. We start from the assumption that the conductor has a certain charge Q, and calculate its potential by finding the work necessary to bring $+1$ charge from infinity or earth to the conductor. The path along which this is done is, as mentioned, immaterial.

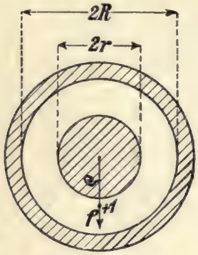


FIG. 336.

As an example, we shall calculate in this way the capacity C of a cylinder of diameter $2r$ (Fig. 336) and length l surrounded by a co-axial earthed hollow cylinder of inside diameter $2R$. The hollow cylinder has zero potential, and the potential of the internal cylinder is the work $\int_{\rho=R}^{\rho=r} -f\,d\rho$, which is required to convey unit charge from the outside cylinder to the inside. For a very long cylinder we had

$$f=\frac{2Q}{\rho},$$

where Q = charge per unit length; hence

$$P=\int_{\rho=R}^{\rho=r} -\frac{2Q}{\rho}\,d\rho = -2Q(\log_\epsilon r-\log_\epsilon R)=2Q\log_\epsilon\frac{R}{r},$$

and the capacity C of the two cylinders is

$$C=\frac{lQ}{P}=\frac{l}{2\log_\epsilon\frac{R}{r}}. \qquad \text{(202)}$$

In a similar manner we find the capacity of a sphere of radius r concentrically surrounded by a hollow sphere of inside radius R. Here

$$P=-\int_{\rho=R}^{\rho=r}\frac{Q}{\rho^2}\,d\rho=Q\left(\frac{1}{r}-\frac{1}{R}\right)=Q\frac{R-r}{Rr}.$$

Hence the capacity $C=\frac{Q}{P}=\frac{Rr}{R-r}$. This may be very different from the capacity of a sphere removed far away from other bodies. The charge on the inner surface of the hollow sphere equals the charge Q on the surface of the inner sphere.

FIG. 337.

If a surface F having the charge Q placed opposite to an earthed surface at a distance r, the field-strength between the two plates is everywhere constant (Fig. 337), when the surfaces are large compared with the distance r. The direction of the field is normal to the plates, and its strength is

$$f=4\pi\sigma=\frac{4\pi Q}{F}. \qquad \text{(203)}$$

At the surface of the charged plate the field-strength f_0 is only one half, since here only the charge of the earthed plate can produce a component of force; thus

$$f_0 = 2\pi\sigma = \frac{2\pi Q}{F}.$$

The potential of the charged plate is

$$P = -\int_{\rho=r}^{\rho=0} f\,d\rho = \frac{4\pi Q}{F} r = fr,$$

and the capacity of the pair of plates

$$C = \frac{Q}{P} = \frac{F}{4\pi r}. \quad \text{.............................}(204)$$

Such systems of two conductors having large surfaces a small distance apart are called *condensers*, the two conductors being termed the *plates* of the condenser. Condensers are used for collecting large electric charges by means of moderate potential differences.

In all practical condensers, the plates are so near together, that they always receive the same charge, which depends only on the potential difference applied to the plates, and is wholly independent of external influences such as the presence of strong electric fields or other condensers. Usually the plates are made of tin-foil, whilst the dielectric consists of paraffin-wax paper or thin mica sheets. Recently, high-pressure condensers with glass tubes and metal plates—similar to Leyden jars—have been placed on the market.

The capacity C of a condenser is numerically equal to the charge Q which collects on one plate when it is raised to unit potential, the other plate being earthed, or in other words, when the potential difference between the plates is unity. If several condensers are placed in parallel, each assumes a charge proportional to its capacity and to the common potential difference, and the total charge of all the condensers equals the sum of the charges of the several condensers. *Thus the capacity of condensers in parallel equals the sum of the capacities of the several condensers*, when these are independent of one another. If several condensers are placed in series, they will all assume the same charge Q, and the potential difference P between the first and last will be divided between the several condensers in inverse proportion to their capacity. Thus,

$$P = P_1 + P_2 + P_3 + \ldots = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots = \frac{Q}{C},$$

whence it follows that the reciprocal value of the capacity of several condensers in series equals the sum of the reciprocal values of the capacities of the several condensers.

(c) We have seen that when other bodies, e.g. the earth, are in the neighbourhood of a conductor, the capacity of the latter alters. Every body at zero potential which is brought into the electric field of the conductor in question raises the charge of the latter, and thereby increases its capacity.

Maxwell defined the capacity of a conductor as the ratio of its charge to its potential, the potential of all neighbouring bodies being zero, as when they are earthed. If there are several conductors K_1, K_2, etc., with charges Q_1, Q_2, etc., in the electric field, the potential at any point equals the sum of the potentials assumed by the same point when each conductor receives its charge separately whilst the others remain uncharged. We have thus a superposition of the electric effects.

If the first conductor K_1 has the charge Q_1, whilst the others remain uncharged and insulated, the potentials of the conductors K_1, K_2, ... will be respectively

$$p_{11}Q_1, \quad p_{12}Q_1, \quad p_{13}Q_1, \quad \text{etc.},$$

where p_{11}, p_{12}, etc., are constant magnitudes depending only on the position and dimensions of the conductors. These constants are known as *potential coefficients.* If conductor K_2 is charged with the quantity Q_2, whilst the others remain insulated and uncharged, the conductors will have the potentials

$$p_{21}Q_2, \quad p_{22}Q_2, \quad p_{23}Q_2, \quad \text{etc.}$$

Hence when the conductors have simultaneously the charges Q_1, Q_2, etc. their potentials will be

$$\left.\begin{array}{l} P_1 = p_{11}Q_1 + p_{21}Q_2 + p_{31}Q_3 + \dots \\ P_2 = p_{12}Q_1 + p_{22}Q_2 + p_{32}Q_3 + \dots \\ \vdots \end{array}\right\} \quad \dots\dots\dots\dots\dots(205)$$

From these equations, we get

$$\left.\begin{array}{l} Q_1 = c_{11}P_1 + c_{21}P_2 + c_{31}P_3 + \dots \\ Q_2 = c_{12}P_1 + c_{22}P_2 + c_{32}P_3 + \dots \\ \vdots \end{array}\right\} \quad \dots\dots\dots\dots\dots(206)$$

The magnitudes c are functions of the magnitudes p, and like the latter are determined by the position and dimensions of the conductors. The magnitudes c are called *capacity coefficients,* when the two suffixes are the same, or simply, the respective capacities. Thus c_{11} is the capacity coefficient or the capacity of the conductor K_1, c_{22} the similar coefficient for conductor K_2, and so on. The magnitudes c, where the two suffixes are different, are called the *mutual capacity coefficients* of the respective conductors. In this case $c_{mn} = c_{nm}$. Thus c_{12} is the mutual capacity coefficient of conductor K_1 relatively to conductor K_2, and so on.

From the last series of equations, it follows: *The capacity or the capacity coefficient of a conductor is equal to the quantity of electricity possessed by the conductor when its potential equals unity, the potential of all other conductors being zero.*

The mutual capacity coefficient of a conductor K_1 relatively to a conductor K_2 equals the quantity of electricity which collects on K_2 when all other conductors except K_1 have zero potential whilst the conductor K_1 is brought to unit potential.

If the conductor K_1 is charged positively whilst the remaining conductors in the field are earthed, the lines of force from conductor K_1 pass into these conductors and away to earth. Obviously, lines of force cannot come from the other conductors, since no point at a lower potential exists in the field. Consequently, there can be no positive charge on any of the other conductors. The sum of the negative charges on the earthed conductors, therefore, can never become numerically greater than the positive charge on the conductor K_1. *From this it is seen that the mutual capacity coefficients must always be negative (or zero), and that the sum of the mutual capacity coefficients is numerically smaller than (or at the most equal to) the capacity coefficient.*

(d) To determine the capacity coefficients experimentally, the method given by Professor Schleiermacher* can be used with advantage. All conductors except the x^{th} are earthed, and the capacity of the x^{th} is then measured; from the above definition this equals the coefficient c_{xx}. Similarly, we proceed with all other conductors, whereby c_{yy}, equal to the capacity of the y^{th} conductor, is obtained. If now all the conductors with the exception of the x^{th} and y^{th} are earthed, whilst these two are joined in parallel, we shall not get the capacity $c_{xx}+c_{yy}$, as would be the case with parallel-connected independent condensers, but a capacity $c_{(x+y)}$ since both conductors mutually affect one another. If we form the system of equations (185) for the two conductors x and y under the assumption that all the remaining conductors are earthed whilst they have the same potential P, then

$$Q_x = c_{xx}P + c_{yx}P,$$

$$Q_y = c_{yy}P + c_{xy}P,$$

and

$$Q_x + Q_y = Pc_{(x+y)}.$$

By eliminating Q_x and Q_y from these three equations, we get

$$c_{xy} = c_{yx} = -\frac{c_{xx} + c_{yy} - c_{(x+y)}}{2}. \qquad (207)$$

If $c_{(x+y)} = c_{xx} + c_{yy}$, as in independent condensers, then $c_{xy} = 0$, which indicates that the two conductors x and y induce no charge on each other.

It follows further that the three capacity coefficients of two conductors can be determined experimentally by three capacity measurements. For three conductors six capacity measurements are necessary and for n conductors $(1+2+3+\ldots+n)$ measurements in order to find all the coefficients.

If one of two conductors acts as a screen to the other, as in two concentric spherical shells, then the lines of force go partly between the two opposing spherical surfaces and partly between the external spherical surface and the outside space. The latter lines are only present, however, when the outer conductor is charged. Hence the outer conductor possesses a capacity equal to the capacity of the

* *E.T.Z.* 1905, p. 1043.

inner sphere increased by the capacity it would have if the internal conductor were not present. With two spherical shells with radii r_1 and r_2 and R_1 and R_2 respectively (Fig. 338), the capacity of the inner shell is

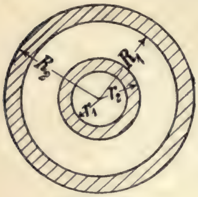


FIG. 338.

$$c_{11} = \frac{r_2 R_1}{R_1 - r_2},$$

and of the outer shell,

$$c_{22} = c_{11} + R_2 = \frac{r_2 R_1}{R_1 - r_2} + R_2.$$

From this the mutual capacity coefficient is

$$c_{12} = -c_{11} = \frac{r_2 R_1}{R_1 - r_2},$$

and $$c_{(1+2)} = 2c_{12} + c_{11} + c_{22} = c_{22} - c_{11} = R_2.$$

If the outer shell is charged, a charge will collect both on its inner and on its outer surfaces, when the inner shell is earthed. On the surface of the inner sphere there will then exist the same charge as on the inner surface of the larger shell.

(e) The formulae (206) for calculating the capacity are inconvenient in many practical cases. Thus in transmission lines, for example, in which there may be several conductors supported by the same poles, each conductor can possess a different potential. In this case it is complicated to calculate the charge on a conductor from formulae (206).

Hence we define in general *the effective capacity of a conductor as the ratio between its charge and its potential.*

Since the effective potential of a conductor depends on the potentials of the other conductors, both the capacities and potentials of the other conductors must always be given. The capacity of a conductor can then in general be found in the same way as above, by calculating the work done in moving unit positive charge from earth to the surface of the conductor.

In calculating this work, not only the charges on the conductor, but also all electric charges in the field must be taken into account.

By way of example, the relation existing between the effective capacity and the capacity coefficients will now be shown in the calculation of the charging current of a double-line of a single-phase alternating-current system with earthed neutral. The potentials of the two lines with respect to earth are p_1 and p_2, where

$$p_1 = -p_2 = \frac{1}{2} P_{\max} \sin \omega t.$$

The charges are

$$q_1 = c_{11} p_1 + c_{21} p_2 = (c_{11} - c_{21}) \frac{1}{2} P_{\max} \sin \omega t,$$

$$q_2 = c_{22} p_2 + c_{12} p_1 = -(c_{22} - c_{12}) \frac{1}{2} P_{\max} \sin \omega t,$$

and the charging currents

$$i_1 = \frac{dq_1}{dt} = (c_{11} - c_{21}) \frac{\omega}{2} P_{\max} \cos \omega t = \frac{\omega}{2} C_1 P_{\max} \cos \omega t,$$

$$i_2 = \frac{dq_2}{dt} = -(c_{22} - c_{12}) \frac{\omega}{2} P_{\max} \cos \omega t = -\frac{\omega}{2} C_2 P_{\max} \cos \omega t,$$

where $C_1 = c_{11} - c_{12}$ and $C_2 = c_{22} - c_{12}$, the effective capacities of each of the two conductors.

If the neutral point of the system is not earthed, the same current $i_1 = -i_2 = i$ will flow in the conductors and

$$i_1 = (c_{11} - c_{21}) \omega P_{1\max} \cos \omega t = \omega C_1 P_{1\max} \cos \omega t,$$

$$i_2 = -(c_{22} - c_{12}) \omega P_{2\max} \cos \omega t = -\omega C_2 P_{2\max} \cos \omega t,$$

whence $$\omega(P_{1\max} + P_{2\max}) \cos \omega t = \frac{i_1}{C_1} - \frac{i_2}{C_2} = i_1\left(\frac{1}{C_1} + \frac{1}{C_2}\right),$$

or $$\omega P_{\max} \cos \omega t = i\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{i}{C},$$

where C is the effective capacity of the double-line. Since

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{c_{11} - c_{12}} + \frac{1}{c_{22} - c_{12}}$$

it follows $$C = \frac{(c_{11} - c_{12})(c_{22} - c_{12})}{c_{11} + c_{22} - 2c_{12}}.$$

In calculating the effective capacities, however, it is not necessary to first determine all the capacity coefficients, but the effective capacity is calculated for the actual conditions, as will be shewn in Chap. XXI.

121. Specific Inductive Capacity. Until now we have assumed that the conductors are surrounded by air. If some other insulator (solid or fluid) other than atmospheric air is brought between the plates of a condenser, it is invariably found that the capacity of the latter is increased. Even in air the capacity is somewhat—although very little—greater than in a vacuum.

(**a**) The ratio of the capacity of a condenser, in which the space between the plates is filled with an insulator, to the capacity of the same condenser when this space is occupied by air (or is a vacuum) is defined as the *specific inductive capacity* of the respective insulator. Since the insulator in this relation is often called the dielectric, the above ratio is frequently referred to as the *dielectric constant* of the particular dielectric.

In what follows, we shall denote this constant by ϵ.

With ordinary gases, ϵ differs only very little from unity, and can therefore be taken as unity for all practical purposes.

All solid and liquid dielectrics have dielectric constants greater than unity.

In the following Table, the dielectric constants for solid and liquid dielectrics in common use are given. The values vary within fairly wide limits—owing to the fact that the materials were of different composition and were investigated under different physical conditions:

Ether	3·4—4·7
Ethyl-alcohol	24·3—27·4
Amyl-alcohol	15
Aniline	7·1
Benzine	1·9
Benzol	2·2—2·4
Methyl-alcohol	32·7
Olive-oil	3—3·16
Ozokerit oil	2·16
Paraffin oil	1·9
Petroleum	2
Rape-seed oil	1·47
Castor oil	4·53
Carbon disulphide	1·7—2·7
Turpentine	2·2
Water (distilled)	76—82
Xylol	2·4
Ebonite	2·1—3·1
Ice	3·0
Glass heavy, easily fusible	2·0—5·0
light, difficult to fuse	5·0—10·0
Mica	5·0—7·0
Rubber	2·35
Vulcanised rubber	2·5—3·5
Gutta percha	3·0—5·0 (usually 4·2)
Impregnated paper or jute	4·3
Colophonium	2·5
Manilla paper	1·8
Marble	6·0
Paper impregnated with turpentine	2·4
Paraffin	2·3
Porcelain	5·3
Shellac	2·75
Sulphur	4·0
Silk	1·6

As the temperature increases, the dielectric constant decreases. Thus if ϵ_0 denote the dielectric constant at t_0°, then at t° we have

$$\epsilon = \epsilon \ + a(t_0^\circ - t^\circ).$$

For the following substances the values of a are

Mica	(between 11° and 110°)	0·0003.
Ebonite	(„ 11° „ 63°)	0·0004.
Glass	(„ 17° „ 60°)	0·0012 to 0·002.
Benzol and Toluol		0·0035.

In the case of some media, the dielectric constant depends on the strength of the electric field.

(**b**) If ϵ is the specific inductive capacity of the dielectric, the potential difference of a condenser is, for the same charge, only $\frac{1}{\epsilon}$ times that of the potential difference in air.

Since (from Eq. 195)
$$P_A - P_B = \int_A^B f_s \, ds,$$
it follows that the strength of the electric field f in a dielectric, for a given charge, is only $\frac{1}{\epsilon}$ times as large as in air. Two electric charges q_1 and q_2, when situated in a dielectric, repel one another with a force
$$K = \frac{1}{\epsilon}\frac{q_1 q_2}{r^2}. \qquad (208)$$

If we represent the field-strength in the dielectric by lines of force, the number of lines leaving positive unit of electricity is $\frac{4\pi}{\epsilon}$.

Between two parallel conducting plates with the surface charge σ, and separated by a dielectric, the field-strength is
$$f = \frac{4\pi\sigma}{\epsilon} = \frac{P}{r}, \qquad (209)$$
where P denotes the potential difference between the plates.

The force acting on unit surface of either of the plates is
$$f_0\sigma = \frac{2\pi\sigma^2}{\epsilon} = \frac{\epsilon}{8\pi}\frac{P^2}{r^2}. \qquad (210)$$

If the surface densities σ are given, the attraction between the plates is therefore inversely proportional to the dielectric constant. On the other hand, for a given potential difference, the attraction between the plates is directly proportional to the dielectric constant.

The capacity for F cm² of the effective surface of a system of plates in a plate condenser is
$$C = \epsilon \frac{F}{4\pi r}, \qquad (211)$$
where ϵ = dielectric constant of the dielectric,—that is ϵ times greater than in air.

(**c**) Gauss's equation (182) for a closed surface surrounded by a dielectric will be
$$\epsilon \int_F f_n \, dF = 4\pi \Sigma q. \qquad (196a)$$

We shall now consider the boundary surface, F, between two dielectrics I and II (Fig. 339) having the dielectric constants ϵ_1 and ϵ_2.

The positive direction of the field-strength f is assumed to be from dielectric I to dielectric II.

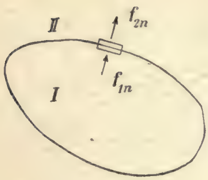


FIG. 339.

It can be deduced from the principle of the conservation of energy, just as in the case of a magnetic field, that the tangential component f_t of the electric field-strength is continuous in passing through the surface F. Let f_{1t} and f_{2t} denote these tangential components at two points very near to one another, but on opposite sides of the boundary surface; then

$$f_{1t}=f_{2t}.$$

Now consider the normal components f_{1n} and f_{2n} of the electric field-strength at two such points. Imagine an extremely short cylinder placed perpendicularly to the surface F with the points at the centres of its end surfaces (see Fig. 339). These end surfaces are parallel to the element dF of the surface considered and both have the same area as dF. Let σ be the surface density on the element, then

$$\epsilon_2 f_{2n}\,dF-\epsilon_1 f_{1n}\,dF=4\pi\sigma\,dF;$$

$$\epsilon_2 f_{2n}-\epsilon_1 f_{1n}=4\pi\sigma.$$

If the surface is uncharged ($\sigma=0$), then

$$\frac{f_{1n}}{f_{2n}}=\frac{\epsilon_2}{\epsilon_1}. \qquad \text{(212)}$$

Thus, in passing from one dielectric to the other, the normal components of the electric field-strength vary inversely as the dielectric constants of the two dielectrics. Thus we have an analogous law for electric lines of force to that for magnetic. Similarly, termini of the electric lines of force occur at the boundary surface, which appear to give electric charges to the surface.

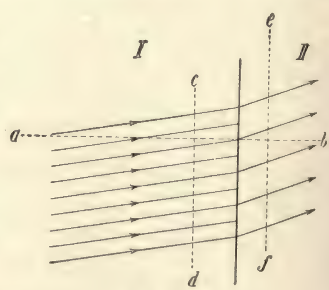


FIG. 340.

Fig. 340 represents the transition of electric lines of force from one medium I to another medium II having double the dielectric constant. One half of the lines terminate at the surface, the other half pass out at an angle which is inclined to the normal, such that its tangent is twice that in medium I. A horizontal plane a—b cuts the same number of lines of force per unit of surface in both media. A vertical plane c—d in medium I will cut twice as many lines per cm² as a vertical plane e—f in II.

At the boundary surface of the two insulators there will be an apparent electric surface charge, whose density σ_s will be given by the following equations:

$$\epsilon_2 f_{2n} - \epsilon_1 f_{1n} = 0,$$

$$f_{2n} - f_{1n} = 4\pi\sigma_s,$$

whence
$$\sigma_s = \frac{\epsilon_1 - \epsilon_2}{\epsilon_2}\frac{1}{4\pi}f_{1n} = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1}\frac{1}{4\pi}f_{2n}.$$

Let an insulator be brought into an insulating medium of smaller dielectric constant, then where the electric lines of force enter, there is an apparent negative, and where they leave, an apparent positive surface charge. Such an apparent electric charge is called the *influence electricity* of the insulator. It corresponds to the magnetic surface charge of paramagnetic substances, and vanishes as soon as the insulator is removed from the electric field. It disappears also when the insulator is divided into two parts while in the field, the one part containing the positive and the other the negative apparent charge, and the individual parts are removed out of the field. The same holds also for the magnetic surface-charge.

On the other hand, a conductor retains its charge in the latter case.

(**d**) By the term *induction flux* through an element of surface dF, we mean the magnitude

$$d\phi = \epsilon f_n dF, \quad \text{.............................(213)}$$

where f_n denotes the electric field-strength normal to the elemental surface. The ratio

$$b_n = \frac{d\phi}{dF} = \epsilon f_n \quad \text{...........................(214)}$$

can be defined as the *induction* or *polarisation* in the direction normal to the surface element dF at the place considered. In air or vacuum the induction coincides with f_n. In dielectrics, b is always greater than f. From positive unit charge there are always 4π induction lines leaving and into negative unit always 4π lines entering, no matter whether the charge is placed in air or in some other insulator. Induction lines only start and finish at actual electric charges, and not at apparent charges on insulators. In passing through the boundary surface between two insulators, the normal components of induction remain continuous, whilst the tangential components are in proportion to the dielectric constants. Thus we have

$$\left.\begin{aligned} b_{1n} &= b_{2n} \\ \frac{b_{1t}}{b_{2t}} &= \frac{\epsilon_1}{\epsilon_2} \end{aligned}\right\} \quad \text{.............................(215)}$$

At such a boundary surface no induction lines will terminate, provided there is no actual electric charge on the same.

(e) Let two conducting plates M_1 and M_2 (Fig. 341), charged with $+Q$ and $-Q$, be separated from one another by insulators of different dielectric constants ϵ_1, ϵ_2, ϵ_3 and of thickness r_1, r_2, r_3. The density of the charge is

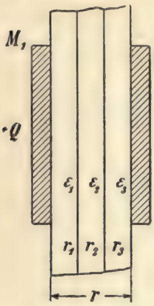


FIG. 341.

$$\sigma = \frac{Q}{F},$$

where F denotes the effective surface of a plate. The induction between the plates can be taken as constant, i.e.

$$b = 4\pi\sigma.$$

Since the electric field-strength is inversely proportional to the dielectric constants, we have

$$f_1 = \frac{b}{\epsilon_1}; \quad f_2 = \frac{b}{\epsilon_2}; \quad f_3 = \frac{b}{\epsilon_3}. \quad \dots\dots\dots\dots\dots(216)$$

Let P be the total potential difference between the two plates, and P_1, P_2 and P_3 the potential differences between the several boundary surfaces; then

$$P = P_1 + P_2 + P_3 = b\left(\frac{r_1}{\epsilon_1} + \frac{r_2}{\epsilon_2} + \frac{r_3}{\epsilon_3}\right) = 4\pi\sigma\left(\frac{r_1}{\epsilon_1} + \frac{r_2}{\epsilon_2} + \frac{r_3}{\epsilon_3}\right).$$

The capacity of the system per unit of effective surface of a plate is therefore

$$C = \frac{1}{4\pi\left(\frac{r_1}{\epsilon_1} + \frac{r_2}{\epsilon_2} + \frac{r_3}{\epsilon_3}\right)}. \quad \dots\dots\dots\dots\dots\dots\dots\dots(217)$$

Putting $\quad \dfrac{4\pi r_1}{\epsilon_1} = \dfrac{1}{C_1}; \quad \dfrac{4\pi r_2}{\epsilon_2} = \dfrac{1}{C_2}; \quad \dfrac{4\pi r_3}{\epsilon_3} = \dfrac{1}{C_3},$

where C_1, C_2 and C_3 represent the capacity per cm² for each of the dielectrics at the given thicknesses, then we have

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}, \quad \dots\dots\dots\dots\dots(217a)$$

i.e. the capacity of a condenser, whose dielectric consists of several parts, equals the resultant capacity obtained when the capacities of the several parts are connected in series. The potential differences P_1, P_2, P_3 between the several boundary surfaces equal the terminal pressures which act across the several condensers C_1, C_2 and C_3 when P is applied at the terminals. Hence the condenser (Fig. 341) can be replaced by the connection shewn in Fig. 342.

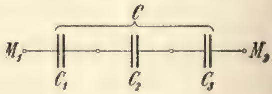


FIG. 342.

Let C_l be the capacity when we have air between the plates, then the ratio between the capacities is

$$\frac{C}{C_l} = \frac{r_1 + r_2 + r_3}{\frac{r_1}{\epsilon_1} + \frac{r_2}{\epsilon_2} + \frac{r_3}{\epsilon_3}} = \frac{r}{\frac{r_1}{\epsilon_1} + \frac{r_2}{\epsilon_2} + \frac{r_3}{\epsilon_3}}.$$

Thus the capacity is increased by introducing the dielectrics into the field.

If we make $\epsilon_2 = \epsilon_3 = 1$ and $r_2 + r_3 = r_0$, i.e. we place only dielectric ϵ_1 of thickness r_1 between the plates, the remainder of the field being in air, then for the same charge Q, the field-strength in the air remains the same as if the whole space were filled with air. The potential difference between the plates is reduced to the value

$$P' = 4\pi\sigma\left(\frac{r_1}{e_1} + r_0\right),$$

which is $4\pi\sigma\left(1 - \frac{1}{\epsilon_1}\right)r$ times less than that existing when the plates are separated by air. The introduction of the dielectric of thickness r_1 has the same effect as if the plates were brought nearer together by the amount

$$\left(1 - \frac{1}{\epsilon_1}\right)r_1.$$

For the same potential difference between the plates, the electric field-strength in the air is increased in the ratio

$$\frac{\dfrac{r}{r_1}}{\dfrac{r}{r_1} - 1 + \dfrac{1}{\epsilon_1}}.$$

The capacity of the plates increases in the same ratio when the dielectric is inserted. The electric field-strength in the inserted dielectric is $\frac{1}{\epsilon_1}$ times that in the air, and is thus:

$$\frac{\dfrac{1}{\epsilon_1}\dfrac{r}{r_1}}{\dfrac{r}{r_1} - 1 + \dfrac{1}{\epsilon_1}}$$

times the field-strength in the air before the dielectric was introduced.

When a conducting plate of thickness r_1 is placed between the two plate conductors, we have only to insert $\epsilon_1 = \infty$ in the above equations. The two charged plates behave in exactly the same way as if they were brought nearer together by the amount r_1. Provided the inserted plate is insulated, its position between the charged plates is quite immaterial, as in the case in which a dielectric is inserted.

122. The Energy in the Electric Field. Similar to the magnetic field energy $\frac{1}{2}\Sigma i\Phi$, the energy required for the production of the electric field is,

$$\left.\begin{aligned} A &= \tfrac{1}{2}\Sigma QP = \tfrac{1}{2}(Q_1P_1 + Q_2P_2 + Q_3P_3 + \ldots) \\ &= \tfrac{1}{2}(p_{11}Q_1^2 + p_{22}Q_2^2 + \ldots) + (p_{12}Q_1Q_2 + p_{13}Q_1Q_3 + \ldots) \\ &= \tfrac{1}{2}(c_{11}P_1^2 + c_{22}P_2^2 + \ldots) + (c_{12}P_1P_2 + c_{13}P_1P_3 + \ldots) \end{aligned}\right\} \quad \ldots(218)$$

If the charged plates are insulated so that their charges remain constant, the work done by a displacement of the plates in the field is equal to the energy lost by the system due to the displacement. The forces exerted by the field on the plates tend to move the latter, so that the energy in the field is a minimum.

If, on the other hand, the potential of the plates is kept constant, as is the case, for example, when the plates are connected to galvanic batteries, the forces acting on the plates tend to displace the latter, so that the energy of the field is a maximum. In this case, the work done by the forces due to the displacement in the field equals the increase of energy in the system. Both the mechanical work done and the increase of energy in the field is taken from the batteries to which the system is connected.

The equation for the energy of a system of conducting plates holds good independently of the dielectric in which the conductors may be situated.

(**a**) If two parallel plates have a surface density σ and a potential difference P, then the energy per cm² of the internal surface of either of the two plates is

$$\tfrac{1}{2}\sigma P.$$

The constant electric field-strength in the space between the plates is

$$f=\frac{P}{r},$$

where r is the distances between the plates.

If the space between the plates is filled by a dielectric whose constant is ϵ, then

$$\sigma=\frac{\epsilon f}{4\pi}, \quad \text{.............................}(197a)$$

and the energy per unit of volume of the dielectric is accordingly

$$\frac{1}{2}\frac{\sigma P}{r}=\epsilon\frac{f_n^2}{8\pi}. \quad \text{.............................}(219)$$

This equation holds quite generally for a field f_n in any dielectric. For a given electric field-strength (or potential difference) the energy in the dielectric is thus proportional to the dielectric constant.

Since the induction

$$b=f\epsilon,$$

it follows that the expression for the energy is

$$\frac{bf}{8\pi}=\frac{b^2}{8\pi\epsilon}. \quad \text{.............................}(219a)$$

For a given induction (or charge) the energy in the dielectric is inversely proportional to the dielectric constant.

The two surfaces of the plate condenser are attracted by a force

$$f_0\sigma = \tfrac{1}{2}\sigma f = \tfrac{1}{2}\sigma\frac{P}{r}$$

per cm², and exert a pressure on the dielectric equal to the energy per unit volume stored up in the same. We thus see that the stored-up energy in the electric field (like the stored-up energy in a magnetic field) causes a mechanical strain between the respective bodies. From this follows that the energies of the electric and magnetic fields do not reside in the magnetic and electric charges—as indicated by the formula from which they are calculated—but, as first pointed out by Maxwell, in the media of the fields.

(b) From the law of minimum field-energy it follows that a small uncharged conductor, exerting no perceptible influence on the field distribution in the neighbouring space, tends to move in that direction in which the field-strength increases.

An uncharged conductor in a uniform field does not experience any resultant transverse force; nevertheless it strives to set itself—just like a piece of iron in a uniform magnetic field—so that its longitudinal axis coincides with the direction of the electric field. This is due to the fact that unit volume of the body in this position can embrace the greatest number of lines of force and neutralise the same.

The following method, which is often used to represent electric lines of force diagrammatically, is based on this phenomenon. It is similar to the representation of magnetic lines of force by means of iron filings. If an insulating liquid is mixed with an insoluble powder possessing a greater dielectric constant than the liquid, and the whole is placed in an electric field, the powder will set itself in lines which run parallel to the electric lines of force.

A positively charged conductor in a uniform field is acted on by a resultant force along the positive direction of the field, since in this direction the field is strong and in the opposite direction weak. When a movement occurs in this direction, the space in which the field is strong is reduced and that in which the field is weak is increased, so that the total energy in the field decreases (see Fig. 343).

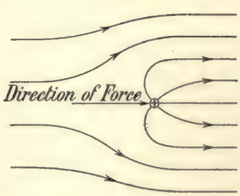


Fig. 343.

(c) The insulator also, like the conductor—in consequence of the principle of minimum energy in the field—tends to embrace as many induction lines as possible when it is surrounded by a medium of smaller dielectric constant.

If it has a longitudinal shape, it tends to set itself with this axis parallel to the electric lines of force. If the field is not uniform, it tends to move in the direction in which the field-strength increases.

When an insulated sphere is brought into a uniform field in a medium having half the induction capacity of the sphere, then we get

a distribution somewhat as shewn in Fig. 344. Lines of force and induction lines are drawn full, whilst induction lines alone are shewn dotted.

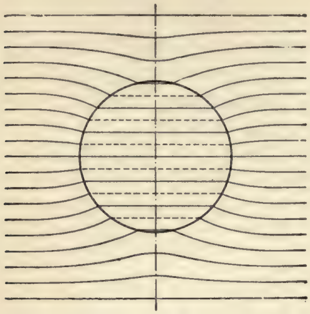


FIG. 344.—Spherical Insulator in a Medium with smaller Dielectric Constant.

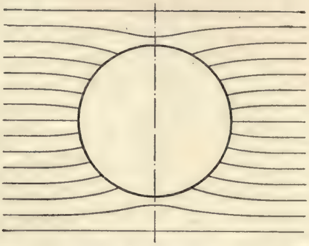


FIG. 345.—Spherical Conductor in Electric Field.

In comparison with this, the influence of a conducting sphere on a uniform field is shewn in Fig. 345. All lines of force and induction terminate at the influenced charges on the surface of the sphere.

123. Electric Displacement.

(a) By the electric displacement at a point in a medium we mean a vector whose absolute value is

$$\dot{j}=\frac{\epsilon}{4\pi}f=\frac{b}{4\pi}, \qquad (220)$$

and whose direction coincides with that of the electric field-strength f.

Just outside a charged surface of a conductor with surface density σ the displacement is

$$\dot{j}=\sigma, \qquad (221)$$

and is directed outwards in the case of a positive charge or inwards in the case of a negative charge.

Inside a conductor $\dot{j}=0$, since here $f=0$.

In passing from one dielectric ϵ_1 to another ϵ_2, the normal components of the electric displacement remain constant, provided there is no real charge on the boundary surface.

$$\dot{j}_{n1}=\dot{j}_{n2}=\frac{\epsilon_1 f_{n1}}{4\pi}=\frac{\epsilon_2 f_{n2}}{4\pi}. \qquad (222)$$

On the other hand, the tangential components are different, for

$$\dot{j}_{t1}=\frac{\epsilon_1 f_{t1}}{4\pi}; \qquad \dot{j}_{t2}=\frac{\epsilon_2 f_{t2}}{4\pi};$$

and since

$$f_{t1}=f_{t2}=f_t,$$

then

$$\frac{\dot{j}_{t1}}{\dot{j}_{t2}}=\frac{\epsilon_1}{\epsilon_2}. \qquad (223)$$

For electric displacement, therefore, the same law of discontinuity holds as for electric field-strength and electric induction.

A unit tube of electric displacement encloses 4π unit tubes of electric induction, and is directed from the positive to the negative unit charge.

The displacement flux through a closed surface F is, from Gauss's law,

$$\phi=\int_F j_n dF=\frac{\epsilon}{4\pi}\int f_n dF=\Sigma q, \quad \text{.................(224)}$$

where Σq equals the quantity of electricity enclosed by the surface.

(b) An electric difference of potential can only produce a constant electric flux, i.e. a continuous-current, in metallic conductors, whilst it places the dielectrics in a state of strain which can be regarded as an elastic displacement. Consequently a continuous current cannot flow in a circuit in which a condenser is connected, when once steady conditions are reached, that is, when the charging current ceases. With alternating-currents it is different, because here the condenser is always being charged and discharged, whereby the dielectric is subjected to displacements pulsating to and fro with the current. Hence, in an alternating-current circuit with a condenser, the charging current of the condenser will flow. Maxwell designated the currents in the condenser as *displacement currents*, and asserted *that such currents obey the same laws as ordinary electric currents*, except that no heating losses occur in the dielectric. This not only holds for the displacement current in the condenser, but also for all the other displacement currents in the dielectrics of the electric fields. The magnitude of the displacement current i is the quantity of electricity which conveys unit quantity to the surface normal to its direction at the instant the polarisation of the dielectric occurs. Consequently, the displacement current $\frac{d\phi}{dt}$ has the dimension $\frac{\text{electric flux}}{\text{time}}$ or $\frac{\text{electric charge}}{\text{time}}$, i.e. $(L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2})$ in the electrostatic system of units. If the displacement current is to be treated like an ordinary current, it must be expressed in electromagnetic units. In this system, current has the dimension $(L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1})$. The ratio of the current in electrostatic units to that in electromagnetic has therefore the dimension (LT^{-1}), that is the dimension of a velocity. The value of this ratio has been experimentally determined, and is approximately 3×10^{10} cm/sec. This agrees with the velocity of light v in a vacuum, which Maxwell explained on the ground that electric charges must move at very high velocities in order to exert the same effect on magnets as ordinary currents.

From this ratio v between currents in the two systems, it follows that the practical unit of current

$$\begin{aligned}1 \text{ ampere} &= 0{\cdot}1 \text{ C.G.S. electromagnetic unit}\\ &= 3 \,.\, 10^9 \text{ C.G.S. electrostatic units.} \quad \text{..........(225)}\end{aligned}$$

The same ratio exists between the units of electric quantity in the two systems:

$$\begin{aligned} 1 \text{ coulomb} &= 0{\cdot}1 \text{ C.G.S. electromagnetic unit} \\ &= 3 \,.\, 10^9 \text{ C.G.S. electrostatic units.} \qquad (226) \end{aligned}$$

The ratio between the units of potential in the several systems of units can be found by considering that the expression for the energy consists of the two factors, electric quantity and potential, i.e. the units of potential must bear to one another the inverse ratio to that of the units of electric quantity.

We have thus:

$$\begin{aligned} 1 \text{ volt} &= 10^8 \text{ C.G.S. electromagnetic units} \\ &= \tfrac{1}{300} \text{ C.G.S. electrostatic units,} \qquad (227) \end{aligned}$$

or $$1 \text{ C.G.S. electrostatic unit} = 300 \text{ volts.}$$

For the units of capacity, we have:

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = \frac{3 \,.\, 10^9}{\frac{1}{300}} = 9 \,.\, 10^{11} \text{ electrostatic units,} \qquad (228)$$

or $$\begin{aligned} 1 \text{ microfarad} &= 9 \,.\, 10^5 \text{ C.G.S. electrostatic units} \\ &= 9 \text{ kilometres,} \qquad (229) \end{aligned}$$

i.e. a sphere of 9 kilometres radius has a capacity of 1 microfarad.

For the displacement flux in the electro-magnetic system of units, we have the expression

$$\phi = \int \frac{j_n dF}{v} = \frac{\epsilon}{4\pi v} \int f_n dF, \qquad (224a)$$

and the displacement current is $i = \dfrac{d\phi}{dt}$.

(c) Starting from the hypothesis that the displacement current obeys the same law as the ordinary current, Maxwell developed the equations for the distribution of the electric and magnetic forces, and the propagation of their variations in space. It will only be mentioned here that Maxwell's equations can be deduced from the fundamental law of electro-magnetism,

$$4\pi i = \int_{C_1} H_l \, dl,$$

where C_1 is a closed curve interlinked with the current i, and from Maxwell's fundamental law of electromagnetic induction.

After inserting the electric field-strength, this is

$$-\frac{d\phi}{dt} = e = \int_{C_2} f_s d_s, \qquad (230)$$

where C_2 is a closed curve embracing the flux ϕ. This method of deducing Maxwell's equations is that given by Galileo Ferraris.* One deduction from Maxwell's equations is that the electric and magnetic

* *Wissenschaftliche Grundlagen der Elektrotechnik.*

forces move in vacuo with the velocity of light. The electric and magnetic forces form an angle of 90° and are both transverse to the direction of propagation; they travel by means of oscillations just like heat and light waves. As a strict consequence of Maxwell's equations, we have the following hypothesis due to Poynting: "The direction in which energy travels through an electromagnetic field is always perpendicular to the directions of the magnetic and electric field-strengths: Through each unit of area of the plane normal to the direction in which the energy is propagated, the quantity of energy passing per second is equal to the area of the parallelogram (Fig. 346) whose sides are measured by the electric and magnetic field-strengths, divided by 4π."

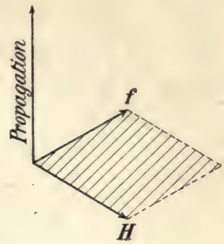


FIG. 346.

From Poynting's hypothesis, the energy in a transmission line is not propagated in the conductors, but in the surrounding dielectrics. The conductor does not represent a channel along which the energy travels, but a space in which a part of the energy converges and in which this part is converted into heat.

CHAPTER XX.

ELECTRIC PROPERTIES OF THE DIELECTRICS.

124. Conductivity and Absorptivity. 125. Energy Losses in the Dielectric. 126. Influence of the Specific Inductive Capacity and Conductivity of the Dielectric on the Distribution of the Electric Field-strength. 127. Dielectric Strength.

IN Chap. XIX. mention was made of the difference in dielectrics in respect of their inductive capacity. Other electrical properties are also possessed by dielectrics, and as these properties are important in practice, they will therefore be shortly dealt with here.

124. Conductivity and Absorptivity.

(a) When the two conductors of a cable or the two plates of a condenser having either a solid or fluid dielectric are connected through a galvanometer with the terminals of a continuous-current machine of constant pressure, it is found that a large current flows at the first instant, thus charging the condenser. This charging current does not sink immediately to zero, but decreases comparatively slowly, until after a fairly considerable time it reaches an almost constant and usually very low value. The explanation of this is partly that the dielectrics have a certain small electric *conductivity*, due to which a current of conduction is added to the charging current. The conduction of the dielectrics may be purely metallic or accompanied by electrolysis. The latter effect is avoided as much as possible on account of damage done to the insulation. Regarding the conductance of the dielectric as constant, then an actual condenser can be replaced by an ideal condenser with a perfectly insulating dielectric and a parallel-connected ohmic resistance. Such an equivalent scheme is shewn in Fig. 347, which can be used for calculating the time of discharge of a condenser when left to itself, i.e. completely insulated. The discharge takes place in accordance with the equation

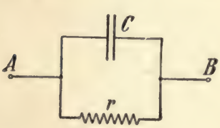


FIG. 347.

$$q = Q\epsilon^{-\frac{t}{rC}}, \quad \text{.............................(231)}$$

where Q is the initial charge and t is the time of discharge in seconds.

The conductance of the dielectric generally increases with the temperature and with the electric tension. Media, which retain their chemical composition at high temperatures, such as glass, porcelain, etc., become comparatively good conductors when raised to their melting temperature. An interesting application of this phenomenon is the Nernst glow lamp. The dielectric forming the glowing filament of the lamp in this case consists of magnesia—the latter is warmed up by a special attachment, whereby the conductance increases to such an extent that an appreciable current begins to flow through the filament which brings the same to incandescence.

Dielectrics have in general a negative temperature coefficient.

Further, the resistance of dielectrics depends largely on the electric conditions (thus on the strength of the electric field)—decreasing as these become more stringent.

The following table gives the specific resistances for several insulating materials at ordinary temperatures, and for average electric conditions:

Material.	Specific Resistance ρ_i in megohms per cm/cm².	Degrees Centigrade.
Gutta-percha	7×10^9	0
	$0{\cdot}45 \times 10^9$	24
Wires insulated with Gutta-percha	$0{\cdot}2 \times 10^9$	24
Pure Rubber	$10{\cdot}9 \times 10^9$	24
Vulcanised Rubber	$1{\cdot}5 \times 10^9$	15
Paper impregnated with Turpentine	3×10^9	15
Jute impregnated with Turpentine	$11{\cdot}9 \times 10^9$	15
Shellac	9×10^9	28
Paraffin wax	24×10^9	
Mica	$0{\cdot}084 \times 10^9$	

The effect of the temperature on the insulation resistance of a transformer (curve A) and of dry cloth (curve B) is shewn in Fig. 348. With the cloth the resistance increases at first with the temperature until the moisture has been driven out, and then for still higher temperatures it falls again to a value of only a few megohms.

(b) Prof. Schleiermacher* has proposed the use of the same expressions for the currents due to conduction as used by Maxwell for the charging currents, when several conductors at different potentials are placed in the electric field. These conduction currents for the several conductors are:

$$i_1 = g_{11}P_1 + g_{12}P_2 + g_{13}P_3 + \cdots,$$
$$i_2 = g_{12}P_1 + g_{22}P_2 + g_{23}P_3 + \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

* *E.T.Z.* 1905, p. 1043.

where the coefficients with like suffixes g_{11}, g_{22}, g_{33} ... denote the ratio of the conduction current to the potential above earth, when all the other conductors are earthed. The coefficients with unlike suffixes

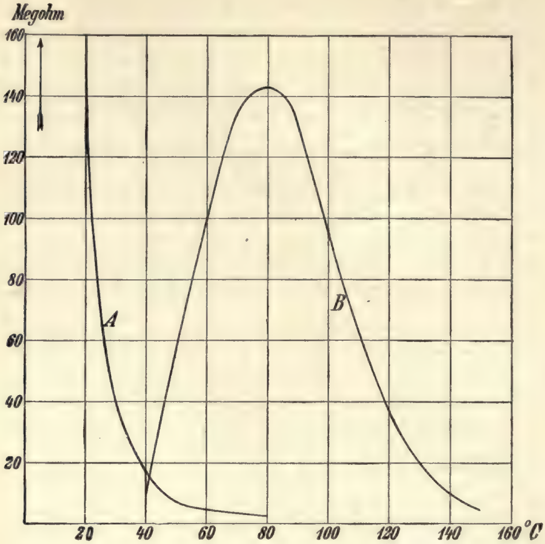


FIG. 348.—Relation between Insulation Resistance and Temperature. A, for Transformer; B, for dry Cloth.

correspond to the mutual capacity coefficients, defined as follows: g_{xy} denotes the current flowing from conductor y to conductor x, when the former has unit potential and all other conductors have zero potential. The experimental determination of these coefficients is quite similar to that adopted for capacities.

To determine g_{xx}, all the conductors except the x^{th} are earthed, and the ratio of the conduction current i of the x^{th} conductor to its potential P is measured.

$$g_{xx}=\frac{i}{P}.$$

In the same way g_{yy} is determined for the y^{th} conductor and $g_{(x+y)}$ for the x^{th} and y^{th} together. Then it follows

$$g_{xy}=g_{yx}=-\frac{g_{xx}+g_{yy}-g_{(x+y)}}{2}. \quad \text{....................(232)}$$

(**c**) The *slow* falling off of the charging current with time is not explained by assuming a constant conductance for the dielectric, but must be considered in connection with the phenomena which occur when a condenser is discharged.

If the two plates of a charged condenser are connected through a galvanometer, at first a large current will flow, which gradually begins to sink, and only after some considerable time vanishes altogether. If the connection is broken after the first rush of current and made again after some time, another but weaker rush of current will ensue in the same direction as the first. The condenser can thus give several such discharges, which gradually become feebler and feebler. This phenomenon is due to the *residual charge* in the dielectric. The explanation of the phenomenon was first given by Maxwell. According to him, the residual charge is due to the heterogeneous nature of most dielectrics.

Fig. 349 shows a section through the dielectric of a condenser, whose plates are A and B. Assume the dielectric consists of the layers D and D', having different properties. As shewn on p. 398, such a condenser can be replaced by two condensers C and C' connected in series (Fig. 350). If the dielectric D' is not a perfect

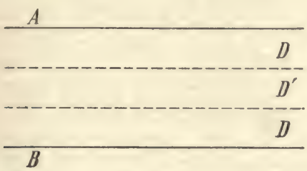


FIG. 349.

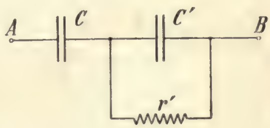


FIG. 350.

insulator, we must suppose an ohmic resistance r' to be connected in parallel with the condenser C'. Fig. 350 thus gives the equivalent scheme of the condenser in which the dielectric D is a perfect insulator.

Whether the above mentioned action of the several layers of a dielectric is the sole cause of the residual charge or whether other influences, e.g. chemical action (similar to that in an electric accumulator) are at work, is not yet certain. Certainly very heterogeneous dielectrics have specially large residual charges, but even quite homogeneous liquid dielectrics appear to shew traces of the same.

Since transmission lines and all electrical apparatus subject to high potential differences act as condensers, the formation of residual charges (or the so-called *absorption* of the dielectrics) must not be left out of account when working with high pressure currents, otherwise serious consequences may follow. If for example a cable or transformer is disconnected from the high-pressure terminals, the disconnected apparatus should be first connected to earth before it is touched. A single earthing, however, is not always sufficient, since charges may afterwards collect and may give dangerous shocks. Special attention should be given to earthing where high direct-current pressures are concerned, since the liability to residual charge is greater in this case.

As a practical case in which all parts of the dielectric possess conductance, the equivalent scheme shewn in Fig. 351 may be taken. According to the above, a residual charge should not occur when the ratio of the dielectric constant to the electric conductivity is the same at all points in the dielectric.

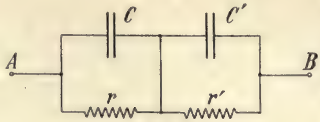


FIG. 351.

125. Energy Losses in the Dielectric.

(a) The energy loss in a dielectric placed in a constant field is given by the leakage current. If, however, the electrification is alternating, as for instance in a condenser to which an alternating pressure is applied, the losses are in general much greater than those corresponding to the insulation resistance. The cause of these additional losses has not yet been thoroughly investigated. It may be due to the absorptivity of heterogeneous dielectrics, discussed in the previous section.* In the dielectric represented in Fig. 350 a loss will occur when an alternating pressure is applied, but not with a continuous pressure. Also in the scheme in Fig. 351, the loss is greater with alternating-current than with continuous when $\frac{C}{r} \gtrless \frac{C'}{r'}$, i.e. when the ratio of specific inductive capacity to resistance of the several parts of the dielectric varies. These losses are often supposed to be due to what is called *dielectric hysteresis*—of a similar nature to magnetic hysteresis.

Steinmetz † found for practical condensers made of paraffined paper, with tin-foil plates dried in a vacuum-drying oven and steeped in paraffin, that the losses at constant frequency increase with the square of the pressure, which corresponds to a constant conductance g for the condenser. Since the dielectric constant, and with it the capacity or the susceptance b of the condenser, is—under normal conditions—independent of the pressure, the phase displacement of the charging current remains constant, at a given frequency. If the thickness of the dielectric of a condenser be increased, the current remains the same for the same electric field-strength, whilst the pressure increases in proportion with the thickness. The loss then increases in proportion with the thickness of the dielectric, so that the phase displacement of the charging current remains constant for the same frequency. Thus for a given frequency, every dielectric has a constant phase displacement. Steinmetz found for the above paper condensers, $\cos\phi = 0{\cdot}0038$ to $0{\cdot}0068$, according to the frequency.

* Hess, *L'Eclairage Electr.* 1895, vol. 4, p. 205.

† *El. World*, 1901, vol. 37, p. 1065.

For the power factor of the charging current in electric cables, we have the following values:

0·01 to 0·025 for paper and jute cables,
0·02 to 0·04 for rubber cables,
0·03 to 0·07 for gutta-percha cables.

(b) The capacity generally decreases somewhat as the frequency increases, which is easily explained by the action of the heterogeneous nature of the dielectric mentioned in the previous action.

In the scheme in Fig. 350, for example, let the capacity for continuous charge be C, then for rapid charge and discharge it will be $\frac{CC'}{C+C'}$. In a paraffined paper condenser it was found by Eisler * that the capacity was 2·5 mfs. for continuous charge; 2·15 mfs. for $c=18$ cycles; and 2·01 mfs. for $c=45$ cycles.

The decrease of the effective capacity of condensers with increasing frequency must be specially noted in measurements. It follows also that the dielectric constant of a dielectric will vary with the frequency at which the determination is made. To eliminate absorption phenomena as far as possible, such determinations are often made with very high frequencies, as with Hertzian waves.

At constant pressure, the losses in the dielectric increase with the frequency. The energy absorbed per cycle usually increases at first as the frequency is increased, attains a maximum, and at higher frequencies may decrease. Eisler found an increase of 17 % in the losses per cycle from 18 to 45 cycles. In the experiment of Steinmetz mentioned on p. 411, the loss per cycle increased up to a frequency of about 100, and began to fall at higher frequencies.

Since the conductance g of a condenser is always small compared with the susceptance b, we can write

$$\cos\phi = \frac{g}{\sqrt{g^2+b^2}} \simeq \frac{g}{b} = \frac{\frac{P^2 g}{c}}{P^2 2\pi C}.$$

Since C only varies slightly with the frequency, the power factor will vary in the same way as the losses per cycle.

The inconstancy of the losses per cycle is explained by many as a kind of viscous hysteresis; or the same phenomena may be deduced from the equivalent scheme for non-homogeneous dielectrics (Fig. 351).

126. Influence of the Specific Inductive Capacity and Conductivity of the Dielectric on the Distribution of the Electric Field-strength.

(a) If layers of various dielectrics are placed between the plates of a condenser, then—provided no conductance is present—the distribution of the electric field-strength will vary inversely as the dielectric constants. Thus a uniform field can by this means be made non-uniform.

* *Zeitschr. f. Elektr.* 1895, H. 12, p. 345.

Conversely, a non-uniform field can be made more or less uniform by the use of various dielectrics.

Considering a long conductor of radius r (Fig. 352), having potential P and surrounded by a co-axial, conducting cylinder of radius R and potential zero; then at distance ρ from the axis, let the dielectric constant be ϵ. The electric induction at this distance is, according to Gauss's theorem,

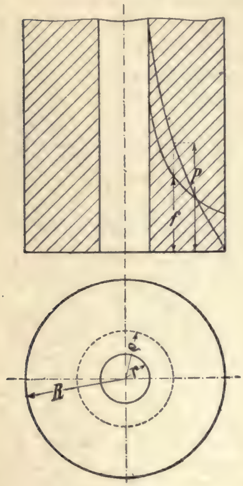


FIG. 352.

$$b=\frac{4\pi Q}{2\pi\rho}=2\frac{Q}{\rho},$$

where Q = electric charge per cm length of conductor.

The electric field-strength is therefore

$$f=\frac{b}{\epsilon}=\frac{2Q}{\epsilon\rho}, \quad \text{..............(233)}$$

i.e. if the dielectric constant ϵ is constant throughout, the field-strength will vary inversely as the distance from the axis of the wire, as the figure shews. The variation of the potential $P=\int_{\rho=R}^{\rho}-f\,d\rho$ is shewn by the second curve P. If, however, we wish to keep the electric field-strength constant, an insulator must be used whose dielectric constant is inversely proportional to the distance away from the axis of the conductor. This can be obtained by using various insulating materials in several layers.

Moreover, it follows from the integration to the limit R, that air-bubbles and other irregularities in the insulating material are to be avoided, both in compound and solid cables. With stranded cables, on account of the small radius of the single wires, the maximum electric field-strength is 25 to 40 % greater than with solid cables or lead-covered stranded cables.

On p. 401 we have seen that particles of a dielectric having a larger dielectric constant than the neighbourhood, tend to move in the direction in which the field increases. In a liquid or semi-liquid dielectric, such particles would assist in forming a uniform distribution of field, which is of importance, as will be shewn later in connection with the piercing strength, and this property can be utilised in cables.

(b) The distribution of the electric field-strength is only determined by the dielectric constants when no conductor is present, or when the field is alternating so rapidly that the conduction currents are negligible compared with the displacement currents. Otherwise the specific resistances determine the distribution. *In a uniform and constant field, the electric field-strength distributes itself according to the specific resistances of the several layers of the dielectric.* If a constant potential difference be applied at the terminals A and B in Fig. 351, the pressures P and

P' of the condensers C and C' will have the ratio of r to r', and are independent of the magnitudes of the capacities C and C'.

In a non-uniform field produced by a direct pressure, a constant electric field-strength can be obtained by giving to each part of the dielectric a specific conductivity, proportional to the induction in the field at the respective point. In Fig. 352, for example, the conductivity of the dielectric at any point is inversely proportional to the distance of the point from the axis of the conductor. Use is made of this in the insulation of cables by saturating the inner layers of the insulation with a liquid of higher conductivity than the outer.*

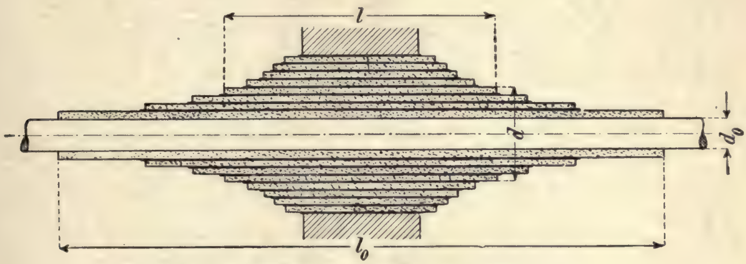


FIG. 353.—Wall-insulator for High-tension Lines.

In some cases an approximately uniform distribution of field-strength throughout the dielectric may be obtained by an arrangement due to the Siemens-Schuckert-Werke.† The insulator is composed of thin layers separated from one another by a conductor (tin-foil). Fig. 353 shews a leading-in tube for high-tension alternating-current made on this principle. The tin-foil is shewn by full lines and the insulating layers dotted.

d_0 = diameter of wire,

l_0 = length of the inner layer of insulation,

d_n = diameter of the hole in the wall,

l_n = length of the hole in the wall.

For a sheet of tin-foil of length l and diameter d, we have

$$ld = l_0 d_0 = l_n d_n.$$

The layers act therefore, neglecting electrical leakage, like a number of condensers of equal capacity connected in series, and each layer takes up the same pressure.

Moreover, with this type of leading-in tube, the harmful discharges between wall and wire disappear. With the ordinary leading-in

* O'Gorman, "Insulation of Cables," *Journ. Inst. E.E.* 1900, xxx. p. 608.

† R. Nagel, *Elektr. Bahnen und Betriebe*, 1906, p. 278.

tube, as shewn in Fig. 354, large surface discharges occur, and are unavoidable even with very long insulators. The following consideration will make this clear. Each element of the conductor with

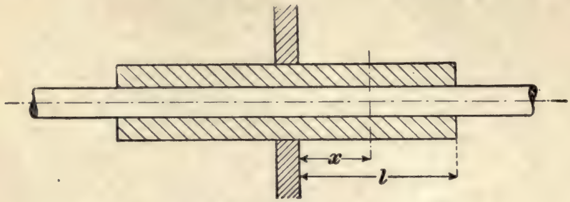


FIG. 354.—Wall-insulator for Low-tension Lines.

its insulation forms a small condenser, of which the primary plates are metallically connected by the conductor and the secondary plates are connected in series through the surface resistance, as shewn diagrammatically in Fig. 355. If r is the surface resistance per unit

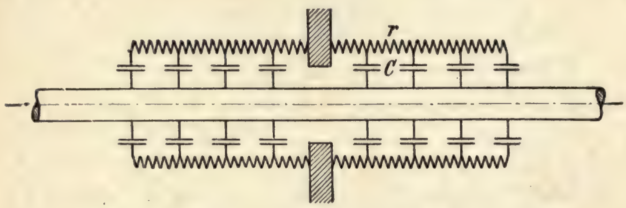


FIG. 355.—Equivalent Circuit of Wall-insulator.

length of the insulator and C the capacity, the potential along the whole surface is distributed according to the same exponential law,

$$P_x = P\frac{\epsilon^{(1-j)\lambda x} - \epsilon^{-(1-j)\lambda x}}{\epsilon^{(1-j)\lambda l} - \epsilon^{-(1-j)\lambda l}}, \qquad (234)$$

in accordance with which the potential is distributed over a long alternating-current cable without conductance or self-induction when one end is earthed. In this equation $\lambda = \sqrt{\frac{rb}{2}} = \sqrt{r\pi c C}$, and the slope of potential $\frac{dPx}{dx}$ is a maximum near the end of the insulator, where $x = l$. The slope is here almost independent of the length of the insulator, so that surface discharges always occur, even with long insulators, when the slope of potential is high, relatively to the surface resistance r per unit length. When the leading-in tube is of the form shewn in Fig. 353, on the other hand, the pressure is distributed according to a straight line over the whole surface of the insulator, and no harmful surface discharges can occur until the pressure is sufficient to produce a spark over the whole surface.

(c) To determine the electric field-strength, it is best to use the same methods as for magnetic fields; viz. that of drawing a diagram of the lines of forces and thence calculating the field-strength f_n for each point, equal to the electric flux $d\phi$ of the tube of force divided by the section dF of the same at the point considered. Thus $f_n = \frac{d\phi}{\epsilon dF}$, where ϵ is the dielectric constant.

As starting-points in drawing out the lines of force, we can apply the law of discontinuity to the lines passing from one medium to

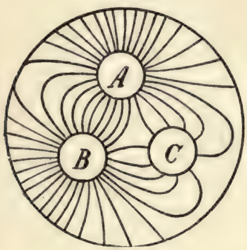


FIG. 356*a*.

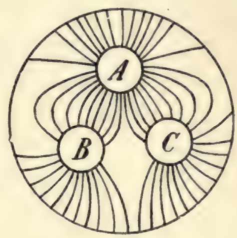


FIG. 356*b*.

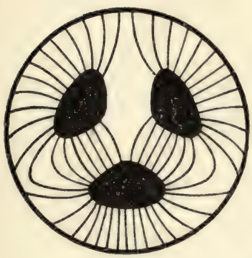


FIG. 356*c*.

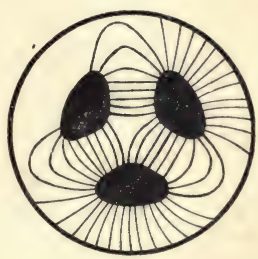


FIG. 356*d*.

FIG. 356*a* to *d*.—Lines of Force in Three-phase Cables (Thornton).

another and the law of maximum field-energy. According to the latter, the lines of force between conductors of given potential arrange themselves, so that the displacement flux between the conductors is a maximum. Owing to the small values of the dielectric constants compared with the magnetic permeability, it is much more difficult to draw electric lines of force accurately than magnetic, when insulating materials of different dielectric constants are present in the field. For this reason Hele-Shaw's method of representing the lines of force by stream-lines between two flat plates, as described on p. 365, is very useful in this connection. From such figures it is easy to determine simply the electric flux in each tube and thus obtain the field-strength at each point. Figs. 356*a* to *d* shew the diagrams for two

three-phase cables, taken by W. M. Thornton and O. J. Williams.* The dielectric constant of the conductor is assumed infinitely large, and the space between the plates where the conductors are is therefore made as large as possible. At the places where the insulation is, the space between the plates is made directly proportional to the cube root of the dielectric constant. The fluid is led in and out at the places where the conductors are, and the quantity of fluid for each conductor is made proportional to the pressure in that wire at the moment considered. Figs. *a* and *b* shew the field when one wire has zero potential and the other two the potentials $\pm\sqrt{\frac{3}{4}}P_{\text{max}}$. Figs. *c* and *d* shew the field when one wire has a potential P_{max}, and the other two the potential $-\frac{1}{2}P_{\text{max}}$. The sheath of the cable has zero potential in all the figures. As is clear from the diagrams, the field-strength alters from point to point, and at every point varies with the time.

127. Dielectric Strength. If the pressure between two insulated conductors (electrodes) is gradually raised, various discharge phenomena occur within the dielectric and along its surface, and finally the pressure is equalised by a sudden discharge through the dielectric. The dielectric is then said to be pierced. If the dielectric is liquid or gaseous, all traces of the passage of electricity immediately vanish; a solid dielectric, however, will remain pierced at the place where the discharge occurred. If sufficient electric energy is supplied to the electrodes, the break-down will continue in the form of an arc, even with comparatively low pressures.

The pressure between the electrodes at which the break-down occurs, is called the piercing pressure. This latter depends on the material, the distance of the electrodes apart, and on the distribution of the electric field in the dielectric (shape of the electrodes). The length of time during which the pressure acts on the dielectric has also a considerable effect on the piercing pressure. For a very short time the insulation can often withstand a much higher pressure than continuously. The piercing pressure of a dielectric for a given distance between the electrodes is a maximum when the field is uniform, as for instance, between two parallel plates at a sufficient distance from the edges, in which case the maximum field-strength is a minimum. Between two points or between a point and a large plate, the electric field is very varied, so that in this case the piercing pressure for a given distance is smaller.

Between the edges of two parallel plates the electric lines of force are curved. The electric field-strength near the surface of the dielectric is consequently increased and on the inside decreased. Since the maximum field-strength in the dielectric is thus increased, the break-down between two such plates generally occurs at the edge. For this reason, high-pressure condensers are often made so that the dielectric is thicker between the edges of the plates than elsewhere.

* *Engineering*, 1909, p. 297.

In the case of alternating-currents, piercing depends chiefly on the amplitude of the wave of pressure.

The dielectric strength of an insulating material can be small even when the specific resistance of the same is high and vice versa. Dry air, for example, is a very good insulator, but compared with most solid and liquid insulators its dielectric strength is very small.

The piercing pressure usually increases somewhat more slowly than the thickness of the insulating medium. With thin layers, however, the converse may be the case.

Fig. 357 shews by way of example the piercing pressure for mica as a function of the thickness of the same, taken from experiments made by Steinmetz. The amplitudes of the pressures are given in kilovolts and the thicknesses in hundredths of a mm.* In this case an alternating-current at 150 cycles was used. Since the material shewed much heating, the pressure could only be applied for ¼ minute.

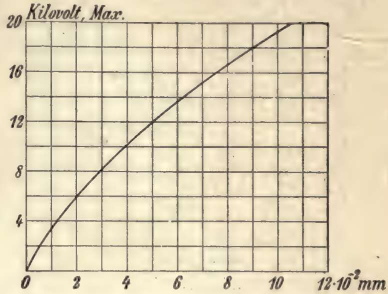


FIG. 357.—Break-down Pressure for Mica.

In the following Table, due to Steinmetz and Dr. Baur, the piercing pressures for 1 mm thickness of various insulating materials are given.

Dielectric.	Piercing Pressure for 1 mm thickness.
Mica	58000
Micanite	about 35000
Paraffined Plates, Paraffined Paper	,, 30000
Dry Wood Fibre	,, 13000
Hard Porcelain	,, 13000
Oiled Linen	,, 12500
Presspahn	,, 12000
Leatheroid	,, 10000
Vulcanised Rubber	,, 10000
Red Vulcanised Fibre	,, 5000
Asbestos Paper	,, 4300
Vulcanised Asbestos	,, 3500
Transformer Oil	,, 9000
Melted Paraffin	,, 8000
Boiled Oil	,, 8000
Oil of Turpentine	,, 6500
Insulating Varnish	,, 5000
Lubricating Oil	,, 1500

* *E.T.Z.* 1893, p. 251.

The figures represent average values taken from experiments with test pieces of various thicknesses and, on the assumption of proportionality between thickness and piercing pressure, are reduced to a thickness of 1 mm. Since, however, no relation exists between the thickness of the insulating material and the piercing pressure, the values can only be taken for plates about 1 mm thick.

Insulating oil under high pressures gives a straight line increase of the piercing pressure with the distance between the electrodes. For a mineral transformer oil with plate electrodes, the alternating pressures $+\frac{P}{2}$ and $-\frac{P}{2}$, at which breakdown occurred, were found, for sparking distances d greater than 5 cm, to be

$$P = 124000 + 9000d.$$

With very unsymmetrical distribution of the electric field the piercing pressure is much less. Between an earthed plate and a pointed electrode at potential P, it was found for the same oil as above

$$P = 37000 + 7000d.$$

If the spark gap d is given in cm, the effective pressure will be in volts. These pressures can act on the oil for about 5 minutes without causing a break-down. If the pressure is quickly raised, much higher pressures can be reached before the oil breaks down—in such cases, however, the results are generally irregular.

The breaking-down strength is considerably weakened by moisture in the case of both solid and liquid insulating materials. Oils are dried for this reason either by heating, or by treating with quicklime and such like. Hygroscopic solid substances must be dried in a vacuum oven and impregnated with varnish of some kind, so that they cannot absorb moisture from the air.

The breaking-down strength of an insulating material is in general reduced when mechanical stresses are simultaneously applied.

With most solid and liquid insulators, the duration of application of the pressure has a considerable influence on the insulation resistance as well as on the dielectric strength. The dielectric strength usually decreases considerably for the first few minutes, while the insulation resistance increases. A well-dried machine usually has a very high insulation resistance at the beginning when cold. At first the insulation resistance decreases very rapidly, even after the temperature has become constant, and often reaches a minimum after several days' work, after which it slowly recovers during a still longer time. Measurements of insulation on machines and apparatus should therefore be carried out after the normal temperature rise has been reached in the process of its work.

The temperature has little effect on the dielectric strength, provided that the same is not sufficient to bring about chemical changes in the material. This is, however, often the case even at comparatively low temperatures.

If the dielectric consists of several layers of different materials perpendicular to the lines of force, the electric field-strength distributes itself—as previously shewn—over the several materials according to their respective specific resistances when the pressure is constant. *With constant pressure therefore, in order to use the several materials to the best advantage, the dielectric strengths of the materials should be proportional to their specific resistances.*

If the pressure is alternating, the electric field-strength distributes itself over the insulating substances inversely as their dielectric constants. *Hence, with alternating-current apparatus, in order to use the insulating material to the best advantage, the dielectric strengths of the several materials should be inversely as their dielectric constants.*

In the construction of insulators for high pressures, attention must be paid not only to the dielectric strength of the insulating material, but also to the phenomena at the boundary surface of two dielectrics. For example, if two conductors at a large difference of pressure are supported in air by solid insulators, it is not sufficient that the distances between the two conductors, in air and through the insulator, correspond to the pressure, but it is most important of all to see that the distance apart measured *along the surface* is sufficient.

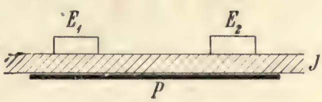


FIG. 358.

Sparking may easily occur through the collection of moisture and dirt on the surface. Moreover, if the capacities of the two electrodes are different, the electrode with the smaller capacity produces surface discharges in the form of rays, which assist the sparking between the electrodes. Also, the capacity of the two conductors under pressure with regard to a third insulated conductor can influence the piercing pressure between the two former conductors to a less extent. For example, if two electrodes E_1 and E_2 (Fig. 358) stand on an insulating plate J under such a pressure that sparking does not yet occur, and an insulated conducting plate P is brought to the other side of the dielectric J, surface discharges occur between the two electrodes and sparking takes places from one to the other. This phenomenon is similar to the surface discharges with leading-in tubes. Sparking occurs with a still smaller pressure when the plate P is connected to one of the electrodes. The surface discharges are then seen only about the electrode not connected to P, in the form of rays. In order to obtain the largest possible distance over the surface with the smallest distance between the electrodes and still avoid sparking, bell and petticoat insulators are used.

In accordance with the standards of the Verband deutscher Elektrotechniker the dielectric strength of electric machines and transformers should be tested for one minute when they are warm.

The testing pressures should be:

Working Pressure.	Test Pressure.
Under 40 volts -	At least 100 volts.
40 to 5000 volts -	$2\frac{1}{2}$ times working pressure, but not less than 1000 volts.
5000 to 7500 volts -	7500 volts above working pressure.
7500 and upwards -	Twice working pressure.

The dielectric strength must be tested between windings and frame and between electrically separated windings. In the latter case, with windings of different pressures, the highest must be used as the test-pressure.

CHAPTER XXI.

CONSTANTS OF ELECTRIC CONDUCTORS.

128. Resistance of Electric Conductors. 129. Self- and Mutual Induction of Electric Conductors. 130. Self- and Stray Induction of Coils in Air and Iron. 131. Increase of Resistance, due to Eddy Currents in Solid Conductors. 132. Stray Fields and Electrodynamic Forces due to Momentary Rushes of Current. 133. Capacity and Conduction of Electric Cables. 134. Capacity of Coils in Air and in Iron. 135. Telegraph and Telephone Lines.

128. Resistance of Electric Conductors. Most conductors consist of copper. With continuous currents and alternating-currents of low frequency, the current is uniformly distributed over the section of the conductor. If l denotes the single length of the line in km and q its section in mm², and $\rho = 0{\cdot}016\ (1 + 0{\cdot}004T^\circ)$, the specific resistance of the copper, then the ohmic resistance of the whole line is

$$r = \frac{2l\rho}{q} 1000 \text{ ohms.}$$

The heating loss in the line is

$$I^2 r = 2lq\rho \frac{I^2}{q^2} 1000 = 1000 \rho V s^2,$$

where $V = 2lq$ denotes the volume of the line in dm³, and s the current density in amperes per mm².

Of late years bare aluminium conductors have also been used for transmission lines. An aluminium line with the same ohmic resistance as copper will have

a diameter	1·3	times,
a section	1·69	times,
a weight	0·513	times,

larger than the copper line. The aluminium wire, however, has only 0·65 times the tensile strength of the copper. According to circumstances, sometimes the aluminium is cheaper and sometimes the copper.

In the following table the specific resistances and weights of the materials most commonly used are given. The specific resistance is for 1 metre length and 1 mm² section. If this is required for 1 cm

length and 1 cm^2 section, as it occurs in many formulae, the values given in the table must be divided by 10^4. The specific weight is given in gms. per cubic cm.

	Specific Resistance at 0° in ohms per m/mm².	Increase of Resistance per 1° C. in %.	Specific Weight.
Silver	0·015	0·36	10·5
Copper	0·016	0·40	8·9
Gold	0·021	0·35	19·3
Aluminium	0·027	0·40	2·75
Zinc	0·056	0·39	7·2
Platinium	0·090	0·24	21·5
Tin	0·10 to 0·13	0·45	7·3
Nickel	0·10 to 0·12	0·4 to 0·3	8·9
Lead	0·19	0·37	11·4
Pure Iron	0·095	0·5	—
Wrought Iron and Mild Steel	0·10	0·5	7·8
Iron Wire Conductor	0·125	0·5	7·8
Cast Steel	0·20	0·4	7·8
Alloyed Stampings	0·54	—	7·8
Cast Iron	1·00	0·1	7·2
Brass (30 % Zinc)	0·065 to 0·085	0·12 to 0·20	8·3
Manganin	0·41 to 0·45	0·001	8·4
Constantin	0·48	0·003	8·8
Nickelin I.	0·41 to 0·43	0·24	8·8
German Silver	0·36 to 0·38	0·27	8·7
Rheotin	0·47	0·21	8·55
Kruppin	0·84	0·07	8·1
Retort Carbon	13 to 100	0·08 to 0·02	2·3 to 1·9

The specific resistance of ordinary fresh water is about 10^4 ohms. For liquids and electrolytes the lowest specific resistances are those given in the following table,* along with the corresponding solutions.

	Specific Resistance.	Percentage Solution.	Specific Weight.
HNO_3	$1{\cdot}36 . 10^4$	29·7	1·185
HCl	$1{\cdot}39 . 10^4$	18·3	1·092
H_2SO_4	$1{\cdot}45 . 10^4$	30·4	1·224
KOH	$1{\cdot}96 . 10^4$	28·0	1·274
$NaCl$	$1{\cdot}70 . 10^4$	25·0	—
$MgSO_4$	$21{\cdot}7 . 10^4$	17·0	1·183
$ZnSO_4$	$22{\cdot}6 . 10^4$	23·5	1·286
$CuSO_4$	$22{\cdot}7 . 10^4$	18·1	1·210

* *Deutscher Kalendar für Elektrotechniker von Uppenborn.*

The resistance of the earth, in so far as it affects electric railways and earthed installations, is very variable. It not only depends on the nature of the soil and on the weather, but chiefly on the arrangement of the earthing plates or rails. The highest value that has been observed for the earth's resistance in the case of railways is 0·2 ohm per km. It may, on the other hand, be also nearly zero. To obtain low contact resistance, it is advisable to have several parallel plates placed at some distance from one another and sunk as deeply as possible, so that they come into contact with underground water.

The contact resistance of a plate is proportional to the specific resistance of the soil surrounding it, and inversely proportional to the mean linear dimensions of the plate. Let r denote the contact resistance in an unlimited medium having a specific resistance ρ.

Then for circular plates of diameter d, $r=\dfrac{\rho}{4d}$,

for square plates with side d, $r=\dfrac{\rho}{2{\cdot}72d}$,

for cylindrical electrodes of diameter d and length l,

$$r=\frac{\rho}{2\pi l}\log_e\left(\frac{2l}{d}\right).$$

129. Self- and Mutual Induction of Electric Conductors.

(a) In the determination of the self-induction of conductors, we shall first start with the case of a single-phase system. The two conductors which serve as the outgoing and return lines are assumed to be fixed to poles and parallel to one another over the whole length. We suppose that the two conductors are connected by wires at both ends instead of by the actual apparatus, so that we have to determine the self-induction of a rectangular loop.

For the time being we assume that the current is distributed uniformly over the section of the conductors, and further that no ferro-magnetic bodies are present in the magnetic field produced by the current in the conductors. It is therefore allowable to superpose the magnetic fluxes produced by the current flowing in each of the wires. As shewn in the introduction, the current flowing in each conductor produces a magnetic field, whose lines of force are circles round the conductor.

The field-strength H at a point P at a distance ρ from the axis of the wire is

$$H=\frac{\int H\,dl}{\int dl}=\frac{\text{M.M.F.}}{\text{length of line of force}},$$

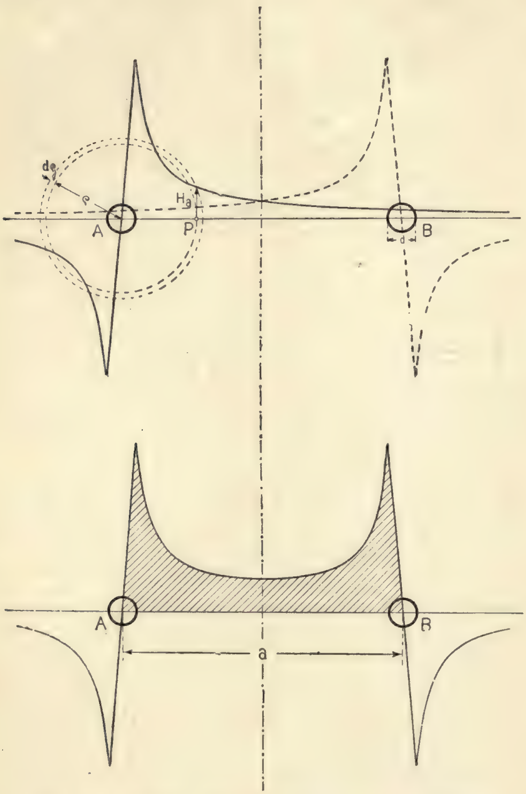
or, when the point P lies outside the wire,

$$H_a=\frac{0{\cdot}4\pi i}{2\pi\rho}=\frac{0{\cdot}2i}{\rho},$$

and, when the point P lies inside the wire,

$$H_i = \frac{0{\cdot}4\pi i\left(\dfrac{2\rho}{d}\right)^2}{2\pi\rho} = \frac{0{\cdot}2i\rho}{(\frac{1}{2}d)^2}.$$

From this we get the diagram of the field-strength for the plane AB, as shewn in Fig. 359.



Figs. 359 and 360.—Magnetic Field of Two-wire System.

If there are two conductors serving as outgoing and return lines, the current produces a field for each of the two wires. Superposing these fields, we get the resultant field-strength of a double line, as shewn in Fig. 360. The shaded surface serves as a measure for the flux per cm length interlinked with the conductors.

Since, however, the total current is not interlinked with the whole flux, we must take this into account.

The energy supplied to the magnetic field during a time interval dt is

$$dA = \Sigma(iw_x\Phi_x) = \frac{L}{2}\,d(i^2).$$

Here iw_x (or w_x, since in the calculation of L, i is put equal to 1 ampere) denotes the current interlinked with the tube of force Φ_x. From Formula 27, p. 41, we get the following expression for the coefficient of self-induction L,

$$L = \Sigma\left(\frac{w_x^2}{R_x}\right)10^{-8} = \Sigma\,(w_x\Phi_x)\,10^{-8}\ \text{henry},$$

where the summation is to be taken over all the tubes of force in the field. Since, however, the field is produced by the superposition of two equal fields, it is sufficient if we integrate the tubes of force in one field and multiply the result thus obtained by 2.

We calculate first the sum for the space between the wires. The flux in this part is interlinked with the whole current in the conductors; hence w_x is here unity, and the sum is

$$\sum_{\rho=\frac{d}{2}}^{\rho=a}(w_x\Phi_{xa}) = \sum_{\rho=\frac{d}{2}}^{\rho=a}\Phi_{xa} = 2\int_{\rho=\frac{d}{2}}^{\rho=a} lH_a\,d\rho,$$

where d = diameter of wires

and a = distance between the axes of the wires.

By assuming the limit $\rho = a$, a small error is introduced, which, however, is negligible for small values of $\frac{d}{a}$.

Hence $$\sum_{\rho=\frac{d}{2}}^{\rho=a}(\Phi_{xa}) = 2l\int_{\rho=\frac{d}{2}}^{\rho=a}\frac{0{\cdot}2d\rho}{\rho} = 0{\cdot}4l\log_e\left(\frac{2a}{d}\right),$$

or, substituting ordinary for natural logarithms,

$$\sum_{\rho=\frac{d}{2}}^{\rho=a}(\Phi_{xa}) = 0{\cdot}92l\log_{10}\left(\frac{2a}{d}\right).$$

For the interior of each wire we consider only that field produced by the current in the wire itself, and since here

$$w_x = \frac{\pi\rho^2}{\pi\left(\frac{d}{2}\right)^2}\cdot 1 = \frac{\rho^2}{\left(\frac{d}{2}\right)^2},$$

we have $$\sum_{\rho=0}^{\rho=\frac{d}{2}}(w_x\Phi_{xi}) = 2\int_{\rho=0}^{\rho=\frac{d}{2}} lH_iw_x d\rho = 2\int_{\rho=0}^{\rho=\frac{d}{2}} lH_i\frac{\rho^2}{\left(\frac{d}{2}\right)^2}d\rho$$

$$= 2l\int_{\rho=0}^{\rho=\frac{d}{2}}\frac{0{\cdot}2\rho^3 d\rho}{\left(\frac{d}{2}\right)^4} = 0{\cdot}4l\,\frac{\left(\frac{d}{2}\right)^4}{4\left(\frac{d}{2}\right)^4} = 0{\cdot}1l.$$

Hence the coefficient of self-induction of a double line is

$$L=\frac{l}{10^8}\left[0{\cdot}92\log_{10}\left(\frac{2a}{d}\right)+0{\cdot}1\right], \qquad \text{(235)}$$

and its reactance

$$x=2\pi cL=\frac{2\pi cl}{10^8}\left[0{\cdot}92\log_{10}\left(\frac{2a}{d}\right)+0{\cdot}1\right],$$

where l is measured in cm. If l is measured in kilometres, the reactance is

$$x=\frac{2\pi cl}{10^3}\left[0{\cdot}92\log_{10}\left(\frac{2a}{d}\right)+0{\cdot}1\right] \text{ ohms.} \qquad \text{(236)}$$

We have seen that the magnetic field inside a conductor is not constant. It follows from this that the current lines in the conductor

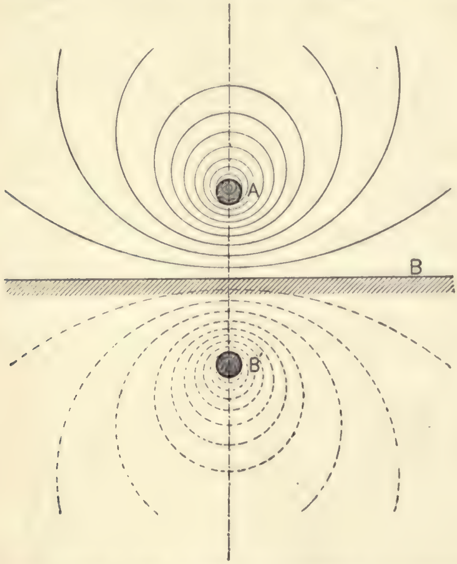


FIG. 361.—Effect of Earth on the Self-induction of a Conductor.

do not all have the same inductance, and that when the alternating-current is of high frequency the current is not uniformly distributed over the section of the conductor. We shall return to this in Section 131.

(b) In a system in which only one overhead conductor is used and the earth acts as a return, the self-induction of the former can be ascertained from the following considerations.

In Fig. 361 the lines of force of the magnetic field represented are those produced by the current flowing in the two conductors A and B'. It is clear that perpendicular B, passing through the middle point

of the line joining the centres of the two circles, represents a line of force. The flux above is interlinked with the conductor A and that below with the conductor B'. If we now substitute for the conductor B' a surface-carrying current (for instance, the surface of the earth) B, then this will have no effect on the diagram of the lines of force and equipotential surfaces above B, so that the self-induction of the conductor A remains the same and that of the conductor B vanishes, since the radius of B is infinite. From this it follows that as regards self-induction the earth return acts like a conductor which is the image of the first conductor with respect to the earth's surface.

If a denotes the distance of the conductor from the surface of the earth, then the summation $\Sigma(w_x\Phi_x)$ must be extended from $\rho=\frac{d}{2}$ to $\rho=2a$, and since we only have one conductor the coefficient of self-induction will be

$$\frac{L}{10^8}\left[0{\cdot}46\left(\frac{4a}{d}\right)+0{\cdot}05\right]. \quad \text{......................}(237)$$

(c) We have still to investigate the influence of a current in a conductor on the neighbouring conductors of other circuits. If, for example, there are four conductors on the same pole, of which A and B belong to one circuit and C and D to another, then some of the tubes of force of the magnetic field produced by the currents in A and B will be interlinked with the loop formed by the conductors C and D, and will therefore induce E.M.F.'s in the latter conductors. It is simplest, however, to calculate the effects of the two fields due to the current in A and due to the current in B separately and afterwards to superpose them.

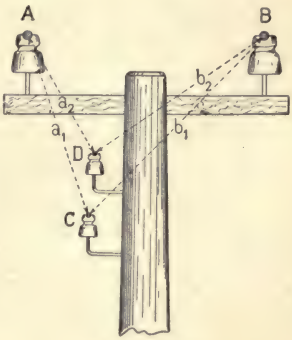


FIG. 362.

The magnetic lines of force produced by the current in A are concentric circles, whence it follows that the mutual induction coefficient of the conductor A and the loop formed by C and D is

$$M_{A-CD}=\sum_{\rho=a_2}^{\rho=a_1}(w_x\Phi_x)=\frac{l}{10^3}0{\cdot}46\log_{10}\left(\frac{a_1}{a_2}\right).$$

In the same way we find the mutual induction coefficient between the conductor B, and the loop CD equals

$$M_{B-CD}=\sum_{\rho=b_2}^{\rho=b_1}(w_x\Phi_x)=\frac{l}{10^3}0{\cdot}46\log_{10}\left(\frac{b_1}{b_2}\right).$$

Since the currents in A and B are equal but of opposite direction, the mutual induction coefficient between the two circuits is

$$M_{AB-CD}=\frac{l}{10^3}0{\cdot}46\left(\log_{10}\frac{a_1}{a_2}-\log_{10}\frac{b_1}{b_2}\right)=\frac{l}{10^3}0{\cdot}46\log_{10}\left(\frac{a_1b_2}{a_2b_1}\right). \quad \text{...}(238)$$

If the circuit CD consists of one overhead wire with an earth return, then a_2 and b_2 are to be taken as the distances of the conductors A and B from a conductor situated symmetrically, with respect to the earth's surface, to the conductor C. Accordingly $a_2 = b_2$, and we get for M_{AB-C}, the simple expression

$$M_{AB-C} = \frac{l}{10^3} 0{\cdot}46 \log_{10}\left(\frac{a_1}{b_1}\right).$$

In general, the mutual induction between neighbouring conductors, (as, for example, between telephone wires on the same poles as a transmission line) is made as small as possible. This is done by crossing the wires A and B or by placing the telephone wires symmetrically with respect to the conductors A and B; for in this case we get $a_1 b_2 = b_1 a_2$ and $M_{AB-CD} = 0$.

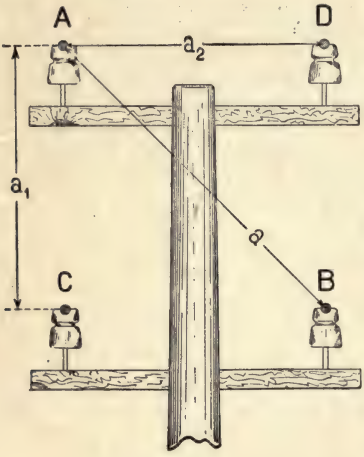


FIG. 363.

(**d**) In an uninterlinked two-phase system, which is the system usually employed for two-phase transmission, the best arrangement for the wires is that shewn in Fig. 363. The mutual induction coefficient between the two phases in this case equals

$$M_{AB-CD} = \frac{l}{10^3} 0{\cdot}46 \log_{10}\left(\frac{a_1 b_2}{a_2 b_1}\right) = 0,$$

since $a_1 = a_2$ and $b_1 = b_2$. The two phases are entirely independent of one another as regards inductive action between the wires, and the resultant coefficient of self-induction for one phase is

$$L = \frac{l}{10^3}\left[0{\cdot}92 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}1\right].$$

(**e**) If the three conductors of a three-phase system are symmetrically arranged, i.e. placed at the three angles of an equilateral triangle (Fig. 364), then equal currents flowing in lines II and III will induce the same E.M.F. in phase I.

Since now two wires can always be considered as the return for the third, the coefficient of self-induction of a phase with the above symmetrical arrangement of the wires is independent of the load in the several phases and equals

$$L = \frac{l}{10^3}\left[0{\cdot}46 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}05\right], \quad \dots\dots\dots\dots\dots(239)$$

since here for one phase only the single length has to be considered.

If the three wires are not symmetrical, but arranged in a straight line, as shewn in Fig. 365, the current in the middle wire cannot exert any inductive effect on the two outer wires and conversely. The coefficient of self-induction of the middle phase is, therefore,

$$L_m = \frac{l}{10^3}\left[0{\cdot}46 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}05\right],$$

while with a symmetrical load in all three phases the coefficient of the two outside phases is

$$L_a = \frac{l}{10^3}\left[0{\cdot}46 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}119\right].$$

To make the coefficients of self-induction of all the phases equal with this arrangement, each of the three phases may in turn occupy

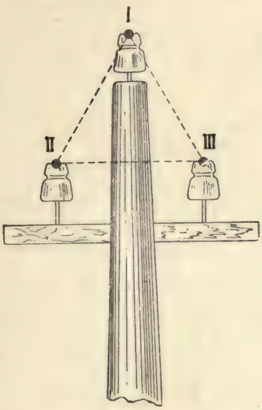


FIG. 364.

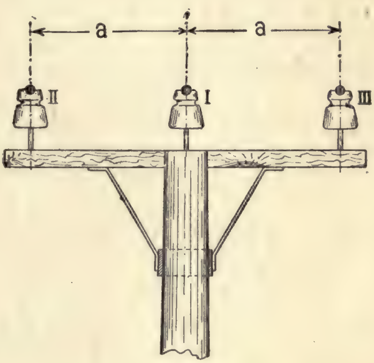


FIG. 365.

a third of the length l as the middle phase. In this case the coefficient of self-induction of each phase will be

$$L = \frac{l}{10^3}\left[0{\cdot}46 \log_{10}\left(\frac{2a}{d}\right) + \frac{2}{3}\log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}05\right]$$

$$= \frac{l}{10^3}\left[0{\cdot}46 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}096\right]. \qquad (240)$$

(**f**) With concentric cables the conductor forming the core is a complete cylinder, whilst the other is a hollow concentric cylinder.

This arrangement of the two conductors as one cable used to be almost exclusively used and was most convenient for manufacture. The capacity of the outside conductor of such a cable, however, with respect to the inner conductor, is so large that in recent years stranded cables, in which the conductors lie side by side, have come more into use. If each conductor is arranged in a cable by itself, an iron sheath

should be avoided, because the latter would considerably increase the self-induction of the conductor. Since the iron covering is only required for giving strength to the cable, stranded cables with several conductors are largely used.

For stranded cables with two and three conductors we get precisely the same formulae as for a double line and a three-phase line. Hence with a double-line cable,

$$L = \frac{l}{10^3}\left[0{\cdot}92 \log\left(\frac{2a}{d}\right) + 0{\cdot}1\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots (235a)$$

and in a three-phase cable, for each phase

$$L = \frac{l}{10^3}\left[0{\cdot}46 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}05\right]. \quad \ldots\ldots\ldots\ldots\ldots (239a)$$

When cables are provided with an iron sheath, the lines of force outside the conductor close through the covering, whereby the self-induction is increased. The eddy-currents produced in the iron covering by these lines of force are however so small that no heating is produced in the covering when the load is symmetrical, and only very little heating when the load is slightly unbalanced.

130. Self- and Stray Induction of Coils in Air and in Iron.

(**a**) Of all coils the simplest is the circular coil formed by a wire of circular cross-section (Fig. 366). Its coefficient of self-induction is

$$L = \frac{l_s}{10^8}\left[0{\cdot}46\left(1 + 1{\cdot}645\,\frac{d^2}{l_s^2}\right)\log_{10}\frac{l_s}{d} + 0{\cdot}37\,\frac{d^2}{l_s^2} - 0{\cdot}163\right], \ldots (241)$$

or, if the value of $\frac{d}{l_s}$ is not too large,

$$L = \frac{l_s}{10^8}\left[0{\cdot}46 \log_{10}\frac{l_s}{d} - 0{\cdot}163\right].$$

This can only be determined by means of a complicated integration.

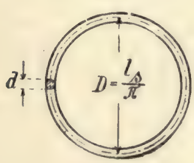


FIG. 366.

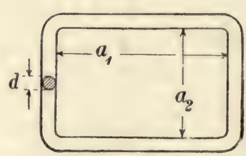


FIG. 367.

Another simple coil is of a circular wire wound in the form of a rectangle of sides a_1 and a_2. Since the coefficient of self-induction per unit length for two parallel round wires of diameter d and distance a apart (Eq. 235) is

$$L = \frac{1}{10^8}\left[0{\cdot}92 \log_{10}\left(\frac{2a}{d}\right) + 0{\cdot}1\right],$$

and since, further, two conductors at right angles can exert no inductive action on each other, the coefficient of self-induction of the rectangle, shewn in Fig. 367, is

$$L=\frac{1}{10^8}\left[0{\cdot}92a_1\log_{10}\left(\frac{2a_2}{d}\right)+0{\cdot}92a_2\log_{10}\left(\frac{2a_1}{d}\right)+(a_1+a_2)\text{ const.}\right].$$

By accurate calculations this constant is found equal to $-0{\cdot}24$ instead of $+0{\cdot}1$, which might have been expected; hence the coefficient of self-induction of a rectangle equals

$$L=\frac{1}{10^8}\left[0{\cdot}92a_1\log_{10}\left(\frac{2a_2}{d}\right)+0{\cdot}92a_2\log_{10}\left(\frac{2a_1}{d}\right)-0{\cdot}24(a_1+a_2)\right] \quad (242)$$

or approximately

$$L\simeq\frac{a_1+a_2}{10^8}0{\cdot}92\left[\log_{10}\frac{2(a_1+a_2)}{d}-0{\cdot}2\right]=\frac{0{\cdot}46l_s}{10^8}\left[\log_{10}\left(\frac{l_s}{d}\right)-0{\cdot}2\right], \quad (242a)$$

where l_s is the mean length of the coil.

If the circular or rectangular coil is not formed of wire of circular section, but say of rectangular section, the calculations may be carried out with sufficient accuracy by taking the diameter d as the diameter of a circle having the same periphery as the section of the conductor (see Fig. 368). This, however, is only permissible when the section is not too flat.

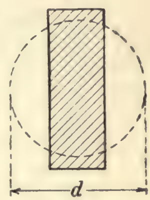


Fig. 368.

If the circular coil consists of several (w) turns, as is shewn in Fig. 369, the formula becomes

$$L=\frac{w^2l_s}{10^8}\left[0{\cdot}46\log_{10}\left(\frac{l_s}{d_s}\right)-0{\cdot}163\right], \quad \text{..............}(243)$$

where d_s is the diameter of a circle of equal periphery to the coil and $l_s=\pi D$ is the mean length of the coil. It is assumed in this formula that l_s is large compared with d_s.

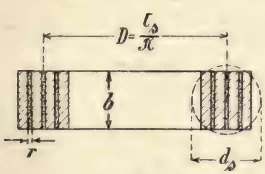


Fig. 369.

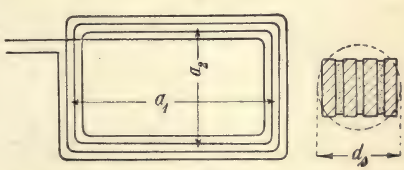


Fig. 370.

From the above formula it follows directly that the coefficient of self-induction is proportional to the square of the number of turns.

Treating a rectangular coil with w turns (Fig. 370) in a similar way, we have

$$L=\frac{w^2}{10^8}\left[0{\cdot}92a_2\log_{10}\left(\frac{2a_1}{d_s}\right)+0{\cdot}92a_1\log_{10}\left(\frac{2a_2}{d_s}\right)-0{\cdot}24(a_1+a_2)\right]$$

$$\simeq\frac{0{\cdot}46l_sw^2}{10^8}\left[\log_{10}\left(\frac{l_s}{d_s}\right)-0{\cdot}2\right]. \quad \text{.................}(244)$$

If such a coil is laid on a flat iron surface, the coefficient of self-induction is approximately doubled, because the magnetic reluctance is practically reduced to half.

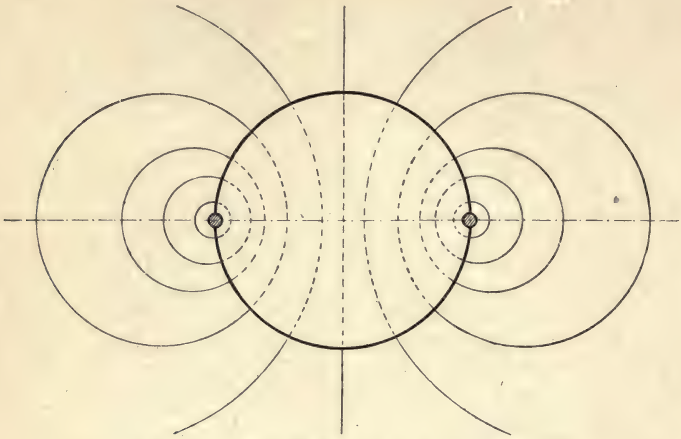


FIG. 371.—Magnetic Field of an Armature Coil.

This is also approximately the case, even when the iron surface is cylindrical, because the lines of force always pass into the iron at right angles; the surface of the iron forms an equipotential surface.

Fig. 371 shows the distribution of the lines of force for a coil of circular section half embedded in an iron cylinder. The lines of force are dotted for the case in which the cylinder is made of non-magnetic material. From the distribution of the lines it is clear that the introduction of the iron cylinder into the field of the coil reduces the magnetic reluctance to half and thereby doubles the self-induction.

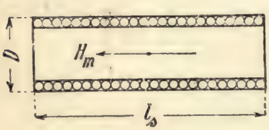


FIG. 372.

The field-strength in the middle of a long thin coil of diameter D and length l_s (Fig. 372) is

$$H_m = \frac{0{\cdot}4\pi iw}{\sqrt{l_s^2 + D^2}} \simeq \frac{iw}{0{\cdot}8 l_s}.$$

Denoting the section of the coil by $q_s = \frac{\pi}{4} D^2$, the flux through the middle part of the coil equals approximately $q_s H_m$; at the ends of the coil, however, the flux is somewhat smaller, so that all the w turns do not embrace the same flux. $\frac{q_s H_m w}{k_s}$ is a measure of the flux-interlinkages with the coil, where the factor k_s is greater than 1 and takes into account the decrease in flux at the ends of the coil.

Hence we obtain the coefficient of self-induction L of such a coil, equal to the sum of the flux-interlinkages for $i=1$ ampere,

$$L=\frac{w^2}{10^8}\frac{q_s}{0 \cdot 8k_s l_s}; \quad \text{.........................(245)}$$

k_s depends on the dimensions of the coil, especially on the ratio $\frac{l_s}{q_s}$. The greater this ratio, the nearer k_s approaches unity. If $\frac{l_s}{D}$ is very large, $\frac{q_s}{0 \cdot 8l_s}$ is the magnetic reluctance of the cylindrical coil and $\frac{q_s}{0 \cdot 8k_s l_s}$ is the magnetic reluctance of the effective flux, which is considered to be interlinked with all w turns.

(b) When dealing with the coils in electric machines and transformers, it is not usual to calculate with self- and mutual induction, (as mentioned in Chap. VII., p. 116), but with the main and leakage fluxes, or the quantities corresponding to these, i.e. the coefficients of mutual and leakage induction. It would carry us too far here to calculate all the coefficients occurring in machines and transformers; and therefore we shall confine ourselves to pointing out the methods by which they may be calculated.

Fig. 373 shews the distribution of the lines of force in a single-phase iron-core transformer with a cylindrical winding. I denotes the primary coils and II the secondary. Both embrace the main flux, which is produced by the difference between the primary and secondary ampere turns. The leakage lines, of which the primary are interlinked with part of the primary winding and the secondary with part of the secondary winding, are squeezed between the primary and secondary coils, in which currents flow in opposite directions. The leakage coefficients S_1 and S_2 are given by the summations (p. 112)

$$S_1=\Sigma\frac{w_{1x}\left(w_{1x}-w_{2x}\frac{w_1}{w_2}\right)}{R_x},$$

$$S_2=\Sigma\frac{w_{2x}\left(w_{2x}-w_{1x}\frac{w_2}{w_1}\right)}{R_x},$$

which extend over all the tubes of force interlinked with the primary and secondary turns respectively.

In general, it is only necessary to know the sum of these two coefficients, and this can easily be approximated as follows.

For each limb of the transformer,

$$S_1+S_2'=\frac{1}{10^8}\frac{w_1^2}{R_s} \text{ henrys},$$

where w_1 is the number of primary turns per limb, S_2' the secondary leakage coefficient reduced to the primary and R_s the effective magnetic reluctance of the space between the two windings. This reluctance can

be expressed in the same way as the reluctance of a cylindrical coil (Eq. 245),

$$R_s = \frac{q_s}{0 \cdot 8 k_s l_s},$$

where q_s is the section of the effective flux between the primary and secondary winding, l_s the mean length of the two windings and k_s a factor which takes into account the magnetic reluctance of the

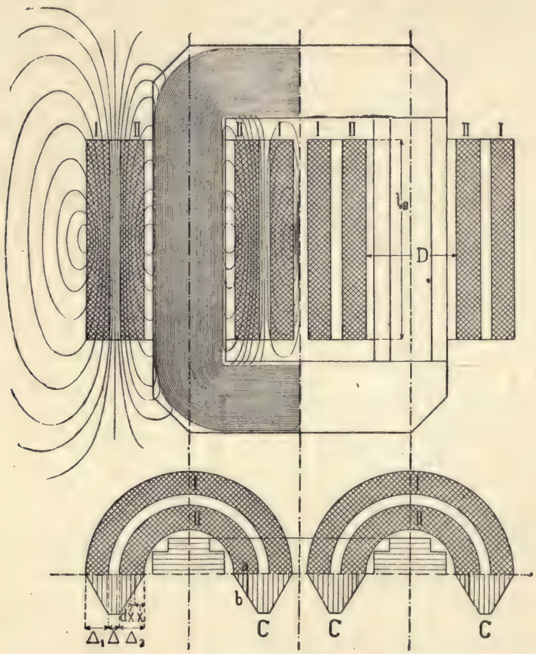


FIG. 373.—Leakage Field of Transformer with Cylindrical Winding.

leakage flux outside the space between the two windings, and the decrease in the leakage field at the ends of the windings. Denoting the radial distance between the two windings by Δ, the depth of the primary and secondary windings by Δ_1 and Δ_2, and the periphery between the two windings by U, we have

$$q_s \simeq U\left(\Delta + \frac{\Delta_1 + \Delta_2}{3}\right).$$

The presence of $\frac{\Delta_1}{3}$ and $\frac{\Delta_2}{3}$ in this expression is due to the fact that the integration has to be carried out for the interlinkages of the tubes of force $\Sigma\left(\frac{w_x^2}{R_x}\right)$ and not for the tubes of force $\Sigma\left(\frac{w_x}{R_x}\right)$. The result of

this is not the mean of Δ_1 and Δ_2, but a third of their sum. Hence the sum of the leakage coefficients of the windings per limb is

$$S_1 + S_2' = \frac{w^2}{10^8} \frac{U\left(\Delta + \frac{\Delta_1 + \Delta_2}{3}\right)}{0 \cdot 8 k_s l_s} \text{ henrys.} \quad \text{.............. (246)}$$

The strength of the leakage field itself for a section in the middle of the windings is shewn by curve C in Fig. 373.

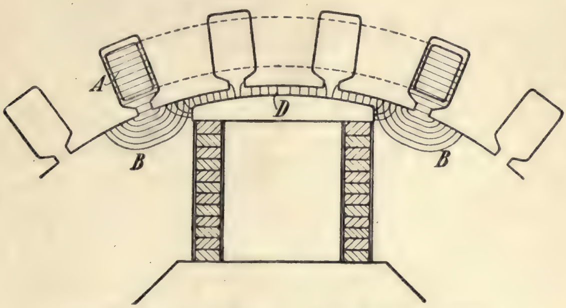


Fig. 374*a*.—Leakage Field of Three-phase Generator.

Prof. G. Kapp has determined experimentally the values of k_s for several transformers; in modern transformers k_s lies between 0·95 and 1·05. In order that no local leakage fields may exist in the transformer, care must be taken that the two windings are as far as possible alike in shape and arranged symmetrically with respect to each other.

The armature coils of electric machines are nowadays nearly always placed in slots. In this case it is of advantage in calculating the leakage coefficient to split up the leakage lines into three groups:

1. Lines A (Fig. 374*a*), which entirely pass through the slots.

2. Lines B, which pass between the tops of the teeth.

3. Lines C (Fig. 374*b*), which are closed round the coil-ends outside the iron.

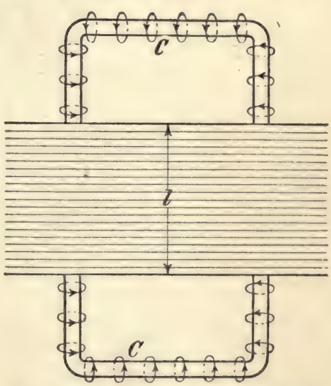


Fig. 374*b*.—Leakage Field of Coil-ends.

In addition to the leakage lines, there are also the lines D of the main flux, which pass through the armature coils and produce in them the E.M.F. of mutual induction. The main flux of a polyphase generator, as shewn in Fig. 374*a*, is produced by the resultant of the field and armature ampere-turns.

As was pointed out in Sect. 118, p. 382, the resultant ampere-turns of an n-phase armature winding having w turns per pole and phase, and having a current of maximum value I_{max}, is equal to $\frac{n}{2}I_{max}w$; this M.M.F. rotates in synchronism with the field, and is displaced from it by a

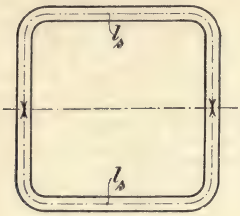


FIG. 374c.

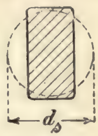


FIG. 374d.

certain angle ψ. This angle is identical with the internal phase displacement ψ of the armature current, if the angle of a pole-pair is just equal to 2π.

Using the same method as employed above, the leakage coefficient of an armature coil can be written

$$S=\frac{1}{10^8}\frac{w_n^2}{R_s},$$

where R_s is the magnetic reluctance of the effective leakage flux, interlinked with all the w_n turns in a slot. It is, however, more convenient for our division of the leakage lines to write

$$\frac{1}{R_s}=2l\lambda_n+2l\lambda_k+2l_s\lambda_s,$$

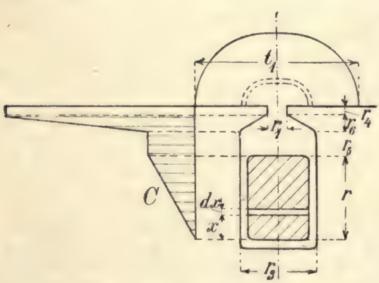


FIG. 375.—Slot Field.

where λ_n is the permeance of the leakage field across the slot for 1 cm length of iron, λ_k the same for the leakage field across the tops of the teeth and λ_s for the coil-ends or overhang. l is the length of the iron and l_s the length of the overhang.

In Fig. 375, the leakage lines A passing through the slots are considered, and curve C shews the strength of the leakage field. The permeance λ_n, calculated from this distribution of the leakage lines, neglecting the magnetic reluctance in the iron, is given by

$$\lambda_n=\int_{x=0}^{x=r}\left(\frac{x}{r}\right)^2\frac{dx}{0{\cdot}8r_3}+\frac{r_5}{0{\cdot}8r_3}+\frac{2r_6}{0{\cdot}8(r_1+r_3)}+\frac{r_4}{0{\cdot}8r_1}$$

$$=1{\cdot}25\left(\frac{r}{3r_3}+\frac{r_5}{r_3}+\frac{2r_6}{r_1+r_3}+\frac{r_4}{r_1}\right). \quad\text{.........................(247)}$$

Here we have again $\frac{r}{3r_3}$ and not $\frac{r}{2r_3}$, because we integrate over the interlinkages of the tubes of force $\Sigma\frac{w_x^2}{R_x}$, i.e. $\left(\frac{x}{r}\right)^2 dx$.

For the leakage lines B we take the distribution as being two quarter-circles and the straight lines joining them, as shewn in Fig. 375. From this we have

$$\lambda_k=\int_{x=0}^{x=\frac{t_1-r_1}{2}}\frac{dx}{0{\cdot}8(\pi x+r_1)}$$

$$=\frac{2{\cdot}3}{0{\cdot}8\pi}\log_{10}\left[\frac{\pi(t_1-r_1)+2r_1}{2r_1}\right]\simeq 0{\cdot}92\log_{10}\left(\frac{\pi t_1}{2r_1}\right). \quad \ldots\ldots(248)$$

The integration is here taken to the limit t_1, which it is best to put equal to the slot-pitch, since all the tubes of force outside this limit usually embrace several slots. To estimate these correctly necessitates complicated constructions, into which we shall not enter further.

To calculate the leakage lines C, it is best to consider the two coil-ends as comprising one rectangular coil (Fig. 374*c*), whose permeance is equal to

$$\lambda_s=0{\cdot}46\left[\log_{10}\left(\frac{l_s}{d_s}\right)-0{\cdot}2\right]. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(249)$$

Hence the leakage coefficient of an armature coil is

$$S=\frac{2w_n^2}{10^8}(l\lambda_n+l\lambda_k+l_s\lambda_s)\text{ henrys}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(250)$$

where λ_n, λ_k and λ_s can be calculated from the above formulae.

If two similar coils, belonging to different circuits, lie side by side in the same slot (Fig. 376), the currents in them are mutually inductive. The coefficient of mutual induction M of two such coils is equal to the leakage coefficient S, assuming the distribution of lines of force in Fig. 375.

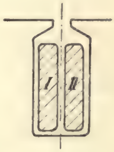


FIG. 376.

The distribution of the lines of force, however, will be quite another thing if the currents in the two coils are very different from each other, and especially if they are oppositely directed. In this case M is somewhat smaller than S.

The above formulae for the calculation of the leakage coefficients of armature coils do not of course give quite accurate values, since the lines of force are not distributed along the assumed geometric lines, but always choose complicated paths, for which the magnetic permeance of the leakage fields is a maximum. For this reason experimental values are usually somewhat greater than calculated.

131. Increase of Resistance, due to Eddy Currents in Solid Conductors. In the previous section we have seen that the magnetic

field in the interior of an electric conductor is not constant, from which it follows that the current lines do not all possess the same self-induction. On this account the distribution of a high-frequency alternating-current over the section of the conductor is not uniform, but such that the variation of the potential energy $L\frac{i^2}{2}$ is as small as possible. For this reason the greatest current-density is obtained in that part of the conductor in which the magnetic field is strongest. Lord Kelvin first demonstrated this phenomenon, which is known as *skin-effect.*

Its action produces an increase in the resistance and reduction in the self-induction of the conductor. When the field in a wire is due to the current in that wire alone, the current-density is dependent on the distance of the point considered from the axis of the wire. The current-density is greatest at the surface and least at the axis.

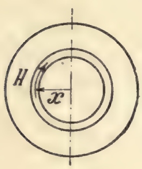


FIG. 377.

(a) We first calculate the distribution of current over the section of a round wire, in which case the approximate equations are similar to those for the distribution of a rapidly alternating magnetic flux in a round iron wire.

Let us consider the element of the wire formed by a cylinder of thickness dx at a distance x from the axis (Fig. 377). Let the maximum current-density be $\dot{I}_x$ and the magnetic field-strength $\dot{H}_x$. This increases from the inside to the surface by the value

$$d\dot{H}_x = \frac{0 \cdot 4\pi \dot{I}_x 2\pi x\, dx}{2\pi x} = 0 \cdot 4\pi \dot{I}_x dx,$$

while the induction, assuming constant permeability, increases by $\mu d\dot{H}_x = d\dot{B}_x$. On the outside of the cylinder a smaller E.M.F. $\dot{E}_x$ is acting than on the inside. The increase in the E.M.F. $\dot{E}_x$, assuming a phase displacement of 90°, is

$$d\dot{E}_x = 2\pi jc\dot{B}_x dx 10^{-8} = 2\pi jc\mu \dot{H}_x dx 10^{-8} \text{ volts.}$$

This increase in the pressure requires an increase in the current-density, according to the equation $\dot{I}_x = \frac{P - \dot{E}_x}{\rho}$, equal to

$$d\dot{I}_x = -\frac{d\dot{E}_x}{\rho} = -2\pi jc\frac{\mu}{\rho}\dot{H}_x dx 10^{-8} \text{ volts.}$$

Hence
$$\frac{d^2\dot{I}_x}{dx^2} = -2\pi jc\frac{\mu}{\rho}\frac{d\dot{H}x}{dx}10^{-8}.$$

Substituting now the value of $\frac{d\dot{H}_x}{dx}$, we have

$$\frac{d^2\dot{I}_x}{dx^2} = -0 \cdot 8\pi^2 jc\frac{\mu}{\rho}\dot{I}x 10^{-8}.$$

Introducing (in the same way as for the distribution of induction in iron wires)

$$\lambda = \frac{2\pi}{10^4}\sqrt{\frac{c\mu}{10\rho}}, \qquad (251)$$

we obtain
$$\frac{d^2 \dot{I}_x}{dx^2} = -2j\lambda^2 \dot{I}_x.$$

The solution of this equation is

$$\dot{I}_x = \dot{A}\epsilon^{(1-j)\lambda x} + \dot{B}\epsilon^{-(1-j)\lambda x},$$

where $\dot{A}$ and $\dot{B}$ are equal, since the same value is obtained for $\dot{I}_x$ for both $+x$ and $-x$. Hence

$$\dot{I}_x = \dot{A}\left[\epsilon^{(1-j)\lambda x} + \epsilon^{-(1-j)\lambda x}\right].$$

At the surface of the wire, where $x = r$, the current-density is a maximum

$$\dot{I}_{\max} = \dot{A}\left[\epsilon^{(1-j)\lambda r} + \epsilon^{-(1-j)\lambda r}\right].$$

Therefore
$$\dot{I}_x = \dot{I}_{\max}\frac{\epsilon^{(1-j)\lambda x} + \epsilon^{-(1-j)\lambda x}}{\epsilon^{(1-j)\lambda r} + \epsilon^{-(1-j)\lambda r}}. \qquad (252)$$

The current-density therefore decreases from the outside to the inside in a curve like the induction in an iron wire. To determine the effective resistance of the wire, the mean of the squares of the current-densities $\int_{x=0}^{x=r} \dot{I}_x^2 2\pi x dx$ must be divided by the square of the mean of the current-density $\left[\int_{x=0}^{x=r} \dot{I}_x 2\pi x\, dx\right]^2$. The real ratio of these two quantities gives the ratio k of the effective resistance r_{eff} to the ohmic resistance r.

Hence
$$k = \frac{r_{\text{eff}}}{r} = \frac{\int_{x=0}^{x=r} \dot{I}_x^2 2\pi x dx}{\left[\int_{x=0}^{x=r} \dot{I}_x 2\pi x dx\right]^2} \text{ (real part).}$$

Since this ratio can only be determined by tedious calculations, the result of exact calculations is here shortly given. For low frequencies, we have for copper wire $\left(\mu = 1 \text{ and } \rho = 0{\cdot}017 \times 10^{-4}\Omega\frac{\text{cm}}{\text{cm}^2}\right)$

$$k = 1 + 0{\cdot}70\left(\frac{cd^2}{1000}\right)^2 - 0{\cdot}40\left(\frac{cd^2}{1000}\right)^4,$$

for aluminium wire $\left(\mu = 1 \text{ and } \rho = 0{\cdot}0285 \times 10^{-4}\Omega\frac{\text{cm}}{\text{cm}^2}\right)$

$$k = 1 + 0{\cdot}25\left(\frac{cd^2}{1000}\right)^2 - 0{\cdot}05\left(\frac{cd^2}{1000}\right)^4,$$

for thin iron wires $\left(\mu = 1000 \text{ and } \rho = 0{\cdot}10 \times 10^{-4}\Omega\frac{\text{cm}}{\text{cm}^2}\right)$

$$k = 1 + 2\left(\frac{cd^2}{10}\right)^2 - 3{\cdot}33\left(\frac{cd^2}{10}\right)^4,$$

where the diameter d of the wire is expressed in cm.

For medium frequencies it is best to use the table calculated by Hospitalier, which gives the values of k for different values of cd^2. This table applies to copper wire with $\rho = 0{\cdot}017$ ohm. To obtain the ratio for wires of other materials, the value of cd^2 must be multiplied by $\frac{\mu}{\rho}0{\cdot}017$, and the value of k corresponding to this new value of cd^2 found from the table:

cd^2	k	cd^2	k
0	1·0000	1520	1·8628
20	1·0000	1880	2·0430
80	1·0001	2280	2·2190
170	1·0258	2710	2·3937
300	1·0805	4820	3·0956
470	1·1747	7500	3·7940
680	1·3180	17000	5·5732
920	1·4920	30000	7·3250
1200	1·6778		

(**b**) For very high frequencies and conductors of magnetic material,

$$\lambda = \frac{2\pi}{10^4}\sqrt{\frac{c\mu}{10\rho}}$$

reaches such high values that $e^{-\lambda x}$ can be neglected compared with $e^{\lambda x}$.

The current-density I_x can then be written

$$I_x = I_{\max}\frac{\epsilon^{(1-j)\lambda x}}{\epsilon^{(1-j)\lambda r}} = I_{\max}\,\epsilon^{(1-j)\lambda(x-r)}. \quad \text{.................. (253)}$$

This, like all the previous equations, serves not only for round wires, but also for bars of rectangular section. For such a bar, x denotes the distance from the middle of the bar and $2r = \Delta$ its thickness. For very high frequencies or permeabilities, the mean current-density in a bar is

$$I_{\text{mean}} = \frac{2}{\Delta}\int_{x=0}^{x=\frac{\Delta}{2}} I_{\max}\,\epsilon^{(1-j)\lambda\left(x-\frac{\Delta}{2}\right)}\,dx$$

$$= \frac{2I_{\max}}{(1-j)\lambda\Delta}\left[1-\epsilon^{-(1-j)\lambda\frac{\Delta}{2}}\right] \simeq \frac{I_{\max}}{(1-j)\lambda\frac{\Delta}{2}}$$

or

$$\rho I_{\max} = (1-j)\lambda\frac{\Delta}{2}\,\rho I_{\text{mean}}. \quad \text{.............................. (254)}$$

When we remember that $\rho \dot{I}_{max}$ denotes the pressure-drop per cm length of the conductor, due to the ohmic resistance and to the field within the conductor, we see that this pressure-drop, based on the mean current-density $\dot{I}_{mean}$ or on the current $\Delta \dot{I}_{mean}$ flowing in the conductor, is composed of two equal components. One of these components is in phase with the current and represents a resistance-drop, while the other leads the current by 90°, and therefore becomes a reactance-drop. Each component is equal to $\frac{1}{2}\lambda\rho$. Hence the same resistance would be obtained, if the current in the conductor was divided into two layers each of thickness $\frac{1}{\lambda}$, since these layers would have an ohmic resistance of $\frac{1}{2}\lambda\rho$ per cm length. For this reason it is said that high-frequency currents only penetrate into the conductor to a thickness $\frac{1}{\lambda}$ or that an outer layer of the conductor of thickness

$$\delta_{eff} = \frac{1}{\lambda} = \frac{10^4}{2\pi}\sqrt{\frac{10\rho}{c\mu}} \text{ cm} \quad \text{.........................(255)}$$

carries the whole current. The effective resistance of the conductor is equal to the resistance of this outer layer, and at the same time this is equal to the effective reactance of the conductor, due to the field within itself. This reactance, however, is usually negligible compared to the reactance due to the field outside the conductor.

The same result is obtained for round wires, where only an outer cylindrical layer of thickness $\delta_{eff} = \frac{1}{\lambda}$ serves to carry the current. For this reason copper tubes are also used as conductors for very high-frequency currents. They not only possess the advantage of utilising the copper better, but they also have a smaller self-induction. Such tubes are used, for example, in switch-gear, and especially for the connections of lightning protectors. The thickness of the conducting layer is as follows:

For copper conductors $\left(\rho = 0{\cdot}017 \times 10^{-4}\Omega\frac{\text{cm}}{\text{cm}^2}\right)$

$$\delta_{eff} = \frac{10^4}{2\pi}\sqrt{\frac{10\rho}{c}} = \frac{6{\cdot}5}{\sqrt{c}} \text{ cm},$$

for aluminium conductors $\left(\rho = 0{\cdot}028 \times 10^{-4}\Omega\frac{\text{cm}}{\text{cm}^2}\right)$

$$\delta_{eff} = \frac{8{\cdot}5}{\sqrt{c}} \text{ cm},$$

for iron conductors $\left(\mu = 1000,\ \rho = 0{\cdot}10 \times 10^{-4}\Omega\frac{\text{cm}}{\text{cm}^2}\right)$

$$\delta_{eff} = \frac{10^4}{2\pi}\sqrt{\frac{10\rho}{c\mu}} = \frac{0{\cdot}5}{\sqrt{c}} \text{ cm}.$$

For railway rails we obtain $\delta_{\text{eff}} = 0{\cdot}1$ cm $= 1$ mm. at 25 cycles. If U is the periphery of the rail in mm, the effective resistance per kilometre length at 25 cycles is

$$r_{\text{eff}} = \frac{0{\cdot}1 \times 10^3}{U} = \frac{100}{U} \text{ ohms.}$$

At 15 cycles the resistance is $\sqrt{\frac{15}{25}} = \sqrt{0{\cdot}6} = 0{\cdot}775$ times as large, i.e. $\frac{77{\cdot}5}{U}$ ohms.

The effective reactance of the rails, due to the field within them, is of course equal to the effective resistance.

(c) If the wires lie near one another as in cables, their mutual induction affects the distribution of current. The highest current-density here occurs in the parts where the wires are near together, and the skin-effect may become very considerable. For this case we can use the formulae given by Prof. G. Mie (*Wied. Ann.* 1900) for non-magnetic wires at low frequencies. The ratio for twin-copper cable is approximately

$$k = 1 + \left[0{\cdot}70 + 8{\cdot}5\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^2 - \left[0{\cdot}40 + 32\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^4$$

and for aluminium cable

$$k = 1 + \left[0{\cdot}25 + 3{\cdot}0\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^2 - \left[0{\cdot}05 + 4{\cdot}1\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^4,$$

where a denotes the distance between the axes of the two conductors. For conductors of magnetic material the distance between the wires has little effect on the current distribution, and in this case the same formulae may therefore be used as for a single conductor.

If the reactance of a cable, due to the field within itself, forms a considerable part of the whole reactance, it is also necessary to correct the coefficient of self-induction at high frequencies. Instead of $0{\cdot}1$ in formula $235a$, we have to put for copper cables

$$0{\cdot}1\left\{1 - \left[0{\cdot}35 + 11{\cdot}2\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^2 + \left[0{\cdot}22 + 45\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^4\right\}$$

and for aluminium cables

$$0{\cdot}1\left\{1 - \left[0{\cdot}125 + 4{\cdot}0\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^2 + \left[0{\cdot}027 + 5{\cdot}7\left(\frac{d}{2a}\right)^2\right]\left(\frac{cd^2}{1000}\right)^4\right\}.$$

If the conductor in the cable consists of several small wires more or less insulated from each other, the skin-effect is considerably reduced, due to thus splitting up the section.

Prof. Mie has given the following formulae for rapid oscillations, in the same place as the above. The ratio k for copper wires is

$$k = \frac{2a}{\sqrt{a^2 - d^2}}\, 1{\cdot}2\sqrt{\frac{cd^2}{1000}} + \frac{1}{8} + \frac{(a - \sqrt{a^2 - d^2})(d^2 + a\sqrt{a^2 - d^2})}{2\sqrt{(a^2 - d^2)^3}}$$

and for aluminium wires

$$k=\frac{2a}{\sqrt{a^2-d^2}}\,0\cdot 92\sqrt{\frac{cd^2}{1000}+\frac{1}{8}+\frac{(a-\sqrt{a^2-d^2})(d^2+a\sqrt{a^2-d^2})}{2\sqrt{(a^2-d^2)^3}}},$$

whilst the coefficient of self-induction approaches the value

$$L=\frac{0\cdot 92}{10^8}\log_{10}\left(\frac{a+\sqrt{a^2-d^2}}{d}\right),$$

as the frequency increases.

(**d**) In a coil consisting of several turns, the distribution of the lines of force of its field is still more complicated than with one or two wires, so that the calculation of the effective resistance is much more difficult. In order to keep the increase in resistance as small as possible, the conductors should be made of flat copper strip, arranged in such a way that the longer side of the section coincides with the direction of the leakage lines. Further, turns which lie in different leakage fields should not be connected in parallel, since heavy local currents might ensue, producing an apparent increase in resistance.

Messrs. Field* have exhaustively treated the distribution of current-density for coils in slots and the increase in resistance due to fields occasioned by the presence of the teeth. Only the main points and the result of these investigations will be given here.

Let us consider two bars placed one above the other, as in Fig. 378, and again assume that the leakage field traverses the slot in straight lines, and that the magnetic reluctance of the iron can be neglected compared with that of the slot. Then it follows that the current-density I_x does not vary in the breadth of the slot, but only in the height. The field-strength increases with the height x according to the following law:

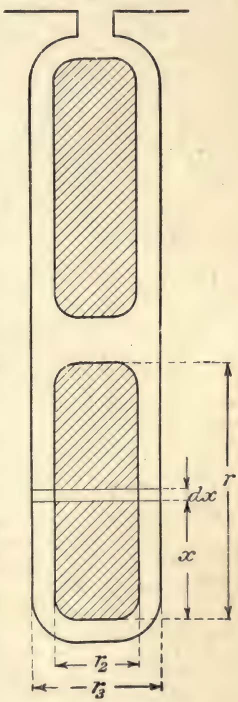


FIG. 378.

$$dH_x=\frac{0\cdot 4\pi r_2 I_x\,dx}{r_3}.$$

In the upper surface of the element of conductor of thickness dx, which we are considering, an E.M.F. is induced, which differs from that induced on the lower surface by dE_x, equal to

$$dE_x=2\pi jc\mu H_x\,dx\,10^{-8}\text{ volts},$$

* *Transactions A.I.E.E.* 1905 and *Proceedings I.E.E.* 1906.

which causes an alteration in the current-density of

$$d\dot{I}_x = -2\pi jc\frac{\mu}{\rho}\dot{H}_x\,dx\,10^{-8}.$$

Hence, from these two differential equations we obtain

$$\frac{d^2\dot{I}_x}{dx^2} = -2\pi jc\frac{\mu}{\rho}\frac{d\dot{H}_x}{dx}10^{-8} = -2\pi jc\frac{\mu}{\rho}\frac{0{\cdot}4\pi r_2}{r_3}\dot{I}_x\,10^{-8}$$

or

$$\frac{d^2\dot{I}_x}{dx^2} = -0{\cdot}8j\pi^2c\frac{\mu r_2}{\rho r_3}\dot{I}_x\,10^{-8},$$

which differs from the equation for wires in air on p. 438 only in the factor $\frac{r_2}{r_3}$. If we substitute

$$\lambda = \frac{2\pi}{10^4}\sqrt{\frac{\mu c r_2}{10\rho r_3}},$$

we have

$$\dot{I}_x = \dot{A}\epsilon^{(1-j)\lambda x} + \dot{B}\epsilon^{-(1-j)\lambda x}$$

and

$$\dot{H} = -\frac{\rho}{2\pi jc\mu}\frac{d\dot{I}_x}{dx} = -\frac{\rho(1-j)\lambda}{2\pi jc\mu}(\dot{A}\epsilon^{(1-j)\lambda x} - \dot{B}\epsilon^{-(1-j)\lambda x}).$$

To determine the constants $\dot{A}$ and $\dot{B}$, we have the following two limits:

Firstly, for $x=0$,

$$\dot{H}_x = \frac{0{\cdot}4\pi(n-1)\dot{I}_{\text{mean}}rr_2}{r_3},$$

where $I_{\text{mean}}rr_2$ denotes the maximum current per conductor, and $(n-1)$ is the number of conductors in the slot underneath the conductor considered. The conductor considered is therefore the n^{th} from the bottom, and $(n-1)\dot{I}_{\text{mean}}rr_2$ is the maximum current-volume lying beneath this conductor.

The second limit is, that the maximum current in a conductor is equal to

$$\int_{x=0}^{x=r}\dot{I}_x r_2\,dx = \dot{I}_{\text{mean}}rr_2.$$

By means of these two limits we can first determine the constants $\dot{A}$ and $\dot{B}$ and then find the ratio k of the effective resistance to the ohmic

$$k = \frac{r_{\text{eff}}}{r} = \frac{\int_{x=0}^{x=r}\dot{I}_x^2\,dx}{\left[\int_{x=0}^{x=r}\dot{I}_x\,dx\right]^2}\ \text{(real part)}.$$

A. B. Field has given the following formula for this ratio:

$$k = \lambda r\frac{4n(n-1)(\cosh\lambda r - \cos\lambda r)(\sinh\lambda r - \sin\lambda r) + (\sinh 2\lambda r + \sin 2\lambda r)}{\cosh 2\lambda r - \cos 2\lambda r}, \tag{256}$$

and this is shewn in Fig. 380 for different values of λr. By means of these curves the ratio k for each turn of the armature coil can now be found, and thus the mean increase in resistance of all the turns easily determined. Fig. 379 shews the current-density and phase-displacement with regard to the main current as functions of the

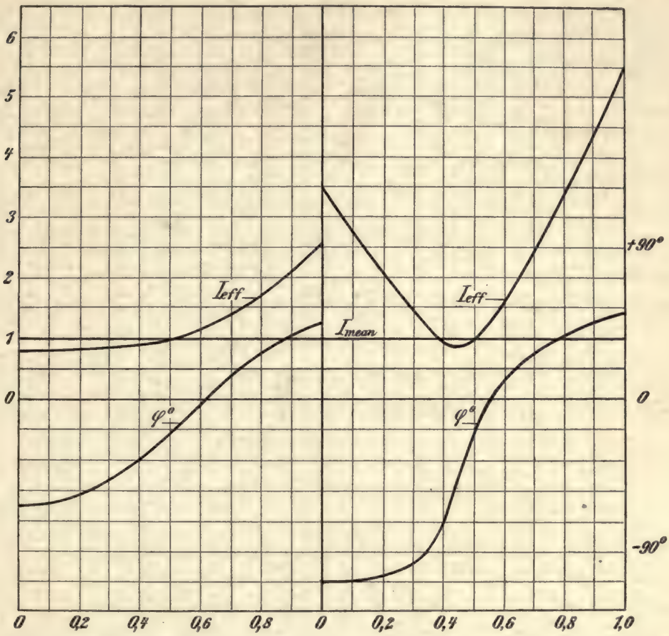


FIG. 379.—Current Density and Phase Displacement in two Armature Conductors.

height of bar. The curves were calculated by A. B. Field for the two conductors shewn in Fig. 378 at 25 cycles. It will be noticed that great variations occur in the current-density. For the lower conductor it is a maximum at the upper corner, while for the upper conductor it is a minimum in the middle. From this, as well as from the curves in Fig. 380, it is clear that the increase in resistance is much greater for the conductor near the armature surface than for the other.

As for wires in air, the skin-effect has not only the effect of increasing the resistance of armature coils, but also of decreasing their self-induction. This is due to the fact that the current is driven upwards in the bars, so that the path of the leakage field across the slot is not straight, as shewn in Fig. 375, but passes chiefly between the bars and through the highest and lowest parts of the bars. If many turns are arranged above one another in the slot, the distortion of the leakage field is not so marked, since the conductors are very

thin and the leakage field varies from the bottom to the top almost according to a straight-line law.

If there are only a few large conductors in the slot, it is advantageous to laminate them parallel to the lines of force or to make them

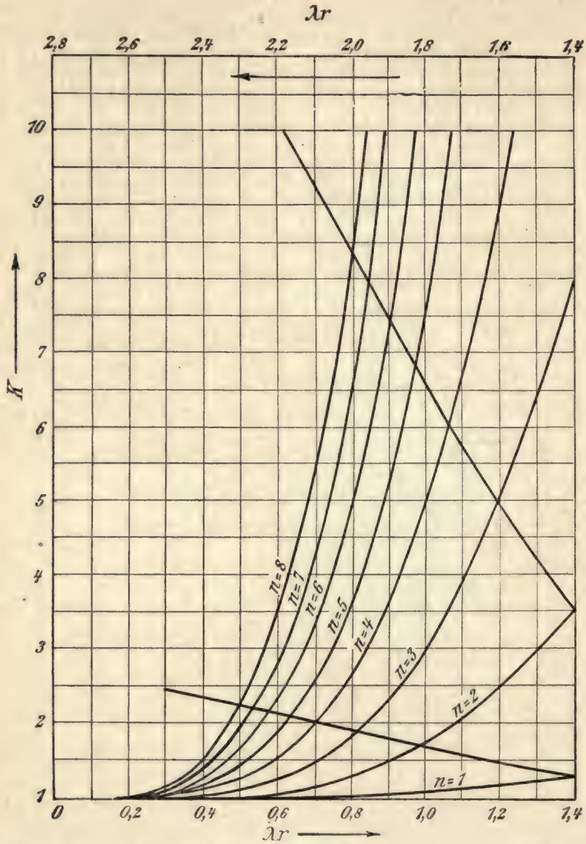


Fig. 380.—Curves for Determining Increase of Resistance in Armature Conductors.

of pressed cable. In many electric machines, such as continuous-current machines, and to a still higher degree in rotary converters, the wave-shape of the currents flowing in the armature conductors is very different from a sine wave. In such cases the current must be resolved into the fundamental and higher harmonics, and the losses on the ratio k calculated for each of these currents. If these ratios are k_1, k_3, k_5, etc. for the currents I_1, I_3, I_5, etc., then for the effective current

$$I = \sqrt{I_1^2 + I_3^2 + I_5^2 + \dots}$$

the effective ratio k is obtained from the equation

$$kI^2 = k_1 I_1^2 + k_3 I_3^2 + k_5 I_5^2 + \dots .$$

Hence $$k = k_1 \left(\frac{I_1}{I}\right)^2 + k_3 \left(\frac{I_3}{I}\right)^2 + k_5 \left(\frac{I_5}{I}\right)^2 + \dots .$$

These considerations and formulae for armature coils can also be used in many other cases with close approximation, so long as the leakage lines run parallel to the surfaces of the conductors, and the path of the lines of force is not appreciably altered through unsymmetrical distribution of current. Such cases occur in transformers and induction coils; but here the paths of the lines of force must be taken into account in choosing the ratio $\frac{r_2}{r_3}$.

(e) Besides the eddy-currents induced in electric conductors by fields within them, there are also currents induced by external fields, which however, do not result in an apparent increase in resistance, but only in a production of heat in the conductor. For these currents the formulae may be used which were developed for the eddy-currents in iron wires and plates. It will be best to demonstrate this by two examples.

On the surface of a smooth armature there is a copper conductor of breadth Δ and thickness r (Fig. 381). The armature has a diameter D and pole-pitch $\tau = \frac{\pi D}{2p}$, and rotates with a peripheral speed of v. We will consider the field in the air-gap as being distributed sinusoidally over the pole-pitch τ. Then the field-strength at any point in the conductor at any moment can be expressed by

$$b = B_l \sin\left(\omega t - \frac{\pi}{\tau} x\right).$$

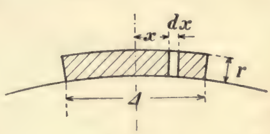


FIG. 381.

The middle of the conductor, where $x = 0$, then falls in the middle of the neutral zone of the magnetic field, where $b = 0$, at time $t = 0$. In an element of the conductor at distance x from the middle, an E.M.F. per cm length is induced equal to

$$e_x = vb\, 10^{-6} \text{ volts},$$

where v is expressed in metres per second. Hence the current-density in this element is

$$i_x = \frac{vb}{\rho} 10^{-6} + C = \frac{v}{\rho 10^6} B_l \sin\left(\omega t - \frac{\pi}{\tau} x\right) + C.$$

The presence of the constant C is due to the fact that the sum of all the internal currents induced in the conductor is equal to zero. Therefore

$$0 = \int_{x=-\frac{\Delta}{2}}^{x=+\frac{\Delta}{2}} i_x dx = -\frac{vB_l}{\rho \frac{\mu}{\tau} 10^6} 2 \cos \omega t \sin \frac{\pi}{\tau} \frac{\Delta}{2} + C\Delta,$$

from which C can be calculated and placed in the expression for i_x.

Hence the current-density is

$$i_x = \frac{vB_l}{\rho 10^6}\left[\frac{\sin\frac{\pi}{\tau}\frac{\Delta}{2}}{\frac{\pi}{\tau}\frac{\Delta}{2}}\cos\omega t + \sin\left(\omega t - \frac{\pi}{\tau}x\right)\right].$$

To find the loss w_w per unit volume, we integrate over $i_x^2 \rho \frac{dx}{\Delta}\frac{dt}{T}$ and obtain

$$w_w = \int_0^T \frac{dt}{T}\int_{x=-\frac{\Delta}{2}}^{x=+\frac{\Delta}{2}} \frac{dx}{\Delta} i_x^2 \rho = \frac{v^2 B_l^2}{2\rho 10^{12}}\left[1 - \frac{\sin^2\left(\frac{\pi}{\tau}\frac{\Delta}{2}\right)}{\left(\frac{\pi}{\tau}\frac{\Delta}{2}\right)^2}\right].$$

Developing the sine into a series and neglecting all terms of the higher orders, we have

$$\left[1 - \frac{\sin^2\left(\frac{\pi}{\tau}\frac{\Delta}{2}\right)}{\left(\frac{\pi}{\tau}\frac{\Delta}{2}\right)^2}\right] = \frac{1}{3}\left(\frac{\pi}{\tau}\frac{\Delta}{2}\right)^2.$$

Further, putting $100v = \frac{\pi D n}{60} = \frac{2p\tau n}{60} = 2\tau c$ and expressing Δ in mm, for a form factor of $f_\epsilon = 1{\cdot}11 = \frac{\pi}{2\sqrt{2}}$, the loss per dm³ is

$$w_w = \frac{4}{3}\frac{10^{-5}}{\rho}\left(\Delta \frac{c}{100}\frac{f_\epsilon B_l}{1000}\right)^2 \text{ watts.} \quad \ldots\ldots\ldots\ldots\ldots\ldots(257)$$

This formula corresponds exactly with the expression given on p. 351 for the eddy-current loss in iron plates. It holds only so long

Fig. 382*a*.

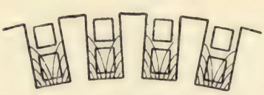


Fig. 382*b*.

Slot Fields.

as the eddy-currents do not appreciably affect the distribution of the lines of force.

If the armature bars lie in slots, E.M.F.'s are also induced in them by the main field. These E.M.F.'s are due mainly to the lines of forces passing between the surface of the pole and the sides of the teeth, which are chiefly present with large open slots and a small air-gap, as is shewn in Fig. 382*a*.

The field-strengths of the slot-leakage field can be resolved into radial and tangential components; the tangential component mainly

induces harmful eddy-currents in the upper conductors. Strongly saturated teeth also raise the field-strength in the slots. If the slots are very deep and the teeth only strongly saturated at the root, the lines of force pass between the sides and bottom of the slots (Fig. 382*b*). They induce eddy-currents in the lower conductors, and in this case the radial as well as the tangential components determine the magnitude of the eddy-current loss.

The eddy-current loss can be determined in this case also by formulae similar to those used for a smooth armature. It is, however, much more difficult to determine, as the calculation is much more complicated, and can only be approximated.

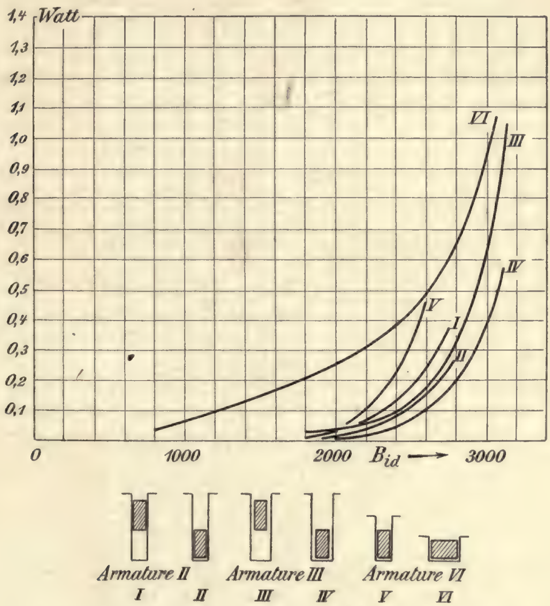


FIG. 383.—Eddy Currents in Armature Conductors.

Dr. Ottenstein* has determined the order of magnitude of this loss by a long series of careful experiments, and has found that maximum tooth-densities of 24–25000 can be employed before large losses occur, due to the lines of force between the sides and bottom of the slots. In Fig. 383 the loss per cm³ is plotted for different slots and different arrangements of the conductors in the slots as a function of the ideal maximum tooth-density B_{id} (i.e. the tooth-density calculated on the assumption that all the lines of force pass through the teeth, which

* "Das Nutenfeld in Zahnarmaturen und die Wirbelstromverluste in massiven Armatur-Kupferleitern." *Sammlung elektrotechnischer Vorträge*, Stuttgart, 1903.

is not actually the case with highly saturated teeth). From this figure it is clear that the lines of force between the pole-face and the surface of the slots may give rise to very high losses.

The highest loss of 1 watt per cm^3 occurring in the curves corresponds to an effective current-density s_w, which is obtained from

$$s_w^2 \rho = 1{\cdot}0.$$

If $\rho = 0{\cdot}02$ is inserted for warm copper, the loss of 1 watt per cm^3 corresponds to an effective current-density $s_w = \sqrt{50}$ amp/mm^2, a value which far exceeds the usual mean density in armature bars. It is therefore advisable, when the copper armature bars lie in open slots, as is usually the case in direct-current machines, not to have the conductors too near the armature surface, that the air-gap should not be too small compared with the breadth of slot (i.e. not less than $\frac{1}{3}$), and that the maximum tooth-saturation is not too high (i.e. not above 25000 on full load). In large alternators with open slots the armature bars near the surface should be laminated tangentially in order to keep the eddy-currents induced by their own field within permissible limits, and the same bars should be laminated radially, in order to destroy the eddy-currents induced by the main field. Since this is not possible in practice, the bars in the neighbourhood of the surface are either made of stranded cable, or they are sunk very deep in the slots and at the same time laminated tangentially.

132. Leakage Fields and Electrodynamic Forces due to Momentary Rushes of Current. During recent years, commercial requirements have led to the building of very large power-stations with large units. At first all the machines were connected to the same bus-bar system and therefore to the same network, since no apparent reasons were forthcoming why the usual practice for small units should be departed from. It had not been considered that with large units working together on the same network, when a short-circuit occurred anywhere in the system an immense amount of energy would act on the short-circuit, and therefore give rise to enormous rushes of current. These rushes produce great mechanical as well as electrical forces, and often lead to destructive explosions in the automatic circuit-breakers, which are provided to cut out the faulty part from the rest of the network. In the following section some formulae will be given for calculating the mechanical forces due to such momentary rushes of current. To determine the mechanical forces, however, the distribution of the leakage fields at the moment of short-circuit must be known, and for this reason the strengths of the leakage fields will be calculated together with the mechanical forces.

To illustrate the forces which act between straight conductors, Fig. 384 shews the switchboard of a 6500 volt motor, destroyed by a short-circuit. The motor was connected to the large network of the Manchester Corporation power-station, and the figure was supplied by C. L. Pearce, Esq., the chief engineer. All the cables were well hung between insulators at a distance of about 12·5 cm apart. The figure

shews clearly how the outgoing and return cables of the same phase were repelled from each other, and the cables of different phases attracted. The insulators a and b were broken and the insulating plate J made of asbestos board was cut clean through. The thin cables, which were for the most part bent, normally carried 10 amperes, but as the following calculations shew, must have carried a very much higher current during the short-circuit. It is clear that the bending

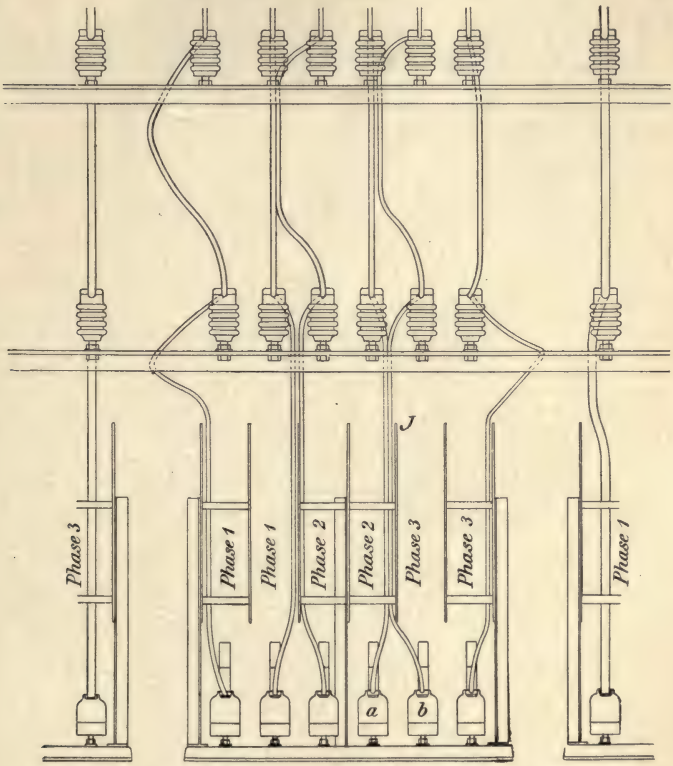


FIG. 384.—Effects of a Short-circuit on the Cable Connections of a Switchboard.

of the cable was greatest near the angle-iron carrying it on account of the magnetic field-strength being greatest there. Also, we may conclude from the figure, that the bending started near the angle-iron, and after the wires had first approached this place the motion proceeded further downwards.

(a) We first calculate the repelling force between two parallel conductors, serving as the outgoing and return lines. The force must be repulsion, since the currents in the two conductors are oppositely directed. It can also be said that the wires tend to move in such a

way that the self-induction of the loop formed by them becomes as large as possible; since the magnetic field-energy is then a maximum. The wires therefore tend to move away from each other. Parallel wires carrying currents in the same direction have the opposite effect. From Ampere's law the repelling or attracting force between two parallel wires per cm length is equal to

$$K = \frac{2i_1 i_2}{a981000 \times 100}\,\text{kg} \simeq \frac{2i_1 i_2}{a10^8}\,\text{kg}, \quad\ldots\ldots\ldots\ldots\ldots(258)$$

where i_1 and i_2 denote the currents in the wires in amperes and a their distance apart in cm. This formula is amplified when we consider that one conductor produces a magnetic field of $H = \frac{2i_1}{10a}$ at the position of the second conductor, and that the mechanical force on the second conductor, from formula (7a), is $\frac{Hi_2}{10}$ dynes. If the two conductors carry the effective current I, the maximum force per cm length is

$$K = \frac{4I^2}{a10^8}\,\text{kg}.$$

Substituting in this $I = 10$ amperes and $a = 12{\cdot}5$ cm, we have

$$K = \frac{4 \times 100}{12{\cdot}5 \times 10^8} = \frac{32}{10^8}\,\text{kg}.$$

For a length of 100 cm the force is thus only about $\frac{32}{10^6}$ kg, and to obtain a force of 1 kg, the rush of current must therefore increase to $\sqrt{\frac{10^6}{32}} = 175$ times its normal value.

Considering further that each cable in Fig. 384 was repelled from one side and attracted from the other, it still requires $\frac{175}{\sqrt{2}} =$ about 125

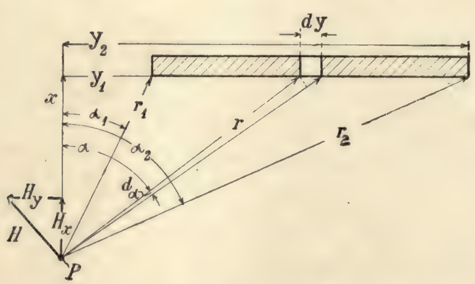


FIG. 385.—Field Intensity of a Long Thin Conductor.

times the normal current to exert a force of 1 kg on a cable 1 metre long. This calculation shews clearly that very considerable rushes of

current are met with in networks of large systems. Short-circuits in such networks act almost like dynamite explosions, in that the forces which occur are sudden shocks, acting momentarily. This accounts for the great damage so often done to the windings of generators and transformers.

In order to calculate the mechanical forces acting on the coils, we shall first consider the field-strength H, produced by a long flat conductor (Fig. 385).

For this purpose we divide H into a component $H_x = H \sin \alpha$ perpendicular to the conductor and a component $H_y = H \cos \alpha$ parallel to the flat side of the conductor. If the conductor, which stands perpendicular to the paper, is very thin and carries the current $i\,dy$ in the element dy, then the field-strength produced by this element at the point P is

$$dH = \frac{2i\,dy}{10r}$$

and its components are

$$dH_x = \frac{2i\,dy \sin \alpha}{10r} \quad \text{and} \quad dH_y = \frac{2i\,dy \cos \alpha}{10r}.$$

Integrating over the whole conductor, we now obtain, since

$$r\,d\alpha = dy \cos \alpha \quad \text{and} \quad dr = dy \sin \alpha,$$

the two resultant components

$$H_x = \int \frac{2i\,dy}{10r} \sin \alpha = \frac{2i}{10} \int \frac{dr}{r} = \frac{2i}{10} \log \frac{r_2}{r_1} = 0{\cdot}46 i \log_{10} \frac{r_2}{r_1} \quad \ldots\ldots(259)$$

and

$$H_y = \int \frac{2i\,dy}{10r} \cos \alpha = \frac{2i}{10} \int d\alpha = 0{\cdot}2 i (\alpha_2 - \alpha_1). \quad \ldots\ldots(260)$$

i is here the current per cm breadth of the conductor. If the length of the conductor is not very great, but considerable with regard to the distance of the point P, the two components H_x and H_y must be multiplied by $\frac{\gamma^\circ}{180^\circ}$, where γ is the angle in degrees which the conductor subtends at the point P. If the conductor is not very thin, the components H_x and H_y (Fig. 386) must be determined by a double integration

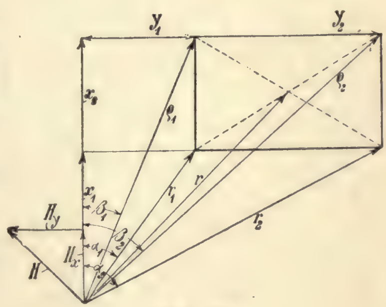


FIG. 386.

$$H_y = \iint \frac{2i\,dx\,dy}{10r} \cos \alpha = \int \frac{2i}{10} (\alpha_2 - \alpha_1)\,dx.$$

Since $\tan\alpha_1 = \frac{y_1}{x}$ and $\tan\alpha_2 = \frac{y_2}{x}$ (Fig. 386),

we have $$H_y = \int_{x=x_1}^{x=x_2} \frac{2i}{10}\left[\tan^{-1}\left(\frac{x}{y_2}\right) - \tan^{-1}\left(\frac{x}{y_1}\right)\right]dx,$$

whence

$$H_y = \frac{2i}{10}\left[x_2(\beta_2-\beta_1) - x_1(\alpha_2-\alpha_1) + 1{\cdot}15y_2\log_{10}\left(\frac{\rho_2}{r_2}\right) - 1{\cdot}15y_1\log_{10}\left(\frac{\rho_1}{r_1}\right)\right] \tag{260a}$$

where i denotes the current-density per cm² and x_1, x_2, y_1 and y_2 are expressed in cm. In the same way we have for H_x,

$$H_x = \frac{2i}{10}\left[y_2(\alpha_2-\beta_2) - y_1(\alpha_1-\beta_2) + 1{\cdot}15x_1\log_{10}\left(\frac{r_1}{r_2}\right) - 1{\cdot}15x_2\log_{10}\left(\frac{\rho_1}{\rho_2}\right)\right], \tag{259a}$$

and the resultant field-strength is

$$H = \sqrt{H_x^2 + H_y^2}.$$

If the conductor is not very long, the factor $\frac{\gamma}{180}$ must be added to this. This formula also holds for a coil-side consisting of several turns, in which case i denotes the current volume per cm², and the lengths are expressed in cm. As a first approximation, the field-strength can also be written

$$H = \frac{2i(x_2-x_1)(y_2-y_1)}{10r}, \tag{261}$$

where r denotes the distance of the point considered from the centre of the coil.

(**b**) Considering two coils placed over one another, as in Fig. 387, then if they are connected in series to oppose each other (or if either coil is short-circuited on itself), the two coils will be repelled by a momentary rush of current. The leakage field, passing between the two coils, tends to spread out as much as possible and thereby exerts a strong repelling force on the upper coil. This repelling force can be calculated from the above formulae for the field-strength. The field-strength is approximately equal to

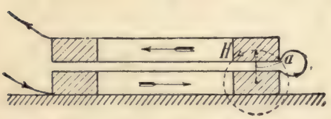


Fig. 387.—Two Mutually-repelling Coils.

$$H = \frac{2iw}{10a}$$

and the repelling force

$$K = l_s\frac{(iw)H}{10^7} = \frac{2(iw)^2 l_s}{a10^8}\ \text{kg}, \tag{262}$$

where l_s is the mean length of the coils, iw the ampere-turns and a the distance between the coils from centre to centre.

The leakage field in all electric machines and transformers strives to attain maximum field-energy, just as do the two coils in Fig. 387.

Since the leakage field is squeezed between the primary and secondary windings, and always tries to expand as much as possible, the windings are driven apart by momentary rushes of current, if they are not fixed securely enough. Rushes of current which exert these forces are chiefly due to short-circuits in the secondary circuit, that is, in the stator circuit in the case of alternators. In this case the field winding is the primary and the stator winding the secondary. Besides this, mechanical forces also occur in machines and apparatus between the several coils of one winding carrying the same or proportional currents. These coils need not belong to the same phase.

In a transformer in which the coils of the primary and secondary windings are sandwiched between one another, as in Fig. 388, the leakage fields are squeezed between each primary and secondary coil, so that these mutually repel one another.

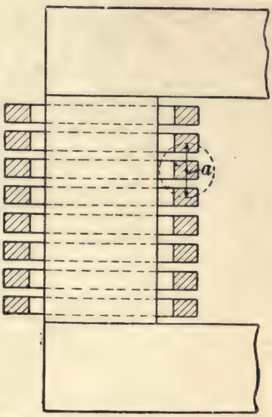


FIG. 388.—Section of Transformer with Disc Winding.

It has even happened that the coils themselves have been blown apart. The mechanical forces acting on the upper and lower coils are of course the largest, since in the neighbourhood of the yoke the permeance of the leakage field is greatest.

To determine the repelling force between two coils, we must first make a calculation of the field-strength produced by one coil at the position of the next coil. On account of the great magnetic permeance of the iron core, this is not $\frac{2iw}{10a}$, but almost double this value. It must of course be considered that the rushes of current occur so rapidly, that the iron partially loses its permeance owing to the eddy-currents induced in the plates. This remains, however, so large on the sides where the leakage lines enter the iron parallel to the laminations, that the field-strength here must be put equal to $\frac{4iw}{10a}$, while on the sides where the leakage field enters at right angles to the plates, $\frac{2iw}{10a}$ must be used. The mean field-strength is therefore somewhat smaller than $\frac{3iw}{10a}$. For this reason the short-circuit reactance of a transformer becomes somewhat smaller during a momentary rush of current than under steady conditions. Denoting the effective value of the momentary short-circuit current by I_{mk} and the number of turns of the outer coil by w_s, the maximum force by which the upper and lower coils are pressed against the yoke is

$$K_{\max} \simeq \frac{6(I_{mk}w_s)^2 l_s}{a10^8} \text{kg}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (263)$$

if all the coils have the same ampere-turns. If the coils in the middle have double as many ampere-turns as the two outer coils, the force is approximately double as large as that given by the formula. The formula is not very accurate, because of the very great difficulty in calculating I_{mk}.

In transformers with cylinder windings, shewn in Fig. 373, the field-strength produced by one winding at the position of the second can be calculated from formulae 259 and 260. It only interests us here to find the maximum field-strength, which occurs at the middle of the windings. Here $H_x = 0$ and

$$H = H_y = \frac{2iw}{10L}(a_2 - a_1),$$

or if $AS = \frac{Iw}{L}$ denote the effective ampere-turns per cm length of the winding, the maximum field-strength is

$$H_{\max} = \frac{2\sqrt{2}AS}{10}(a_2 - a_1).$$

Hence the force exerted outwards on a coil of w_s turns per cm length of coil is

$$K_{\max} = \frac{4ASIw_s}{10^8}(a_2 - a_1)\,\text{kg.} \qquad (264)$$

If the coil is circular, the force, distributed uniformly over the whole coil, exerts a bursting action on it. If, on the other hand, the coil is rectangular, which is usually the case in large transformers, the long sides of the rectangle tend to bend out, so that the shape becomes elliptical.

Mechanical forces do not only, however, act between the primary and secondary coils on the same core, but also the outer coils on neighbouring cores are mutually attracted, since currents flow in the same direction in the adjacent coil-sides. These forces of attraction can be calculated from the same formulae.

In addition to short-circuits, rushes of current also occur in transformers when they are switched on to the network. These rushes are heavier, the more strongly the iron is saturated. In this case the secondary circuit is open, and therefore carries no current; the primary coil then tends to move towards the position of highest reactance. For this reason care must be taken with cylinder windings that the coils are at equal distances from the two yokes, while in all transformers the upper coils must be well fixed relatively to the yoke, so that they are not drawn against the yoke on switching in.

(c) The argument for generators is similar to that for transformers. The primary and secondary leakage fields strive to press between the stator and field windings and to drive them apart. Here the field winding is fixed so well on the inner rotating member that it cannot be displaced. For this reason the repelling forces tend to drive the coil-ends of the stator winding away from the field system. Forces of repulsion or attraction also occur between the coil-ends of the several

phases, according to the direction of current in the phases at the moment of short-circuit. If a coil-end is very near the iron, it is usually drawn against the iron. With the arrangement of the coil-ends of a three-phase generator shewn in Fig. 389, the coil-ends of phase I are usually bent outwards by the leakage fields between the stator and field windings, while those of the second and third phases are mutually repelled. To calculate the repelling force on phase I, it must be borne in mind that at the moment of short-circuit the main field cannot suddenly vanish despite the demagnetising effect of the stator current and that a greater current is induced in the field coil, which strives to maintain the field. In this way a large primary leakage field crosses over to the pole-shoe, and bends the coil-end of phase I outwards. To determine the forces present it is necessary to know the momentary current in the field coils as well as the magnitude of the main field. If this momentary exciting current is known to be i_{me}, the magnetomotive force $i_{me} w_e - aw_m$ acts on all the tubes of force between pole-shoe and yoke, where aw_m denotes the ampere-turns necessary to send the flux through the field system. The field-strength about phase I can be calculated approximately by drawing the lines of force, and we have

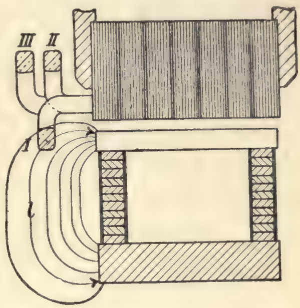


FIG. 389.—Section of Three-phase Generator.

$$H \simeq \frac{i_{me} w_e - aw_m}{0{\cdot}8l}.$$

The maximum mechanical force per cm length of the coil-end is then

$$K = \frac{H i_{a\,\max} w_s}{10^7} = \frac{i_{me} w_e - aw_m}{0{\cdot}8l10^7} i_{a\,\max} w_s \text{ kg}, \quad \text{.............(265)}$$

where $i_{a\,\max}$ is the effective momentary short-circuit current in phase I and w_s is the number of turns in the coil-end. Since $i_{me} w_e$ may in the case of large machines attain a value of 100,000 ampere-turns at the moment of short-circuit, while $i_{a\,\max}$ at the same time reaches a value of 150,000, we have

$$K = \frac{10^5 \times 1{\cdot}5 \times 10^5}{0{\cdot}8l10^7} = \frac{1500}{0{\cdot}8l} \text{ kg}.$$

Thus if $l = 36$ cm, $K = 52$ kg. If the pole-arc of the machine is 60 cm and the length of the coil-end 80 cm, we can reckon on a force on the coil-end of about

$$52 \frac{60 + 80}{2} = 3600 \text{ kg}.$$

Evidently very considerable forces may occur in large machines. For this reason the arrangement shewn in Fig. 389 is not used, and

when possible, the coil-ends are arranged in two planes, as shewn in Fig. 390. The coil-ends are now so far removed from the field coils, that these have little effect. In this latter winding there are chiefly repelling forces between the coil-ends, since at any moment the currents are almost always oppositely directed in the two planes. In the part of the coils running axially, where they come straight out of the slots, the same direction of current occurs in groups, so that attracting as well as repelling forces are here present. The latter are the largest, since

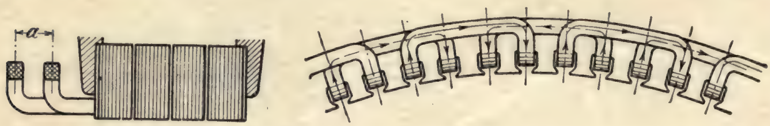


FIG. 390.—Current Distribution in the Coil-ends of Three-phase Generator.

the leakage field between the coils is the greatest, where the current changes its direction. In order to make the repelling forces between the coil-ends of the several phases harmless, they must be fixed as firmly as possible; and further, care must be taken that the coil-ends are sufficiently far from the iron. It is possible to calculate the field strength of the leakage field, which one coil produces where the other is situated, for various positions. To calculate it accurately, the formulae on p. 454 must be used, but we can write as an approximation

$$H=\frac{2i_{a\max}w_s}{10a}$$

and

$$K=\frac{2i_{a\max}w_s}{10a}\,\frac{\frac{i_{a\max}}{2}w_s}{10^7}=\frac{(i_{a\max}w_s)^2}{a10^8}. \quad \text{.................(266)}$$

This holds for the moment when the current is a maximum in one phase and half as large in the other two. For $i_{a\max}w_s=150{,}000$ and $a=10$ cm, we have

$$K=\frac{2\cdot25\times10^{10}}{10\times10^8}=22\cdot5 \text{ kg per cm.}$$

With an active length of 60 cm, the total force on a coil-end becomes

$$K=22\cdot5\times60=1350 \text{ kg,}$$

which is certainly a considerable force. It is clear from the foregoing that it is of the utmost importance to keep the momentary short-circuit current in electric generators and transformers as small as possible. This, however, is not possible without allowing an undue fluctuation in pressure, due to alterations in the working load. In this matter, as so frequently happens in practice, a compromise has to be made between two evils.

133. Capacity and Conduction of Electric Cables.

(a) In order to begin with the simplest case, the capacity of a concentric cable (Fig. 391) will first be calculated. The two conductors may be considered as the plates of a condenser consisting of a pair of cylinders. Denoting the electric charge of the inner conductor by Q, its potential by P, the dielectric constant of the dielectric between the two conductors by ϵ, the diameter of the inner conductor by d and the inside diameter of the outer conductor by $2a$, formula 202 (for the capacity of a pair of cylinders) gives the capacity of the concentric cable per unit length (1 cm) in electrostatic units, thus

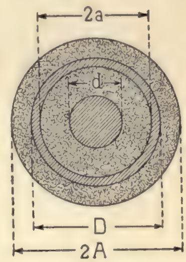


Fig. 391.—Section of a Concentric Cable.

$$C = \frac{Q}{P} = \frac{\epsilon}{2 \log\left(\frac{2a}{d}\right)},$$

or for the length l in kilometres and C in electromagnetic units

$$C = \frac{1}{9 \times 10^{20}} \frac{\epsilon l 10^5}{2 \log\left(\frac{2a}{d}\right)}.$$

Since capacity is usually measured in microfarads (mfd), where $1 \text{ mfd} = \frac{1}{10^{15}}$ times the electromagnetic unit, we have

$$C = \frac{1}{9 \times 10^{20}} \frac{\epsilon l 10^5 10^{15}}{2 \log\left(\frac{2a}{d}\right)} \text{ mfd}$$

or

$$C = \frac{\epsilon l}{9 \times 2 \times 2{\cdot}3 \log_{10}\left(\frac{2a}{d}\right)} = \frac{0{\cdot}0242 \epsilon l}{\log_{10}\left(\frac{2a}{d}\right)} \text{ mfd.} \quad \text{.........(267)}$$

The susceptance b_0 due to the capacity of a cable is

$$b_0 = 2\pi c C,$$

where C is the capacity measured in practical units (farads). The capacity susceptance of a concentric cable is therefore equal to

$$b_0 = 2\pi c \frac{0{\cdot}0242 \epsilon l}{10^6 \log_{10}\left(\frac{2a}{d}\right)} \text{ mho.} \quad \text{......................(268)}$$

Denoting the effective alternating pressure between the conductors of the cable by P, the capacity gives rise to a wattless displacement current

$$I_{wl\,0} = P b_0,$$

which leads the pressure by 90°.

Since the insulation between the conductors is never perfect, and on account of the dielectric hysteresis, a current in phase with the pressure also flows into the cable. This watt-current is equal to

$$I_{w0}=Pg_0.$$

Of this we shall calculate the part due to imperfect insulation, i.e. the *conduction-current* Pg_a. g_a is the electric conductance, or the reciprocal of the resistance, between the two conductors, and is called the *conduction of the cable.* It is given by

$$\frac{1}{g_a}=\int_{x=\frac{d}{2}}^{x=a}\frac{\rho_i dx}{2\pi\rho l}=\frac{\rho_i}{2\pi l}\log\left(\frac{2a}{d}\right)$$

or

$$g_a=\frac{2\pi l}{\rho_i \log\left(\frac{2a}{d}\right)}, \quad\quad (269)$$

where ρ_i is the specific resistance per $\frac{\text{cm}}{\text{cm}^2}$ and l is the length of the cable in cm. Substituting l in kilometres and as is usual ρ_i in megohms per $\frac{\text{cm}}{\text{cm}^2}$, we have

$$g_a=\frac{2\pi l 10^5}{2\cdot 3\times 10_6\rho_i \log_{10}\left(\frac{2a}{d}\right)}=\frac{0\cdot 272 l}{\rho_i \log_{10}\left(\frac{2a}{d}\right)}\text{ mho.}$$

g_a, however, is strongly affected by the junctions in the surface at the ends and connecting-points of the cable, and therefore *in a network with many branches the conductance g_a is much greater than the value calculated from the above formula.*

In the above calculation it is assumed that the insulation between the two conductors consists of a homogeneous material with a constant dielectric constant ϵ. If this is not the case, the calculation becomes very complicated, for the dielectric must then be considered as several condensers in series with different insulation resistances. The capacity of the cable in this case may be approximated as follows:

$$C=\frac{0\cdot 0242 l}{\frac{1}{\epsilon_1}\log_{10}\left(\frac{d_1}{d}\right)+\frac{1}{\epsilon_2}\log_{10}\left(\frac{d_2}{d_1}\right)+\dots+\frac{1}{\epsilon_n}\log\left(\frac{2a}{d_n}\right)}\text{ mfd,} \quad (270)$$

where d_x is the outside diameter of the x^{th} layer of insulation. Similarly the conduction is approximately

$$g_a=\frac{0\cdot 272 l}{\rho_i \log_{10}\left(\frac{d_1}{d}\right)+\rho_2\log\left(\frac{d_2}{d}\right)+\dots+\rho_n\log\left(\frac{2a}{d_n}\right)}\text{ mho.} \quad (271)$$

In addition to the capacity between the two conductors, the capacity between one conductor and earth must be considered.

If the inner conductor is disconnected while the outer still remains under pressure, the capacity of the outer conductor (Fig. 391) with regard to earth is

$$C=\frac{0{\cdot}0242\epsilon l}{\log_{10}\left(\frac{2A}{D}\right)}\text{ mfd.}$$

If the inner conductor is earthed, the capacity of the outer conductor, with regard to the inner and to earth, is

$$C=0{\cdot}0242\epsilon l\left[\frac{1}{\log_{10}\left(\frac{2a}{d}\right)}+\frac{1}{\log_{10}\left(\frac{2A}{D}\right)}\right]\text{ mfd.}$$

If, on the other hand, the outer conductor is disconnected, the capacity of the inner conductor, with regard to the outer, is in series with that of the outer with regard to earth. Hence the capacity of the inner conductor with regard to earth is

$$C=\frac{0{\cdot}0242\epsilon l}{\log_{10}\left(\frac{2a}{d}\right)+\log_{10}\left(\frac{2A}{D}\right)}\simeq\frac{0{\cdot}0242\epsilon l}{\log_{10}\left(\frac{2A}{d}\right)}\text{ mfd.}$$

This is much smaller than the capacity of the outer conductor with regard to earth.

Further, the capacity of the inner conductor with regard to earth, when the outer is earthed, is

$$C=\frac{0{\cdot}0242\epsilon l}{\log_{10}\left(\frac{2a}{d}\right)}\text{ mfd.}$$

(**b**) We now proceed to calculate the capacity of an air-line in a system, using the earth as a return.

In Fig. 392, the electric lines of force (current curves) x and the equipotential surfaces y of the electric field are shewn as they are produced by the conductors A and B charged with equal quantities of electricity, but of opposite sign. The curves x and y represent only the intersections of the current and equipotential surfaces with the plane of the paper. The electric resistance of any element of a tube of force is proportional to $\frac{dy}{dx}$.

By means of a mathematical transformation,* we can now replace the diagram in Fig. 392 by another simpler geometric diagram, in which each elemental tube of force has exactly the same resistance as the corresponding tube in the original system.

The capacity and conduction are thereby unaltered, and their calcula-

* Steinmetz, *E.T.Z.* 1893, S. 477.

tion is considerably simplified. Denoting the new system of current and equipotential curves by v and u, then, in order to satisfy the above condition, we must have

$$\frac{du}{dv}=\frac{dx}{dy}.$$

As is well known, this condition is fulfilled by any equivalent transformation from one plane to another; any transformation being

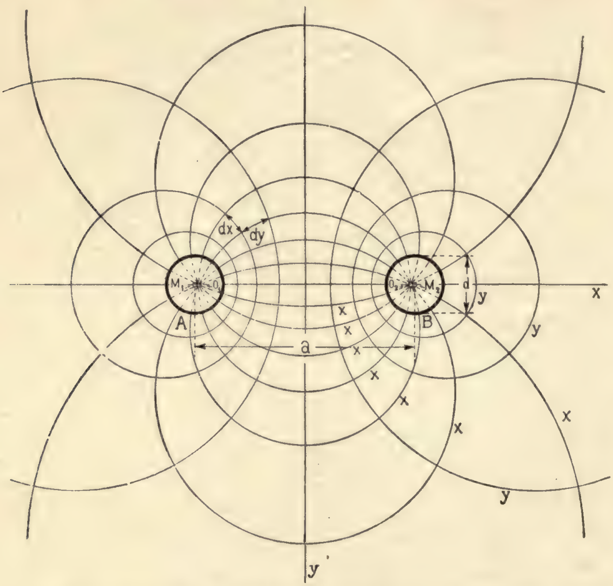


FIG. 392.—Current and Equi-potential Curves of Two Parallel Conductors.

called equivalent or equiangular, when any two curves of the one plane make the same angle as the corresponding curves of the second plane.

We have already had recourse several times to a transformation of this kind, namely inversion, or, as it also called, transformation by reciprocal radii. Since the problem can be solved very simply with this transformation, we make use of it here.

If a conductor A is given, as above, with the earth serving as return, the system of current-lines and equipotential curves given by the circle A and line B (the surface of the earth) may be transformed into another equivalent system. We may, for example, convert the circle A and the line B (Fig. 393) into two concentric circles. To do this, we mark off the inversion centre O, the perpendicular to B drawn through the centre of circle A, and further choose the inversion

coefficient in such a way that circle A corresponds to itself and line B to a circle concentric with A. We then have

$$\overline{OM} = \overline{ME}_1$$

and
$$\overline{OT}^2 = I = \overline{OP} \cdot \overline{OP}_1,$$

where I is the constant of inversion.

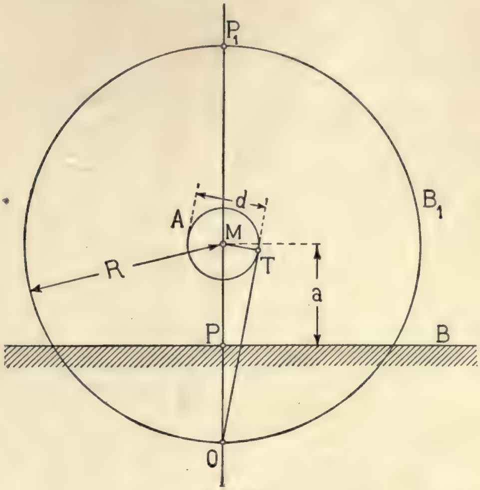


FIG. 393.

$\overline{MP} = a$ is the height of the conductor A above the earth, $\overline{MT} = \frac{d}{2}$ its radius and $\overline{OM} = R$ the radius of the large circle.

Hence
$$\overline{OT}^2 = R^2 - \left(\frac{d}{2}\right)^2 = I = (R-a)\,2R$$

or
$$R^2 - 2Ra + \left(\frac{d}{2}\right)^2 = 0,$$

that is,
$$R = a + \sqrt{a^2 - \left(\frac{d}{2}\right)^2}.$$

If d is negligible compared with a, then

$$R = 2a,$$

that is, the capacity and conduction between a conductor at a height a above the surface of the earth and the earth are the same as between the conductor and a concentric cylinder, of which the radius R is approximately double the distance of the conductor from the earth. The capacity in this case is therefore

$$C = \frac{0{\cdot}0242\epsilon l}{\log_{10}\left(\frac{2R}{d}\right)} = \frac{0{\cdot}0242\epsilon l}{\log_{10}\frac{2a + \sqrt{4a^2 - d^2}}{d}},$$

or very closely

$$C = \frac{0{\cdot}0242\epsilon l}{\log_{10}\left(\frac{4a}{d}\right)} \text{ mfd,} \quad \text{.......................(272)}$$

and the conductance for determining the conduction current is

$$g'_0 = \frac{0{\cdot}272 l}{\rho_i \log_{10}\left(\frac{4a}{d}\right)}. \quad \text{...........................(273)}$$

(c) In calculating the capacity of a double line, where the two conductors are arranged near one another as overhead lines or placed underground, either together in one cable or as separate cables, it must be remembered that the earth affects the electric distribution.

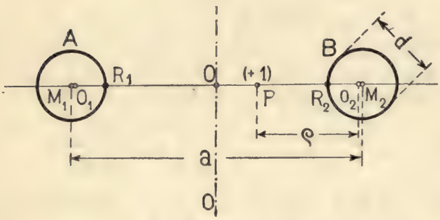


FIG. 394*a*.

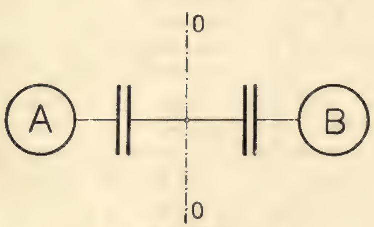


FIG. 394*b*.

We shall first consider the simple case, in which the effect of the earth on the capacity of the double line can be neglected. If the two conductors are represented by the circles A and B in Fig. 394*a*, we know that the line $\overline{OO}$ perpendicular to the line joining the centres of A and B represents an equipotential surface of zero potential. The electric field between conductor A and the surface $\overline{OO}$ and between conductor B and the surface $\overline{OO}$ can therefore each be replaced (Fig. 394*b*) by a condenser of capacity

$$C = \frac{0{\cdot}0242\epsilon l}{\log_{10}\left(\frac{a+\sqrt{a^2-d^2}}{d}\right)}$$

and of conductance

$$g'_0 = \frac{0{\cdot}272 l}{\rho_i \log_{10}\left(\frac{a+\sqrt{a^2-d^2}}{d}\right)}.$$

Connecting these two equal condensers in series, we obtain a capacity and conductance equal to half of each condenser. The capacity of a double line, neglecting the influence of the earth, is therefore equal to

$$C = \frac{0{\cdot}0242\epsilon l}{2\log_{10}\left(\frac{a+\sqrt{a^2-d^2}}{d}\right)} \simeq \frac{0{\cdot}0242\epsilon l}{2\log_{10}\left(\frac{2a}{d}\right)} \text{ mfd,} \quad \text{..........(274)}$$

and the conductance equals

$$g_0' = \frac{0{\cdot}272l}{2\rho_i \log_{10}\left(\frac{a+\sqrt{a^2-d^2}}{d}\right)} \simeq \frac{0{\cdot}272l}{2\rho_i \log_{10}\left(\frac{2a}{d}\right)} \text{ mho. } \ldots\ldots(275)$$

From this we come to the conclusion, as Steinmetz first shewed, that an earth-return, as regards capacity and conduction, behaves like a conductor symmetrical to the overhead line with respect to the earth, whose distance and potential are the same below the earth as the air-line is above it. *The conductor, equivalent to the earth, is therefore the image of the overhead-line in the earth's surface.*

In Fig. 392 are shewn the electric lines of force and the equipotential curves of the electric field of a double line. All the lines of force are arcs of circles, which, if produced inside the conductors, intersect at the points O_1 and O_2. It is further known that

$$\overline{O_1O_2} = 2\sqrt{\left(\frac{a}{2}\right)^2 - \left(\frac{d}{2}\right)^2} = \sqrt{a^2-d^2}.$$

The physical meaning of this is that the electric field produced by the charges on the cylindrical conductors A and B is the same, as if the charges of the conductors were concentrated on the straight lines O_1 or O_2, running parallel to the axis of the conductors.

We can now determine the capacity of a double line in the same way as for a concentric cylinder (p. 387). Thus we calculate the work done in moving unit positive electric mass from the surface of a conductor to the neutral zone. This work is equal to the potential of the respective conductor, and is equal to half the pressure between the conductors. The force acting on unit positive mass at the point P (Fig. 394*a*) is

$$\frac{1}{\epsilon}\frac{2Q}{OP} + \frac{1}{\epsilon}\left(\frac{-2Q}{O_1P}\right) = \frac{1}{\epsilon}\left(\frac{2Q}{\rho}\right) - \frac{1}{\epsilon}\left(\frac{2Q}{\overline{O_1O_2}-\rho}\right).$$

Multiplying this equation by $d\rho$ and integrating from $\rho = \overline{R_2Q_2}$ to $\rho = \overline{OO_2}$, we obtain the work for half the pressure equal to

$$\tfrac{1}{2}P = \frac{2Q}{\epsilon}\log\frac{\overline{OO_2}}{\overline{R_2O_2}} - \frac{2Q}{\epsilon}\log\frac{\overline{OO_1}}{\overline{R_2O_1}} = \frac{2Q}{\epsilon}\log\frac{\overline{R_2O_1}}{\overline{R_2O_2}}.$$

It follows from Fig. 394*a*, that

$$\overline{R_2O_1} = \overline{OO_1} - \overline{R_2O} = \tfrac{1}{2}(\sqrt{a^2-d^2}+a-d)$$

and

$$\overline{R_2O_2} = \overline{OO_2} - \overline{R_2O} = \tfrac{1}{2}(\sqrt{a^2-d^2}-a+d),$$

and therefore

$$\frac{\overline{R_2O_1}}{\overline{R_2O_2}} = \frac{\sqrt{a^2-d^2}+a-d}{\sqrt{a^2-d^2}-a+d} = \frac{a+\sqrt{a^2-d^2}}{d}.$$

Hence the capacity of a double line per cm length, in electrostatic units, is

$$\frac{Q}{P} = \frac{\epsilon}{4\log\left(\frac{a+\sqrt{a^2-d^2}}{d}\right)},$$

which formula corresponds to the previous ones. In this case we have moved the point P along the central line $\overline{O_1O_2}$, but since the potential difference between R_2 and O is independent of the path of P, the same result is always obtained, whatever the motion of P. From this it follows, in general, that *the work done by the electric charge of a straight line O_2, when unit mass is moved from R to S, is proportional to* $\log \frac{\overline{O_2S}}{\overline{O_2R}}$.

To determine the capacity of a double line, taking the earth's influence into account, we substitute for the earth, two equivalent conductors A' and B', forming the images of A and B in the earth's surface. If A and B have the charges $-Q$ and $+Q$, then A' and B' will have the charges $+Q$ and $-Q$ respectively.

To obtain the effective capacity of the double line, including the effect of the earth, we calculate, as shewn on p. 392, the work done in moving unit positive mass from the earth to the surface of the conductor B. The work done by the charge on B itself is equal to (cf. Fig. 394)

$$\frac{2Q}{\epsilon}\log\frac{\overline{OO_2}}{\overline{R_2O_2}},$$

by charge A
$$-\frac{2Q}{\epsilon}\log\frac{\overline{OO_1}}{\overline{R_2O_1}},$$

by charge B'
$$-\frac{2Q}{\epsilon}\log\frac{\overline{OO_2'}}{\overline{R_2O_2'}}$$

and by charge A'
$$+\frac{2Q}{\epsilon}\log\frac{\overline{OO_1'}}{\overline{R_2O_1'}}.$$

Since the dielectric constant is here equal to 1, the total work equals

$$\tfrac{1}{2}P = 2Q\left(\log\frac{\overline{R_2O_1}}{\overline{R_2O_2}} - \log\frac{\overline{R_2O_1'}}{\overline{R_2O_2'}}\right)$$
$$= 2Q\left[\log\left(\frac{a+\sqrt{a^2-d^2}}{d}\right) - \log\left(\frac{\sqrt{4h^2+a^2}}{a}\right)\right].$$

The capacity of the double line therefore equals

$$C = \frac{0{\cdot}0242l}{2\left[\log_{10}\left(\frac{a+\sqrt{a^2-d^2}}{d}\right) - \log_{10}\sqrt{1+\left(\frac{a}{2h}\right)^2}\right]} \text{ mfd.} \quad \ldots\ldots(276)$$

(d) To determine the capacity of the conductors of a three-phase system, we proceed in the same way, by moving unit positive mass from one conductor to the neutral. The work done in this way is equated to the phase-pressure P_p. If conductor I, from which the mass is moved, has the charge $Q \sin \omega t$, the other two conductors will

have charges $Q\sin(\omega t - 120°)$ and $Q\sin(\omega t - 240°)$. The work done (Fig. 395a) is therefore equal to

$$P_p = \frac{2Q}{\epsilon}\sin\omega t \log\frac{\overline{O_1O}}{O_1R_1} + \frac{2Q}{\epsilon}\sin(\omega t - 120°)\log\frac{\overline{O_2O}}{O_2R_1}$$
$$+ \frac{2Q}{\epsilon}\sin(\omega t - 240°)\log\frac{\overline{O_3O}}{O_3R_1}$$
$$= \frac{2Q}{\epsilon}\sin\omega t \log\frac{\overline{O_2R_1}}{O_1R_1}.$$

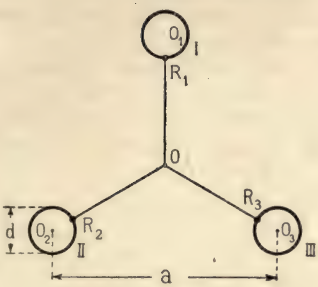


FIG. 395a.

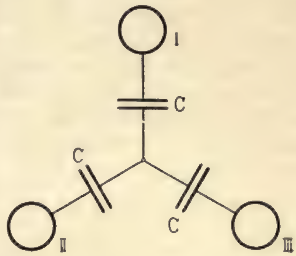


FIG. 395b.

Neglecting the effect of the earth, the capacity per phase of a three-phase line is

$$C = \frac{0{\cdot}0242\epsilon l}{\log_{10}\dfrac{O_2R_1}{O_1R_1}} \text{ mfd.}$$

If further, as in the case of overhead lines, the distance a between the wires is very great compared with their diameter, the capacity may be written with close approximation

$$C = \frac{0{\cdot}0242 l}{\log_{10}\left(\dfrac{2a}{d}\right)} \text{ mfd.} \quad \text{........................(277)}$$

The capacity of the mains of a three-phase system can thus be considered as three condensers connected in star, each of which has the capacity C.

Since three-phase concentric cables introduce dissymmetry into the system (and possess a higher capacity), cables for three-phase work are almost always made stranded. Each phase of a concentric cable has a different capacity to the others.

With stranded cables the effect of the earth on the capacity of each phase must be considered. This can be done approximately in a simple way. In Fig. 396a, the circle A represents the conductor of one phase, and the circle B, the surface of the cable-sheath. This system, consisting of two eccentric circles, is replaced by inversion by

a system consisting of the circle A, which corresponds to itself and the straight line B'. We have

$$\overline{OP_1} = \frac{\overline{OT}^2}{\overline{OP}} = \frac{\overline{OT}^2}{D}.$$

The system consisting of the circle A and the straight line B' is again replaced by an equivalent system, consisting of the circle A and its image B'' with respect to B'. The circle B'' has the opposite electric charge to A, that is $-Q \sin \omega t$. Carrying out this transformation for each phase, we obtain Fig. 396b. Assuming, for the sake of

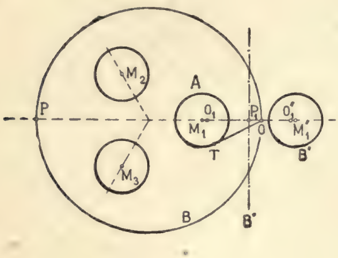


FIG. 396a.

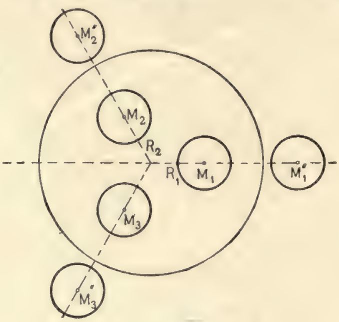


FIG. 396b.

simplicity, that O_1 coincides with M_1, O_2 with M_2, etc., the capacity of each phase becomes

$$C = \frac{0{\cdot}0242\epsilon l}{\log_{10} \dfrac{\overline{M_2R_1}}{\overline{M_1R_1}} - \log \dfrac{\overline{M''_2R_1}}{\overline{M''_1R_1}}}$$

$$= \frac{0{\cdot}0242\epsilon l}{\log_{10} \dfrac{\overline{M_2R_1}}{\overline{M_1R_1}} \cdot \dfrac{\overline{M''_1R_1}}{\overline{M''_2R_1}}} \text{ mfd.} \qquad (278)$$

We have thus reduced the capacity of a three-phase cable with separate conductors to that of three condensers of capacity C connected in star.

(e) In a two-phase system, without connection between the phases, it is found that the two phases are independent of each other as regards capacity and conduction; the same formulae therefore hold as in the case of a single-phase system. The capacity of each phase of a four-phase system (Fig. 397a) is obtained from the equivalent arrangement shown in Fig. 397b. For phase I III and phase II IV, the capacity is the same, and equals

$$C = \frac{0{\cdot}0242\epsilon l}{\log_{10} \dfrac{\overline{M_3R_1}}{\overline{M_1R_1}} \dfrac{\overline{M''_1R_1}}{\overline{M''_3R_1}}} \text{ mfd.}$$

For an interconnected two-phase system, two concentric cables are frequently used, of which the outer conductors are earthed and serve

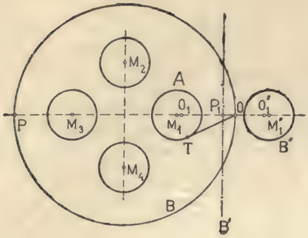


FIG. 397*a*.

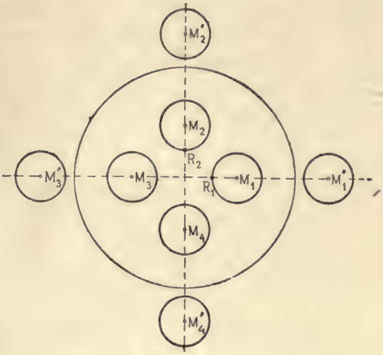


FIG. 387*b*.

as the middle wire. The capacity of such a cable can be determined by the above methods.

(f) As already mentioned, conduction alone is not a measure of the losses in cables and conductors. Losses are also present in the

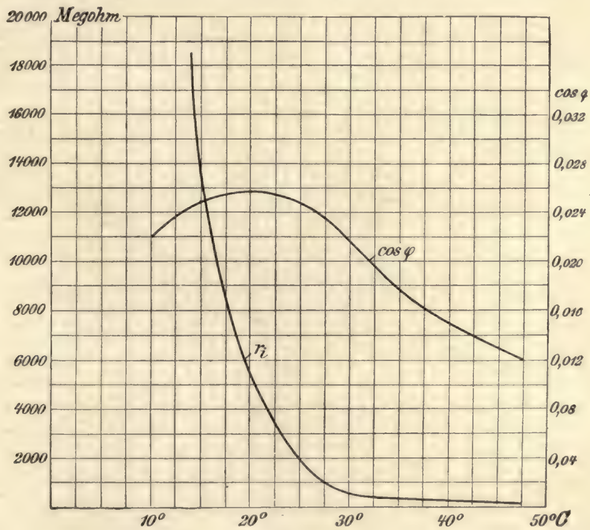


FIG. 398.

dielectric, which are much greater than those due to conduction, and act like an increase of the latter. Usually the losses in cables are estimated by assuming some definite power-factor. This was given

on p. 411 for different cables. It is evident that this method can only give approximate results, since the power-factor of a cable varies with the temperature, and to a certain extent with the frequency. Fig. 398 shows the variation of the power-factor as a function of the temperature from tests carried out by Dr. P. Human.* It might be thought that the variation of the power-factor is due to the variation of the insulation resistance r_i. This, however, is not the case; for the curve r in Fig. 398, giving the insulation resistance, falls very rapidly with increasing temperature, while the power-factor does not show a corresponding increase, but rises only for the lower temperatures and then falls as the temperature increases. *The power-factor*

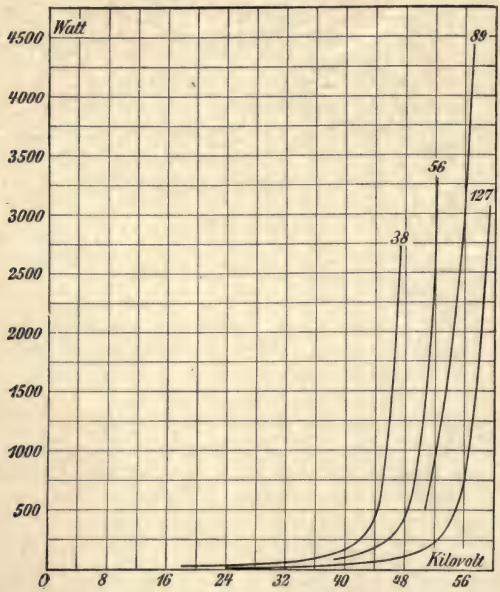


FIG. 399.

of a cable, therefore, bears no direct relation to its insulation resistance. In addition to a sufficiently high insulation resistance, it is usually required of a good alternating current cable that the power-factor at temperatures up to 50° C. must not rise appreciably above the value measured with the cable cold; also the ratio between the capacities measured with continuous and alternating currents at any temperature must not be very different from unity.

With bare overhead conductors also, the losses are considerably greater than those due to conduction. The extra losses here are due to the passage of current over the insulators, and to the

* *Elektr. Bahnen und Betriebe*, 1906, S. 518.

dielectric losses in the insulators and in the other electric fields. In damp weather a part of the electricity is also conducted directly by the moisture and rain. With high pressures, this latter loss may become very large, if the *critical pressure* is exceeded.

Fig. 399 shows the relation between the loss of power between two wires in air, at distances of 38, 56, 89 and 127 cm apart, and the effective alternating pressure, being the results of tests carried out by C. F. Scott and R. D. Mershon. The diameter of the wires was 4·1 mm. The losses are taken over 1 km double line. It is seen here that the critical pressure occurs at about 50,000 volts, since the curves bend sharply upwards at this point. The losses for other lines can be estimated from these curves. A double line of 8·2 mm wires, 250 cm apart at 100,000 volts pressure, for example, will have approximately the same losses as one of 4·1 mm wires, 127 cm apart at 50,000 volts.

134. Capacity of Coils in Air and in Iron. The capacity relations of coils in electric machinery and apparatus are very complicated. It is, however, possible to arrive at simple practical formulae, if we calculate with the capacity between elements of the conductor, as well as between the conductor and the earth. Theoretically this is not quite free from objection, but since an approximate formula is better than none at all, we shall now proceed to obtain such an expression. For the sake of brevity we shall denote, in the following, the expressions deduced on p. 390 for the mutual capacity coefficient by the term "capacity of a conductor-element."

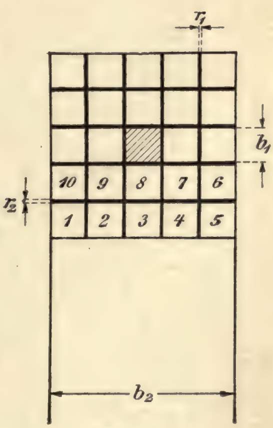


FIG. 400.

(a) Firstly, the capacity of a conductor-element will be calculated with regard to the neighbouring turns. In Fig. 369 a circular coil of flat copper strip is shewn. Such coils are frequently used. Each element of such a coil possesses capacity with regard to all the other turns of the coil, but only the capacities of the adjacent turns are of importance. If the insulation between the turns is thin compared with the thickness of the strip and has the dielectric constant ϵ, the capacity of an element of length 1 cm and breadth b cm equals

$$C_a = \frac{b\epsilon}{4\pi r} \text{ electrostat. units} = \frac{b\epsilon}{11 \cdot 3r} 10^{-6} \text{ mfd.}, \quad \ldots\ldots\ldots\ldots(279)$$

in which each element and the adjacent turn is considered as a plate-condenser with a thickness of insulation of r. This formula for the capacity of an element also holds for the case in which the coil is wound with flat copper strip on edge. If the coil consists of several layers of rectangular bars with n turns per layer, as shewn in section in Fig. 400,

then we estimate first the capacity of an element with regard to the adjacent turns in the same layer, which has the value

$$C_1 = \frac{b_1 \epsilon_1}{11 \cdot 3\, r_1} 10^{-6} \text{ mfd.},$$

and then the capacity of unit-length of a layer with regard to the next layer, which is approximately

$$C_2 = \frac{b_2 \epsilon_2}{11 \cdot 3\, r_2} 10^{-6} \text{ mfd.}$$

If the coil is wound as numbered in Fig. 400, the mean pressure between two adjacent layers is n times as great as that between two adjacent conductors. Since, however, on the other hand the capacity of a conductor with regard to the adjacent layer is only $\frac{1}{n}$ times the capacity between two layers, the capacity of an element with regard to the next turns can be written

$$C_d = C_1 + C_2. \quad \text{..........................(280)}$$

The capacity with regard to the turns on the other side is naturally of the same magnitude. For several coils arranged near one another, the above formulae for the capacity of an element hold very closely. This holds not only for round coils, but also for other shapes; the chief requirement for the accuracy of the formulae is that the distances r_1 and r_2 are small compared with the breadths b_1 and b_2. These formulae can even be used for stator coils with sufficient accuracy. Somewhat smaller values are obtained for the capacity with round wires than for rectangular, with equal thicknesses of insulation r_1 and r_2.

(b) The calculation of the capacity per element between coil and earth appears more difficult. For this reason we shall here also restrict ourselves to a mean value and put the mean capacity of an element with regard to the earth equal to the total capacity of the whole winding with regard to earth divided by the total length of winding. This has the advantage that magnitudes that can be directly measured are used in the calculation.

In a machine with Z slots of periphery U and length l, the capacity of the whole winding with regard to earth is

$$C \geqq \frac{ZUl\epsilon}{11 \cdot 3\, r} 10^{-6} \text{ mfd.} \quad \text{......................(281)}$$

where r is the thickness of the slot insulation (i.e. the distance between copper and iron) and ϵ its dielectric constant. The capacity is not much greater than that given by the right-hand side of the formula, since the coil-ends have very little capacity with regard to earth.

With transformer windings and choking-coils, the capacity with regard to earth is more difficult to calculate and depends so much on

the shape of the surface, that general formulae are too inaccurate and have therefore no value. The capacities of these windings can be calculated in any particular case with some accuracy, however, by using the formulae for plate and cylinder condensers.

(c) Losses, like those in the dielectrics of cables, also occur in electric machines; but still fewer measurements of these are available than of the foregoing. Skinner measured the dielectric losses in two 5000 K.W. generators made by the Westinghouse El. Mfg. Co., Pittsburg, for 11,000 volts maximum and 25 cycles. These values are plotted in Fig. 401

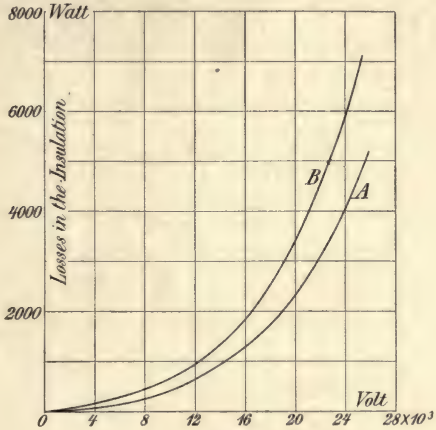


FIG. 401.

as a function of the test pressure. The lower curve A was measured on one machine with the winding at a temperature of about 21° C. and curve B on the other machine with the winding at about 31° C. At 25,000 volts the maximum loss was 0·021 watts per cm^3 of insulation, and this was not sufficient to raise the temperature of the insulation appreciably in 30 minutes.

Dr. P. Hollitscher * measured the dielectric losses on two machines made by the Lahmeyerwerke, Frankfurt, for 500 H.P. and 400 K.W., 10,000 volts, 50 cycles. These are shewn by curves A and B in Fig. 402. This test shews that the losses increase practically proportionally to the cube (instead of the square) of the pressure, which may be due to a certain extent to a discharge of electricity from the coil-ends at higher pressures. Dr. Hollitscher found further, that the losses increase proportionally to the frequency. Also the test shews that the capacity increases with the pressure, i.e. with the electric field-strength; this corresponds to an increase in the dielectric constant. The slot insulation of the machines consisted of micanite

* *E.T.Z.* 1903, S. 635.

tubes, and tests upon these gave the figures shewn in Fig. 403, the dielectric constant increasing from 2·8 in one case and 2·2 in the other at normal pressure to about 5 at double pressure. On the

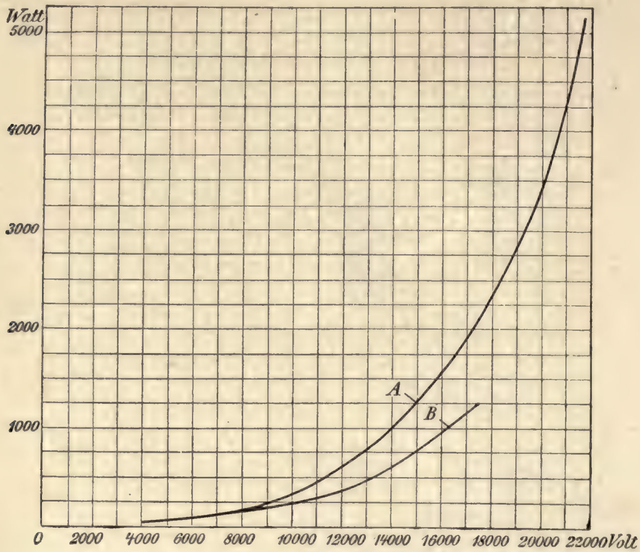


FIG. 402.

other hand a variation of frequency shewed no appreciable effect on the dielectric constant.

Care must also be taken in electric machines and transformers, that the electric field-strength is at no place so great that the insulating material is injured thereby, an effect which may happen even if no

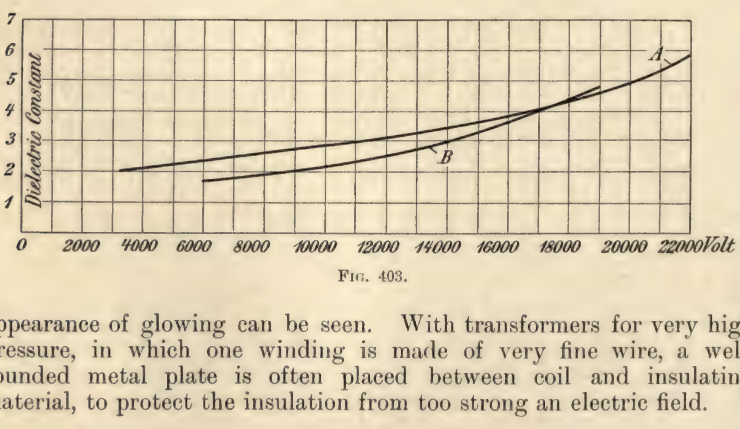


FIG. 403.

appearance of glowing can be seen. With transformers for very high pressure, in which one winding is made of very fine wire, a well-rounded metal plate is often placed between coil and insulating material, to protect the insulation from too strong an electric field.

Besides this, care must be taken in the choice of insulating material in high tension machines, to see that they can withstand the mechanical forces of attraction between the copper and iron, which form the two plates of a condenser. For this reason soft materials should always be avoided.

Up to the present no insulating material has yet been found which can wholly withstand continuously the simultaneous effects of heat and electro-mechanical stresses, as well as the chemical effect of the nitrates formed in high tension machines. Most insulating materials change their structure in time; nevertheless, they still come up to the requirements, because initially they have been rated very liberally.

135. Telegraph and Telephone Lines. As is well known, the transmission of signs in telegraphy is effected by means of unidirectional currents, obtained from any source. The telephonic transmission of speech on the other hand makes use of alternating-currents, induced in the secondary windings of induction coils. The differences in the construction of the lines, especially of cables, is due to this difference in the kind of current. For the same reason the influences of power cables on telephone and telegraph lines are different.

(**a**) *Telegraph lines.* Air-lines are usually made of galvanised iron wire of 3 to 7 mm diameter or of 3 mm bronze wire. Cables placed underground usually contain many wires, and are insulated with either gutta-percha or jute and paper. The strands of gutta-percha cables are made up of several (up to 14) twisted copper wires 0·7 mm diameter, while the wires of cables with fibrous insulation are 1·5 mm diameter.

Submarine cables are always made with a single core, insulated with gutta-percha and heavily armoured against the great mechanical stresses. The resistance of these cables varies between 2 and 6 ohms, the insulation resistance between 500 and 1250×10^6 ohms and the capacity between 0·2 and 0·15 mfd, per km length. With overhead conductors and short cables, which require only very small charging currents, the current at the receiving station follows immediately on the closing of the circuit by the key, and up to 1000 words of five letters can be transmitted per minute. With long submarine cables, the charging current is so great, that an appreciable time elapses before the cable is fully charged, and the rush of current is noticeable at the receiving station. With long submarine cables, therefore, the charging waves are used as signals. The number of possible signals, i.e. current-waves, per minute depends chiefly on the capacity and the resistance of the cable, and only to a small extent on the conduction and self-induction. As a first approximation the product (rC) of resistance and capacity per km length of line serves as a measure of the signalling-speed of a telegraph line. With underground cables the greatest signalling-speed is obtained, when the outside diameter over the insulation of each conductor is 1·65 times the diameter of the bare conductor. Taking mechanical strength into consideration, however, the outside diameter is made 2 to 4 times the bare diameter.

(b) *Telephone lines.* Air-lines are usually made of silicon-bronze wire 1·5 to 5 mm in diameter, according to the distance. Recently, also, for very long lines, double lines are frequently used to eliminate external disturbances. When several double lines are fixed to the same poles, they are arranged as shewn in Fig. 404, as suggested by Christiani. In this way adjacent double lines do not induce any currents in each other.

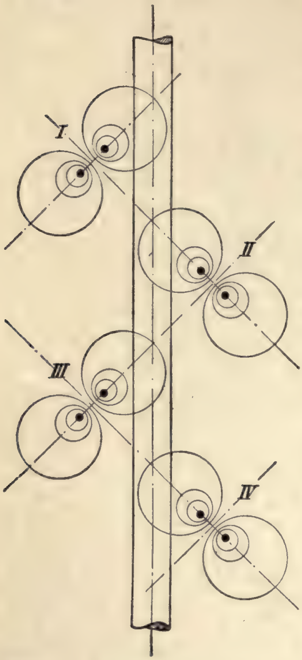


FIG. 404.—Non-inductive Arrangement of Telephone Lines.

Telephone cables consist of many conductors and are usually insulated with paper. Since the capacity must be as small as possible—in modern double-line cables it should not be more than 0·05 mfd per km—the paper is either perforated or arranged in such a way that there are air-spaces round the conductors. On account of the capacity, the diameter of the wire is chosen larger, the longer the cable is, and the usual diameter ranges from 0·8 to 2·0 mm. Telephone cables are laid either in iron tubes or cement troughs. In order to further eliminate the effect of the capacity in very long lines, small induction coils are connected in the lines at certain distances as suggested by Pupin, or the self-induction of the line is increased by wrapping it round with iron wire. The damping of an alternating-current in a long line is proportional to ϵ^{-at}, where the damping-factor

$$a = \frac{r_d}{2L_d} + \frac{g_l}{2C_l},$$

and

$$t = l_2\sqrt{L_d C_l}$$

is the time the current takes to traverse the length l_2 of the line. Hence we have

$$at = \frac{l_2 r_d}{2}\sqrt{\frac{C_l}{L_d}} + \frac{l_2 g_l}{2}\sqrt{\frac{L_d}{C_l}} = \frac{R}{2}\sqrt{\frac{C_l}{L_d}} + \frac{G}{2}\sqrt{\frac{L_d}{C_l}}, \quad \text{.........(282)}$$

where R is the total resistance and G the total conduction of the telephone line, while C_l is the capacity and L_d the self-induction per km length. To make at and therefore the damping of the telephone currents as small as possible, we must have the following relation between the four constants of the line

$$\frac{r_d}{L_d} = \frac{g_l}{C_l},$$

which is also the condition for a line free from distortion (p. 137). Since the self-induction of an ordinary telephone line is smaller than the value given by the formula, Pupin's coils are connected or the line is bound with iron wire, in order to raise the self-induction artificially. In this calculation the constants of the line r_a, L_a, g_l and C_l, measured with continuous current, will not serve. The frequency of the alternating-currents, occurring in telephony, varies over a fairly wide range. Usually 1000 cycles per second is reckoned as a mean value, and hence the constants of the line are measured at a frequency of 1000.

(c) *Effect of power-circuits on telegraph and telephone lines.* If power and signalling lines run close together, the heavy currents may disturb the weak currents. These disturbances are of different kinds and are due either to (1) direct conduction of current, (2) electromagnetic induction, or (3) electrostatic induction. To avoid direct conduction of current, both lines must be carefully insulated. With electric railways in which the rails serve as return, it is desirable on this account to use double lines for parallel telegraph lines, in order to avoid as far as possible a transference of current, due to the pressure-drop in the rails.

The pressures induced in the low-current lines by the electromagnetic fields of the heavy currents are usually small, and can be calculated from the formulae on p. 427. To make the E.M.F.'s induced by electromagnetic induction harmless, it is of advantage to cross the feeble-current lines on every third or fifth pole.

In general telephone lines are disturbed by static charges. These can be calculated from the formulae given in Section 134 as the product of the electric potential and the mutual capacity of the line.

These charging currents, however, can easily be eliminated from telephone lines, by leading them to earth through a special choking-coil connected between the two lines. The terminals of the choking-coil are connected to the two telephone lines and the middle point is earthed. The choking-coil offers a high inductive resistance to a current from line to line, whilst it provides only a very small inductive resistance from the line to earth. Such a choking-coil cannot be used for telegraph lines, since in this case the current is continuous and can therefore pass through the choking-coil to earth without any high resistance. By using high-pressure continuous current (120 volts) for telegraphy, the disturbance from electrostatic charging currents can be made almost entirely harmless.

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